14.- Details of the Extensive Form

- In our study of rational behavior and equilibrium we have focused on normal form games.
- This includes extensive form games as special cases since we can always write them down in normal form
- However, the normal form can obscure some key details of the game that are captured by its extensive form. Specifically: The sequence of moves and the features of the information sets.

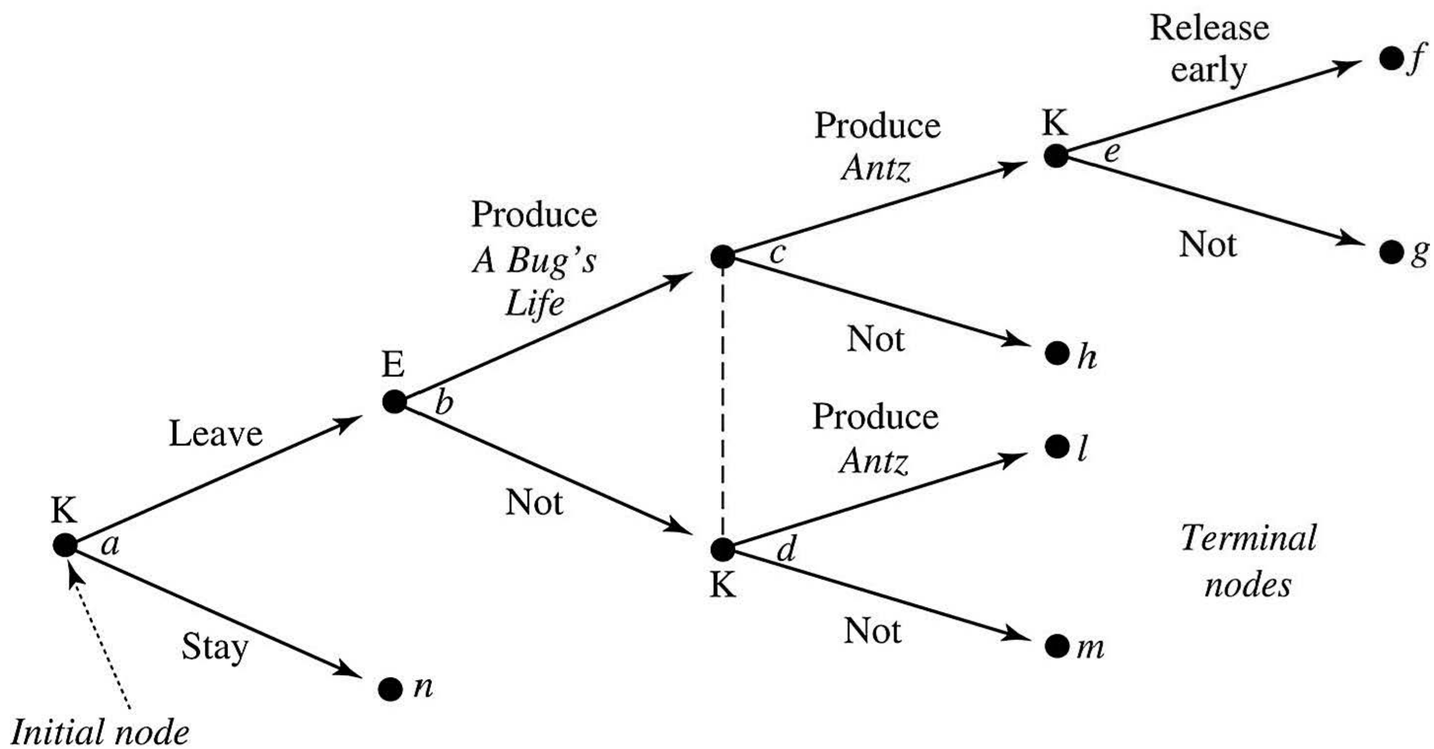
- Analyzing the extensive form can help us identify which normal-form Nash equilibria “make more sense than others”.

- What we mean by “make more sense” will become clear in Chapter 15.

- First, let us outline some key features that must be satisfied by well-defined extensive form games.

- <u>**Recall from Chapter 2:**</u> An extensive form game is defined by **nodes** and **branches**.
- **Nodes:** Represent places in the game where individual players must make decisions.
- **Branches:** They emanate from nodes and indicate the different actions that players can choose.
- **Successor nodes:** Let x be a node in the tree. The successors of x are all the nodes that can be reached from x by progressing forward along the decision tree.
- **Predecessor nodes:** Let x be a node in the tree. The predecessors of x are all the nodes that can be reached from x by going backward along the decision tree.

