Supplement to "Nonparametric probability bounds for Nash equilibrium actions in a simultaneous discrete game": Proofs and extensions to nonequilibrium behavior

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Appendix A: Proofs

Proof of Result 2. Take a given ω. In any NE π, the *expected utility* for agent p of choosing $Y_p = y_p$ (for any $y_p \in \mathcal{A}_p$) is given by

$$\sum_{\mathbf{y}_{-p} \in \mathcal{A}_{-p}} \pi_{-p}(\mathbf{y}_{-p}) \cdot \nu_p(y_p, \mathbf{y}_{-p}; \omega_p).$$

Using Assumption 1(i), for any well defined probability function $\pi_{-p} : \mathcal{A}_{-p} \longrightarrow [0, 1]$, we have

$$\sum_{\mathbf{y}_{-p} \in \mathcal{A}_{-p}} \pi_{-p}(\mathbf{y}_{-p}) \cdot [\nu_p(y_p, \mathbf{y}_{-p}; \omega_p) - \nu_p(y_p - 1, \mathbf{y}_{-p}; \omega_p)]$$
$$> \sum_{\mathbf{y}_{-p} \in \mathcal{A}_{-p}} \pi_{-p}(\mathbf{y}_{-p}) \cdot [\nu_p(y_p + 1, \mathbf{y}_{-p}; \omega_p) - \nu_p(y_p, \mathbf{y}_{-p}; \omega_p)]$$

for any $y_p \in \mathcal{A}_p$. It follows that in any NE π, agent p can only be optimally indifferent between *at most two actions, which must be adjacent*. This occurs if and only if π_{-p} is such that $\sum_{\mathbf{y}_{-p} \in \mathcal{A}_{-p}} \pi_{-p}(\mathbf{y}_{-p}) \cdot [\nu_p(y_p, \mathbf{y}_{-p}; \omega_p) - \nu_p(y_p + 1, \mathbf{y}_{-p}; \omega_p)] = 0$. In addition to strict concavity, this result relies crucially on the independent mixing nature of NE. To see why, consider an alternative equilibrium concept where agents can coordinate in their mixing distributions (e.g., correlated equilibrium) and thus independent mixing no longer holds. The expected utility of choosing y_p is of the more general form

$$\sum_{\mathbf{y}_{-p} \in \mathcal{A}_{-p}} \pi_{-p}(\mathbf{y}_{-p} | y_p) \cdot \nu_p(y_p, \mathbf{y}_{-p}; \omega_p).$$

Consider any subset $\mathcal{C}_p \subseteq \mathcal{A}_p$. Since $\pi_{-p}(\cdot | y_p)$ is allowed to differ from $\pi_{-p}(\cdot | y'_p)$, we can find payoff functions that satisfy our assumptions and a collection of mixing distribu-

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tions such that

$$\sum_{\mathbf{y}_{-p}\in\mathcal{A}_{-p}} \pi_{-p}(\mathbf{y}_{-p}|y_p)\cdot \nu_p(y_p,\mathbf{y}_{-p};\omega_p) = \sum_{\mathbf{y}_{-p}\in\mathcal{A}_{-p}} \pi_{-p}(\mathbf{y}_{-p}|y'_p)\cdot \nu_p(y'_p,\mathbf{y}_{-p};\omega_p)$$
$$\forall y_p, y'_p \in \mathcal{C}_p$$

and

$$\sum_{\mathbf{y}_{-p}\in\mathcal{A}_{-p}} \pi_{-p}(\mathbf{y}_{-p}|y_p)\cdot \nu_p(y_p,\mathbf{y}_{-p};\omega_p) \geq \sum_{\mathbf{y}_{-p}\in\mathcal{A}_{-p}} \pi_{-p}(\mathbf{y}_{-p}|y)\cdot \nu_p(y,\mathbf{y}_{-p};\omega_p)$$
$$\forall y_p \in \mathcal{A}_p, \forall y \notin \mathcal{C}_p.$$

Therefore, without independent mixing, we can find payoff functions consistent with our assumptions such that agents are optimally indifferent across an arbitrary subset of actions in $\mathcal{A}_p$. Strict concavity is key to our result. If we relax it to weak concavity, we can characterize payoff functions consistent with our remaining assumptions which generate complete-information NE where agents choose more than two actions with positive probability. □

Proof of Proposition 1. If $b_p = a_p + 1$, any NE with support $\mathcal{S}$ requires p to randomize across a_p and $a_p + 1$. This in turn requires that

$$\sum_{\mathbf{y}_{-p}\in\mathcal{S}} \pi_{-p}(\mathbf{y}_{-p})\cdot [\nu_p(a_p,\mathbf{y}_{-p};\omega_p) - \nu_p(a_p+1,\mathbf{y}_{-p};\omega_p)]$$
$$\text{for some } \pi_{-p}: \quad \sum_{\mathbf{y}_{-p}\in\mathcal{S}} \pi_{-p}(\mathbf{y}_{-p}) = 1. \tag{A-1}$$

That is, it must be possible for p to be optimally indifferent between a_p and $a_p + 1$ for some mixing distribution $\pi_{-p}:\mathcal{A}_{-p} \longrightarrow [0,1]$ with support $\mathcal{S}$. If the conditions in part (i) of Proposition 1 are not satisfied, Assumptions 1 and 2 imply that no such π_{-p} can exist. For instance, if $\nu_p^u(a_p,\mathcal{S};\omega_p) > \nu_p^u(a_p+1,\mathcal{S};\omega_p)$ (we can ignore weak inequalities by Assumption 1(ii)), then Assumption 2 and the definition of ν_p^u imply that $\nu_p(a_p,\mathbf{y}_{-p};\omega_p) > \nu_p(a_p+1,\mathbf{y}_{-p};\omega_p)\ \forall \mathbf{y}_{-p}\in\mathcal{S}$. Consequently, the left-hand side of (A-1) is strictly positive for any π_{-p} with support $\mathcal{S}$. If $\nu_p^\ell(a_p,\mathcal{S};\omega_p) < \nu_p^\ell(a_p+1,\mathcal{S};\omega_p)$, the definition of ν_p^ℓ and Assumption 2 imply that the left-hand side of (A-1) must be strictly negative for any such π_{-p}. Therefore, both restrictions in (i) must hold for each p for whom $b_p = a_p + 1$. If $b_p = a_p$, any NE with support $\mathcal{S}$ requires p to play a_p as a pure strategy. This requires that

$$a_p \in \underset{y_p\in\mathcal{A}_p}{\arg\max}\left\{\sum_{\mathbf{y}_{-p}\in\mathcal{S}} \pi_{-p}(\mathbf{y}_{-p})\cdot \nu_p(y_p,\mathbf{y}_{-p};\omega_p)\right\}$$
$$\text{for some } \pi_{-p}: \quad \sum_{\mathbf{y}_{-p}\in\mathcal{S}} \pi_{-p}(\mathbf{y}_{-p}) = 1. \tag{A-2}$$

That is, a_p must be an optimal choice for some mixing distribution π_{-p} with support $\mathcal{S}$. If the conditions in part (ii) of Proposition 1 are not satisfied, Assumptions 1 and 2 imply that no such π_{-p} can exist. For instance, suppose $\nu_p^u(a_p - 1, \mathcal{S}; \omega_p) > \nu_p^u(a_p, \mathcal{S}; \omega_p)$. By Assumption 1(i) (concavity), we must also have $\nu_p^u(a_p, \mathcal{S}; \omega_p) > \nu_p^u(a_p + 1, \mathcal{S}; \omega_p)$. From here, Assumption 2 and the definition of ν_p^u imply that $\nu_p(a_p - 1, \mathbf{y}_{-p}; \omega_p) > \nu_p(a_p, \mathbf{y}_{-p}; \omega_p)$ and $\nu_p(a_p, \mathbf{y}_{-p}; \omega_p) > \nu_p(a_p + 1, \mathbf{y}_{-p}; \omega_p)$ for any $\mathbf{y}_{-p} \in \mathcal{S}$. Therefore, by concavity, any expected-utility maximizing choice as described in (A-2) is bounded above by $a_p - 1$. Finally, suppose $\nu_p^\ell(a_p, \mathcal{S}; \omega_p) < \nu_p^\ell(a_p + 1, \mathcal{S}; \omega_p)$ for some p such that $a_p = b_p$. Assumptions 1 and 2 and the definition of ν_p^ℓ now imply that $\nu_p(a_p - 1, \mathbf{y}_{-p}; \omega_p) < \nu_p(a_p, \mathbf{y}_{-p}; \omega_p)$ and $\nu_p(a_p, \mathbf{y}_{-p}; \omega_p) < \nu_p(a_p + 1, \mathbf{y}_{-p}; \omega_p)$ for any $\mathbf{y}_{-p} \in \mathcal{S}$. From here, concavity implies that any expected-utility maximizing choice as described in (A-2) must be bounded below by $a_p + 1$. Therefore, both restrictions in (ii) must hold for each p for whom $b_p = a_p$. □