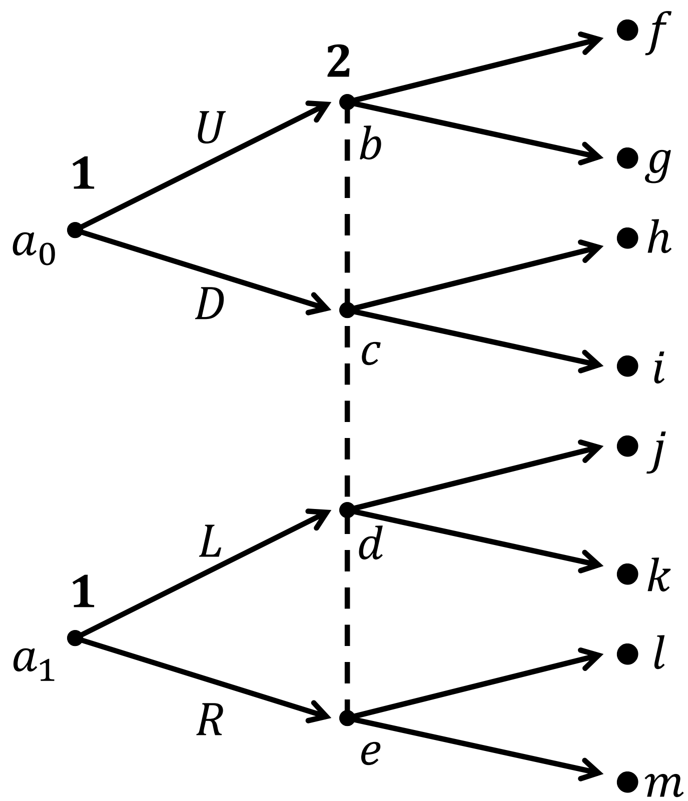
- **Terminal nodes:** They represent final outcomes of the game –places where the game ends.



- Terminal nodes: $\{f, g, h, l, m, n\}$
- Successors to node b: $\{c, d, e, f, g, h, l, m\}$
- Predecessors to node e: $\{c, b, a\}$

- **Tree Rule 1: Every game must have a <u>unique initial node</u>.-** Every node is a successor of the initial node, and the initial node is the <u>only one</u> with this property.
- The following tree violates Rule 1:

This game violates Rule #1. It has two "initial nodes": a_0 and a_1.



Nodes b, c, f, g, h, i are successors of a_0

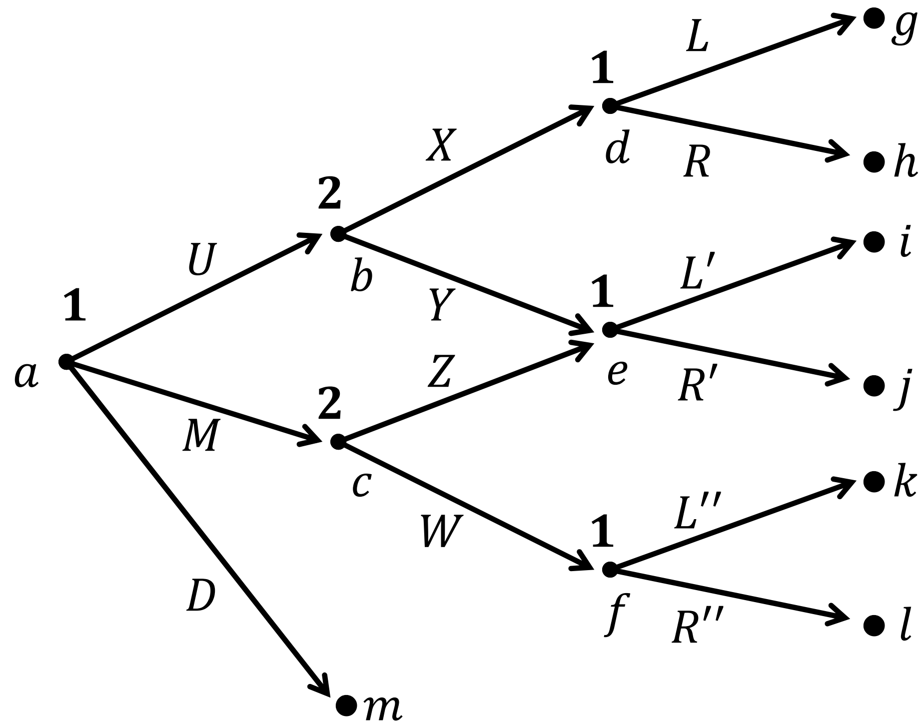
Nodes d, e, j, k, l, m are successors of a_1

- A **path** through the tree is a sequence of nodes that starts at the initial node and ends at one of the terminal nodes.

- The following rule asserts that paths should be mutually exclusive, that they should not “cross”.