Proof of Proposition 2. The restrictions in Proposition 2 follow straightforwardly from Proposition 1 and Assumptions 1 and 2. Consider, for example, Case I, where p is required to randomize between a_p and $a_p + 1$ in the first NE, and between a'_p and $a'_p + 1$ in the second NE. From part (i) of Proposition 1, we must have $\nu_p^u(a_p, \mathcal{S}; \omega_p) < \nu_p^u(a_p + 1, \mathcal{S}; \omega_p)$ and $\nu_p^\ell(a'_p, \mathcal{S}'; \omega_p) > \nu_p^\ell(a'_p + 1, \mathcal{S}'; \omega_p)$. If $\nu_p^\ell(\cdot, \mathcal{S}'; \omega_p) \geq \nu_p^u(\cdot, \mathcal{S}; \omega_p)$ and payoffs are concave, then having $a'_p \leq a_p$ implies a violation of Assumption 2. Thus, in this case we must have $a'_p > a_p$. Part (i) of Proposition 1 also requires $\nu_p^u(a'_p, \mathcal{S}'; \omega_p) < \nu_p^u(a'_p + 1, \mathcal{S}'; \omega_p)$ and $\nu_p^\ell(a_p, \mathcal{S}; \omega_p) > \nu_p^\ell(a_p + 1, \mathcal{S}; \omega_p)$. Thus, if $\nu_p^u(\cdot, \mathcal{S}'; \omega_p) \leq \nu_p^\ell(\cdot, \mathcal{S}; \omega_p)$, concavity now requires $a'_p < a_p$ or Assumption 2 is violated. Now consider Case II, where p is required to randomize between a_p and $a_p + 1$ in the first NE, and play a'_p as a pure strategy in the second NE. The restrictions just described above for the first NE must still hold, while part (ii) of Proposition 1 now requires that $\nu_p^u(a'_p - 1, \mathcal{S}'; \omega_p) < \nu_p^u(a'_p, \mathcal{S}'; \omega_p)$ and $\nu_p^\ell(a'_p, \mathcal{S}'; \omega_p) > \nu_p^\ell(a'_p + 1, \mathcal{S}'; \omega_p)$. Combining both sets of restrictions, we can see that if payoffs are concave and $\nu_p^\ell(\cdot, \mathcal{S}'; \omega_p) \geq \nu_p^u(\cdot, \mathcal{S}; \omega_p)$, Assumption 2 is satisfied only if $a'_p > a_p$, while if $\nu_p^u(\cdot, \mathcal{S}'; \omega_p) \leq \nu_p^\ell(\cdot, \mathcal{S}; \omega_p)$, Assumption 2 requires $a'_p \leq a_p$. Parallel arguments can be used for Case III. In Case IV, p is required to play a_p as a pure strategy in the first NE and a'_p in the second NE. Using the restrictions from part (ii) of Proposition 1, we see that if payoffs are concave and $\nu_p^\ell(\cdot, \mathcal{S}'; \omega_p) \geq \nu_p^u(\cdot, \mathcal{S}; \omega_p)$, Assumption 2 holds only if $a'_p \geq a_p$. If $\nu_p^u(\cdot, \mathcal{S}'; \omega_p) \leq \nu_p^\ell(\cdot, \mathcal{S}; \omega_p)$, we must have $a'_p \leq a_p$. This proves the restrictions described in Cases I–IV. Finally, it is not difficult to show that if either $\nu_p^\ell(\cdot, \mathcal{S}'; \omega_p) < \nu_p^u(\cdot, \mathcal{S}; \omega_p)$ and $\nu_p^u(\cdot, \mathcal{S}'; \omega_p) > \nu_p^\ell(\cdot, \mathcal{S}; \omega_p)$ or if the restrictions established above are satisfied, then satisfaction of the conditions described in Proposition 1 for both $\mathcal{S}$ and $\mathcal{S}'$ is entirely compatible with the payoff shape restrictions implied by Assumptions 1 and 2. Thus, if the set of restrictions in Proposition 2 is satisfied, then the necessary conditions for the coexistence of NE with supports $\mathcal{S}$ and $\mathcal{S}'$ are compatible with Assumptions 1 and 2. □

Proof of Proposition 4. Part (i) follows directly from Proposition 3. The upper bound of 1 in part (ii) is attained, for example, if each agent has a dominant action with probability 1 (see Remark 3 and Equation (3)). If this is the case, the game possesses a unique

NE w.p.1, and it must be in pure strategies. Thus, having $\mathbf{y} \in \mathscr{E}(\omega)$ implies trivially that $\mathcal{M}_{\mathscr{E}}$ must select $\mathbf{y}$ and the upper bound of 1 in part (ii) follows. Next, let

$$\overline{\pi}^*(\mathbf{y}) = \max\{\pi^*(\mathbf{y}) : \mathbf{y} \in \mathscr{E}(\omega) \text{ and } \pi^* \text{ is a NE}\},$$

that is, the largest probability of playing $\mathbf{y}$ in any NE that includes $\mathbf{y}$ in its support. We have

$$\begin{aligned}\Pr[\mathbf{Y} = \mathbf{y} | \mathbf{y} \in \mathscr{E}(\omega), X] &\leq \overline{\pi}^*(\mathbf{y}) \cdot \Pr[\mathcal{M}_{\mathscr{E}} \text{ selects a NE } \pi^* : \pi^*(\mathbf{y}) > 0 | \mathbf{y} \in \mathscr{E}(\omega), X] \\ &\equiv \overline{\pi}^*(\mathbf{y}) \cdot Q_{\mathscr{E}}(\mathbf{y}, X).\end{aligned}$$

Take any $\tau \in (0, 1)$. Given the inferential setting assumed here, if $E[\mathbb{I}^{\mathscr{E}}(\mathbf{Y}, \mathbf{y}; \omega) | X] > 0$, it is impossible for the researcher to determine whether $\overline{\pi}^*(\mathbf{y}) \leq \tau$. In other words, it is impossible in our inferential environment to determine a *valid* upper bound (other than 1) for the probability of playing $\mathbf{y}$ in a NE. From here we conclude that $\Pr[\mathbf{Y} = \mathbf{y} | \mathbf{y} \in \mathscr{E}(\omega), X]$ is a sharp lower bound for $Q_{\mathscr{E}}(\mathbf{y}, X)$. Since $\Pr[\mathbf{Y} = \mathbf{y} | X] = \Pr[\mathbf{Y} = \mathbf{y} \text{ and } \mathbf{y} \in \mathscr{E}(\omega) | X]$ (by Assumption 3), we have $\Pr[\mathbf{Y} = \mathbf{y} | \mathbf{y} \in \mathscr{E}(\omega), X] = \frac{\Pr[\mathbf{Y}=\mathbf{y}|X]}{\Pr[\mathbf{y} \in \mathscr{E}(\omega)|X]} \equiv \frac{\Pr[\mathbf{Y}=\mathbf{y}|X]}{P_{\mathscr{E}}(\mathbf{y}, X)}$. The lower bound in (ii) follows from here and the upper bound in part (i). □

Appendix B: Extensions beyond complete-information NE behavior

If we maintain Assumptions 1, 2, and 4, along with expected-utility maximizing behavior, we can extend our approach to a non-NE behavior setting *as long as we are willing to* take the following steps:

- Impose a prespecified bound on the width of the support of agents' beliefs (see below).
- Maintain that agents' beliefs are still consistent with independent mixing.
- Maintain that beliefs assign positive probability to the actual outcome $\mathbf{Y}$ of the game.

Let $\|\mathbf{w}\|_\infty = \max(|w_1|, \ldots, |w_n|)$ and fix $\kappa \geq 0$. For each p, define

$$\begin{aligned}\Pi^\kappa_{-p} = \big\{&\pi_{-p} : \mathcal{A}_{-p} \longrightarrow [0, 1] : \pi_{-p} \text{ is a probability function on } \mathcal{A}_{-p}, \text{ and} \\ &\|\mathbf{y}_{-p} - \mathbf{y}'_{-p}\|_\infty \leq \kappa \ \forall \mathbf{y}_{-p}, \mathbf{y}'_{-p} : \pi_{-p}(\mathbf{y}_{-p}) > 0 \text{ and } \pi_{-p}(\mathbf{y}'_{-p}) > 0\big\}.\end{aligned}$$

Beliefs in Π^κ_{-p} imply that p thinks that opponents do not randomize across actions that are more than κ units apart. By Result 2, complete-information NE behavior is a special case with $\kappa = 1$. Let

$$\begin{aligned}\mathcal{R}^\kappa(\omega) = \Bigg\{&\mathbf{y} \equiv (y_p)_{p=1}^P \in \mathcal{A} : y_p \in \arg\max_{y \in \mathcal{A}_p} \sum_{\mathbf{v}_{-p} \in \mathcal{A}_{-p}} \widetilde{\pi}_{-p}(\mathbf{v}_{-p}) \cdot \nu_p(y, \mathbf{v}_{-p}; \omega_p) \\ &\text{for some collection of beliefs } (\widetilde{\pi}_{-p})_{p=1}^P \text{ such that } \widetilde{\pi}_{-p} \in \Pi^\kappa_{-p} \\ &\text{and } \widetilde{\pi}_{-p}(\mathbf{y}_{-p}) > 0 \ \forall p\Bigg\}.\end{aligned}$$

That is, $\mathcal{R}^\kappa(\omega)$ is the collection of profiles $\mathbf{y}$ in $\mathcal{A}$ that can be simultaneously rationalized as expected-utility maximizing choices for some collection of beliefs (not necessarily correct) which, for each p, (i) are consistent with independent mixing ($\widetilde{\pi}_{-p}$ does not depend on p's own choice), (ii) belong in Π^κ_{-p}, and (iii) assign nonzero probability to $\mathbf{y}_{-p}$. Note that $\mathcal{R}^\kappa(\omega) \subseteq \mathcal{R}^{\kappa'}(\omega)$ for any $\kappa' > \kappa$ by construction. Also note that $\mathscr{E}(\omega) \subseteq \mathcal{R}^\kappa(\omega)$ for any $\kappa \geq 1$. Thus, $\mathcal{R}^\kappa(\omega)$ includes every NE action profile for any $\kappa \geq 1$. Consider the following assumption.

Assumption 3$''$. *$\mathbf{Y} \in \mathcal{R}^\kappa(\omega)$ w.p.1 for some κ assumed to be known by the researcher.*

Assumption 3$''$ is similar to NE behavior[20] in two ways. First, while they can be incorrect, beliefs are still assumed to include others' ex post choices in their support. Second, beliefs are still essentially required to satisfy independent mixing. However, for relatively large values of κ, Assumption 3$''$ is capable of encompassing important deviations from Assumption 3. For instance, it can be compatible with complete- as well as incomplete-information settings. For any $\mathbf{y}_{-p} \in \mathcal{A}_{-p}$ and given κ, let

$$\nu_p^{u_\kappa}(\cdot, \mathbf{y}_{-p}; \omega_p) = \max\{\nu_p(\cdot, \mathbf{v}_{-p}; \omega_p) : \|\mathbf{v}_{-p} - \mathbf{y}_{-p}\|_\infty \leq \kappa\},$$
$$\nu_p^{\ell_\kappa}(\cdot, \mathbf{y}_{-p}; \omega_p) = \min\{\nu_p(\cdot, \mathbf{v}_{-p}; \omega_p) : \|\mathbf{v}_{-p} - \mathbf{y}_{-p}\|_\infty \leq \kappa\}.$$

Result 3. *Take any $\mathbf{y} \equiv (y_p)_{p=1}^P$ in $\mathcal{A}$. If Assumptions 1 and 2 hold, then $\mathbf{y} \in \mathcal{R}^\kappa(\omega)$ only if*

$$\nu_p^{u_\kappa}(y_p - 1, \mathbf{y}_{-p}; \omega_p) < \nu_p^{u_\kappa}(y_p, \mathbf{y}_{-p}; \omega_p) \quad \textit{and}$$
$$\nu_p^{\ell_\kappa}(y_p, \mathbf{y}_{-p}; \omega_p) > \nu_p^{\ell_\kappa}(y_p + 1, \mathbf{y}_{-p}; \omega_p) \quad \forall p.$$

Proof. By Assumptions 1 and 2, having $\nu_p^{u_\kappa}(y_p - 1, \mathbf{y}_{-p}; \omega_p) > \nu_p^{u_\kappa}(y_p, \mathbf{y}_{-p}; \omega_p)$ implies

$$y^* \leq y_p - 1 \quad \forall y^* \in \left\{ \underset{y \in \mathcal{A}_p}{\arg\max} \sum_{\mathbf{v}_{-p} \in \mathcal{A}_{-p}} \widetilde{\pi}_{-p}(\mathbf{v}_{-p}) \cdot \nu_p(y, \mathbf{v}_{-p}; \omega_p) \right\},$$
$$\forall \widetilde{\pi}_{-p} \in \Pi^\kappa_{-p} : \widetilde{\pi}_{-p}(\mathbf{y}_{-p}) > 0.$$

Additionally, y_p cannot be expected-utility maximizing for any beliefs in Π^κ_{-p} that assign positive probability to $\mathbf{y}_{-p}$. Thus, $\mathbf{y} \notin \mathcal{R}^\kappa(\omega)$. If $\nu_p^{\ell_\kappa}(y_p, \mathbf{y}_{-p}; \omega_p) < \nu_p^{\ell_\kappa}(y_p + 1, \mathbf{y}_{-p}; \omega_p)$, every expected-utility maximizing action y^* for any such set of beliefs must satisfy $y^* \geq y_p + 1$. Thus $\mathbf{y} \notin \mathcal{R}^\kappa(\omega)$. $\square$

This is analogous to the equations in Proposition 1. If we add Assumption 3$''$, we obtain the following proposition:

Proposition 6. *Take any $\mathbf{y} \equiv (y_p)_{p=1}^P$ in $\mathcal{A}$. If Assumptions 1, 2, and 3$''$ hold, then $\mathbf{y} \in \mathcal{R}^\kappa(\omega)$ only if, for each p, either $\nu_p^{u_\kappa}(\cdot, \mathbf{y}_{-p}; \omega_p) > \nu_p^{\ell_\kappa}(\cdot, \mathbf{Y}_{-p}; \omega_p)$ and $\nu_p^{\ell_\kappa}(\cdot, \mathbf{y}_{-p}; \omega_p) <$*

[20] A previous draft of this paper referred to this as *approximate equilibrium* behavior.

$\nu_p^{u_\kappa}(\cdot, \mathbf{Y}_{-p}; \omega_p)$ *or one of the following conditions holds*:

(i) *If* $\nu_p^{u_\kappa}(\cdot, \mathbf{y}_{-p}; \omega_p) \leq \nu_p^{\ell_\kappa}(\cdot, \mathbf{Y}_{-p}; \omega_p)$, *then* $y_p \leq Y_p$.

(ii) *If* $\nu_p^{\ell_\kappa}(\cdot, \mathbf{y}_{-p}; \omega_p) \geq \nu_p^{u_\kappa}(\cdot, \mathbf{Y}_{-p}; \omega_p)$, *then* $y_p \geq Y_p$.

Furthermore, for any given ω, *if the above restrictions are satisfied, then Assumptions* 1 *and* 2 *are entirely compatible with the necessary conditions described in Result* 3. *Let*

$$\begin{aligned}\mathbb{I}_p^{\mathcal{R}^\kappa}(\mathbf{Y}, \mathbf{y}; \omega_p) &= 1 - \mathbb{1}\{\nu_p^{u_\kappa}(\cdot, \mathbf{y}_{-p}; \omega_p) \leq \nu_p^{\ell_\kappa}(\cdot, \mathbf{Y}_{-p}; \omega_p)\} \cdot \mathbb{1}\{y_p > Y_p\} \\ &\quad - \mathbb{1}\{\nu_p^{\ell_\kappa}(\cdot, \mathbf{y}_{-p}; \omega_p) \geq \nu_p^{u_\kappa}(\cdot, \mathbf{Y}_{-p}; \omega_p)\} \cdot \mathbb{1}\{Y_p > y_p\}, \\ \mathbb{I}^{\mathcal{R}^\kappa}(\mathbf{Y}, \mathbf{y}; \omega) &= \min_{p=1,\ldots,P}\{\mathbb{I}_p^{\mathcal{R}^\kappa}(\mathbf{Y}, \mathbf{y}; \omega_p)\},\end{aligned}$$

where $\mathbb{I}^{\mathcal{R}^\kappa}(\mathbf{Y}, \mathbf{y}; \omega)$ *is the indicator function for the event that the conditions in Proposition* 6(i) *and* (ii) *are satisfied for each* p. *We have*

$$\mathbb{1}\{\mathbf{Y} = \mathbf{y}\} \leq \mathbb{1}\{\mathbf{y} \in \mathcal{R}^\kappa(\omega)\} \leq \mathbb{I}^{\mathcal{R}^\kappa}(\mathbf{Y}, \mathbf{y}; \omega). \tag{B-1}$$

Our assumptions are compatible with the conditions needed for either of these bounds to be attained.

Proof. The first inequality in (B-1) follows directly from Assumption $3''$. The second inequality follows from Assumptions 1 and 2 along with Result 3. Having $\mathbb{1}\{\mathbf{Y} = \mathbf{y}\} = \mathbb{1}\{\mathbf{y} \in \mathcal{R}^\kappa(\omega)\}$ arises under (3), where each agent has a strictly dominant action w.p.1. The fact that our assumptions are compatible with a model where $\mathbb{1}\{\mathbf{y} \in \mathcal{R}^\kappa(\omega)\} = \mathbb{I}^{\mathcal{R}^\kappa}(\mathbf{Y}, \mathbf{y}; \omega)$ w.p.1 follows from the first part of the proposition (which establishes that having $\mathbb{I}^{\mathcal{R}^\kappa}(\mathbf{Y}, \mathbf{y}; \omega) = 1$ is compatible with the conditions in Result 3 under our payoff assumptions) and the discussion in Remark 3. □

Function $\mathbb{I}^{\mathcal{R}^\kappa}$ is unobservable. We construct feasible bounds from (B-1) through Assumption 4. Let

$$\begin{aligned}\mathbb{H}_p^\kappa(\mathbf{Y}_{-p}, \mathbf{y}_{-p}) &= \mathbb{1}\{f_p(\mathbf{u}_{-p}) \geq f_p(\mathbf{v}_{-p})\ \forall \|\mathbf{u}_{-p} - \mathbf{Y}_{-p}\|_\infty \leq \kappa, \|\mathbf{v}_{-p} - \mathbf{y}_{-p}\|_\infty \leq \kappa\}, \\ \mathbb{H}_p^\kappa(\mathbf{y}_{-p}, \mathbf{Y}_{-p}) &= \mathbb{1}\{f_p(\mathbf{u}_{-p}) \geq f_p(\mathbf{v}_{-p})\ \forall \|\mathbf{u}_{-p} - \mathbf{y}_{-p}\|_\infty \leq \kappa, \|\mathbf{v}_{-p} - \mathbf{Y}_{-p}\|_\infty \leq \kappa\}.\end{aligned}$$

Let

$$\begin{aligned}\widetilde{\mathbb{I}}_p^{\mathcal{R}^\kappa}(\mathbf{Y}, \mathbf{y}) &= 1 - \mathbb{H}_p^\kappa(\mathbf{y}_{-p}, \mathbf{Y}_{-p}) \cdot \mathbb{1}\{y_p > Y_p\} - \mathbb{H}_p^\kappa(\mathbf{Y}_{-p}, \mathbf{y}_{-p}) \cdot \mathbb{1}\{Y_p > y_p\}, \\ \widetilde{\mathbb{I}}^{\mathcal{R}^\kappa}(\mathbf{Y}, \mathbf{y}) &= \min_{p=1,\ldots,P}\{\widetilde{\mathbb{I}}_p^{\mathcal{R}^\kappa}(\mathbf{Y}, \mathbf{y})\}.\end{aligned} \tag{B-2}$$

By Assumption 4, we have $\mathbb{I}^{\mathcal{R}^\kappa}(\mathbf{Y}, \mathbf{y}; \omega) \leq \widetilde{\mathbb{I}}^{\mathcal{R}^\kappa}(\mathbf{Y}, \mathbf{y})$ w.p.1. Thus, the best feasible valid bounds that can be derived from (B-1) in our inferential environment are given by

$$\mathbb{1}\{\mathbf{Y} = \mathbf{y}\} \leq \mathbb{1}\{\mathbf{y} \in \mathcal{R}^\kappa(\omega)\} \leq \widetilde{\mathbb{I}}^{\mathcal{R}^\kappa}(\mathbf{Y}, \mathbf{y}). \tag{B-3}$$

This expression is analogous to (15) (for the case of mixed or pure-strategy NE) and (21) (for the case of pure-strategy NE). It is easy to verify[21] that $\widetilde{\mathbb{I}}^{\mathscr{E}^*}(\mathbf{Y},\mathbf{y}) \leq \widetilde{\mathbb{I}}^{\mathscr{E}}(\mathbf{Y},\mathbf{y}) \leq \widetilde{\mathbb{I}}^{\mathcal{R}^\kappa}(\mathbf{Y},\mathbf{y})\ \forall \kappa \geq 1$. From (B-3), probability bounds analogous to those in Section 5 follow.

EXAMPLE 1 (Continued). We construct $\widetilde{\mathbb{I}}_p^{\mathcal{R}^\kappa}(\mathbf{Y},\mathbf{y})$ and $\widetilde{\mathbb{I}}^{\mathcal{R}^\kappa}(\mathbf{Y},\mathbf{y})$ for the two cases in the example of Section 3.

(i) We have $f_p(\mathbf{y}_{-p}) = (\sum_{q\in\mathscr{S}_p} y_q, -\sum_{q\in\mathscr{C}_p} y_q)$, where $\mathscr{S}_p$ and $\mathscr{C}_p$ denote the group of substitutes and complements of Y_p. Let $\#\mathscr{S}_p$ and $\#\mathscr{C}_p$ denote the cardinalities of each group. For any profile $\mathbf{y} \equiv (y_p)_{p=1}^P$, Equation (B-2) becomes

$$\begin{aligned}\widetilde{\mathbb{I}}_p^{\mathcal{R}^\kappa}(\mathbf{Y},\mathbf{y}) = 1 - \mathbb{1}\Bigg\{&\sum_{q\in\mathscr{S}_p}\max\{y_q-\kappa,0\} \geq \sum_{q\in\mathscr{S}_p} Y_q + (\#\mathscr{S}_p)\cdot\kappa,\\ &\sum_{q\in\mathscr{C}_p} y_q + (\#\mathscr{C}_p)\cdot\kappa \leq \sum_{q\in\mathscr{C}_p}\max\{Y_q-\kappa,0\}\Bigg\}\cdot\mathbb{1}\{y_p > Y_p\}\\ &-\mathbb{1}\Bigg\{\sum_{q\in\mathscr{S}_p}\max\{Y_q-\kappa,0\} \geq \sum_{q\in\mathscr{S}_p} y_q + (\#\mathscr{S}_p)\cdot\kappa,\\ &\sum_{q\in\mathscr{C}_p} Y_q + (\#\mathscr{C}_p)\cdot\kappa \leq \sum_{q\in\mathscr{C}_p}\max\{y_q-\kappa,0\}\Bigg\}\cdot\mathbb{1}\{Y_p > y_p\}.\end{aligned}$$

(ii) We have $f_p(\mathbf{y}_{-p}) = ((y_q)_{q\in\mathscr{S}_p}, (-y_q)_{q\in\mathscr{C}_p})$. For any $\mathbf{y} \equiv (y_p)_{p=1}^P$, Equation (B-2) yields

$$\begin{aligned}\widetilde{\mathbb{I}}_p^{\mathcal{R}^\kappa}(\mathbf{Y},\mathbf{y}) = 1 - \mathbb{1}\big\{&\max\{y_q-\kappa,0\} \geq Y_q+\kappa\ \forall q\in\mathscr{S}_p \text{ and}\\ &y_q+\kappa \leq \max\{Y_q-\kappa,0\}\ \forall q\in\mathscr{C}_p\big\}\cdot\mathbb{1}\{y_p > Y_p\}\\ &-\mathbb{1}\big\{\max\{Y_q-\kappa,0\} \geq y_q+\kappa\ \forall q\in\mathscr{S}_p \text{ and}\\ &Y_q+\kappa \leq \max\{y_q-\kappa,0\}\ \forall q\in\mathscr{C}_p\big\}\cdot\mathbb{1}\{Y_p > y_p\}.\end{aligned}$$

In all cases, from (B-2), we have $\widetilde{\mathbb{I}}^{\mathcal{R}^\kappa}(\mathbf{Y},\mathbf{y}) = \min_{p=1,\dots,P}\{\widetilde{\mathbb{I}}^{\mathcal{R}^\kappa}(\mathbf{Y},\mathbf{y})\}$.

APPENDIX C: EXPRESSIONS FOR $\widetilde{\mathbb{I}}^{\mathscr{E}}(\mathbf{Y},\mathbf{y})$ AND $\widetilde{\mathbb{I}}^{\mathscr{E}^*}(\mathbf{Y},\mathbf{y})$ IN SECTION 8

Since Y_1 and Y_2 are mutual strategic substitutes, we can use Example 1 in Section 4.3 to see exactly how $\widetilde{\mathbb{I}}^{\mathscr{E}}(\mathbf{Y},\mathbf{y})$ is constructed. For any pair of sets $\mathcal{S} = \{a_1, b_1\}\times\{a_2, b_2\}$ and $\mathcal{S}' = \{a_1', b_1'\}\times\{a_2', b_2'\}$ such that $a_p \geq 0$ and $b_p = a_p$ or $b_p = a_p + 1$ (and the same for a_p', b_p'), the indicator function $\widetilde{\mathbb{I}}_p(\mathcal{S},\mathcal{S}')$ is given as follows:

Case I. If $a_p \neq b_p$ and $a_p' \neq b_p'$, then

$$\widetilde{\mathbb{I}}_p(\mathcal{S},\mathcal{S}') = 1 - \max\big\{\mathbb{1}\{a_{-p} \geq b_{-p}'\}\cdot\mathbb{1}\{a_p' \leq a_p\}, \mathbb{1}\{a_{-p}' \geq b_{-p}\}\cdot\mathbb{1}\{a_p' \geq a_p\}\big\}.$$

[21] It is also easy to see that $\widetilde{\mathbb{I}}^{\mathcal{R}^\kappa}(\mathbf{Y},\mathbf{y}) \leq \widetilde{\mathbb{I}}^{\mathcal{R}^{\kappa'}}(\mathbf{Y},\mathbf{y})\ \forall \kappa' > \kappa$.

Case II. If $a_p \neq b_p$ and $a'_p = b'_p$, then

$$\widetilde{\mathbb{I}}_p(\mathcal{S},\mathcal{S}') = 1 - \max\Big\{\mathbb{1}\{a_{-p} \geq b'_{-p}\} \cdot \mathbb{1}\{a'_p \leq a_p\}, \mathbb{1}\{a'_{-p} \geq b_{-p}\} \cdot \mathbb{1}\{a'_p > a_p\}\Big\}.$$

Case III. If $a_p = b_p$ and $a'_p \neq b'_p$, then

$$\widetilde{\mathbb{I}}_p(\mathcal{S},\mathcal{S}') = 1 - \max\Big\{\mathbb{1}\{a_{-p} \geq b'_{-p}\} \cdot \mathbb{1}\{a'_p < a_p\}, \mathbb{1}\{a'_{-p} \geq b_{-p}\} \cdot \mathbb{1}\{a'_p \geq a_p\}\Big\}.$$

Case IV. If $a_p = b_p$ and $a'_p = b'_p$, then

$$\widetilde{\mathbb{I}}_p(\mathcal{S},\mathcal{S}') = 1 - \max\Big\{\mathbb{1}\{a_{-p} \geq b'_{-p}\} \cdot \mathbb{1}\{a'_p < a_p\}, \mathbb{1}\{a'_{-p} \geq b_{-p}\} \cdot \mathbb{1}\{a'_p > a_p\}\Big\}.$$

The function $\widetilde{\mathbb{I}}(\mathcal{S},\mathcal{S}') = \min\{\widetilde{\mathbb{I}}_1(\mathcal{S},\mathcal{S}'), \widetilde{\mathbb{I}}_2(\mathcal{S},\mathcal{S}')\}$ and, $\forall \mathbf{y} = (y_1, y_2)$, we have $\widetilde{\mathbb{I}}^{\mathscr{E}}(\mathbf{Y},\mathbf{y}) = \max_{\mathcal{S},\mathcal{S}'}\{\widetilde{\mathbb{I}}(\mathcal{S},\mathcal{S}')\}$, where

$$\begin{aligned}
\mathcal{S} \in \Big\{&\{\max\{0, Y_1 - 1\}, Y_1\} \times \{\max\{0, Y_2 - 1\}, Y_2\},\\
&\{Y_1, Y_1 + 1\} \times \{\max\{0, Y_2 - 1\}, Y_2\}, \{\max\{0, Y_1 - 1\}, Y_1\} \times \{Y_2, Y_2 + 1\},\\
&\{Y_1, Y_1 + 1\} \times \{Y_2, Y_2 + 1\}, \{Y_1, Y_1\} \times \{Y_2, Y_2\}\Big\},\\
\mathcal{S}' \in \Big\{&\{\max\{0, y_1 - 1\}, y_1\} \times \{\max\{0, y_2 - 1\}, y_2\}, \{y_1, y_1 + 1\} \times \{\max\{0, y_2 - 1\}, y_2\},\\
&\{\max\{0, y_1 - 1\}, y_1\} \times \{y_2, y_2 + 1\},\\
&\{y_1, y_1 + 1\} \times \{y_2, y_2 + 1\}, \{y_1, y_1\} \times \{y_2, y_2\}\Big\}.
\end{aligned}$$

For the pure-strategy NE case, Example 1 in Section 6 yields

$$\begin{aligned}
\widetilde{\mathbb{I}}^{\mathscr{E}*}(\mathbf{Y},\mathbf{y}) = \min\Big\{&1 - \mathbb{1}\{y_2 \leq Y_2\}\mathbb{1}\{y_1 < Y_1\} - \mathbb{1}\{y_2 \geq Y_2\}\mathbb{1}\{y_1 > Y_1\},\\
&1 - \mathbb{1}\{y_1 \leq Y_1\}\mathbb{1}\{y_2 < Y_2\} - \mathbb{1}\{y_1 \geq Y_1\}\mathbb{1}\{y_2 > Y_2\}\Big\}.
\end{aligned}$$