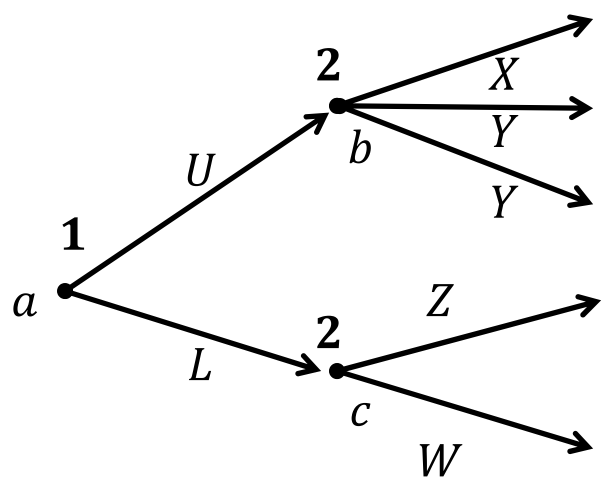
- **Tree Rule 2: Different paths are mutually exclusive and do not “cross”.-** The initial node has no predecessors. Each node except the initial node has exactly one immediate predecessor.

- The following game violates Rule 2:



- Node e has two different immediate predecessor nodes: b and c. This makes the paths $a \rightarrow b \rightarrow e \rightarrow j$ and $a \rightarrow c \rightarrow e \rightarrow i$ cross each other.

- The next rule is very simple: Each branch in a tree is meant to indicate a particular action available to a player.
- **Tree Rule 3: Actions available at a given node are indicated by a unique branch.-** Multiple branches extending from the same node have different action labels.
- The following is a violation to Rule 3:

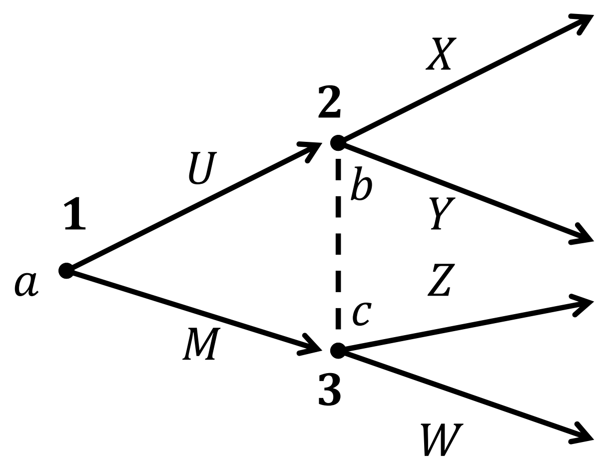


Two branches stemming from node b correspond to the same action by player 2: "Y"

- **Rules about information sets:** Information sets are a main component of extensive form games. Information sets are sets of nodes that are *indistinguishable* for some player when making a decision.
- **Tree Rule 4: Information sets belong to individual players only.-** Each information set contains decision nodes for one and one player only.

- A violation of Rule 4 would indicate that two or more players are not sure who among them is supposed to move in some part of the game. This is ruled out.

- The following example violates Rule 4:



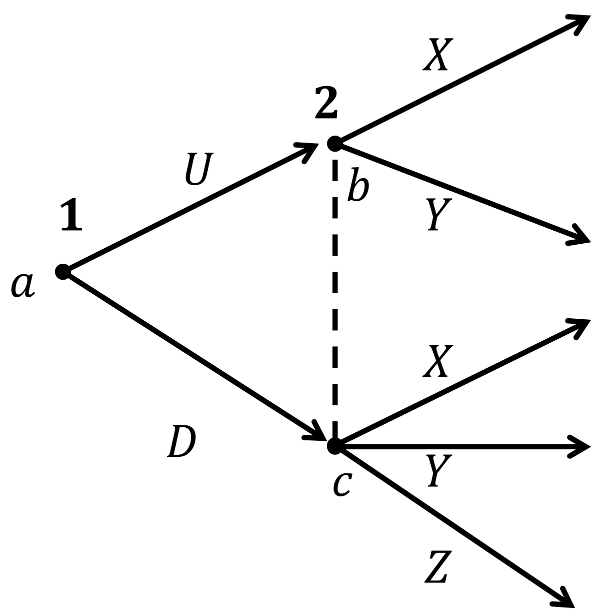
This game has an information set that includes decision nodes b and c.

Decision node b belongs to player 2 and decision node c belongs to player 3

- The above example describes a situation in which, following the move of Player 1, Players 2 and 3 are unsure which of them is supposed to move.

- By definition, all decision nodes in an information set must be indistinguishable to the player who is supposed to move there.
- There must be no information available to the player that helps distinguish between individual nodes in an information set.
- In particular, every decision node in an information set must have the same actions available to the player. Otherwise this would help the player distinguish between these nodes.

- **Tree Rule 5: Decision nodes in an information set must be indistinguishable.-** All nodes in a given information set must have the same number of immediate successors and they must have the same set of action labels on the branches leading to these successors.
- The following example violates Rule 5:

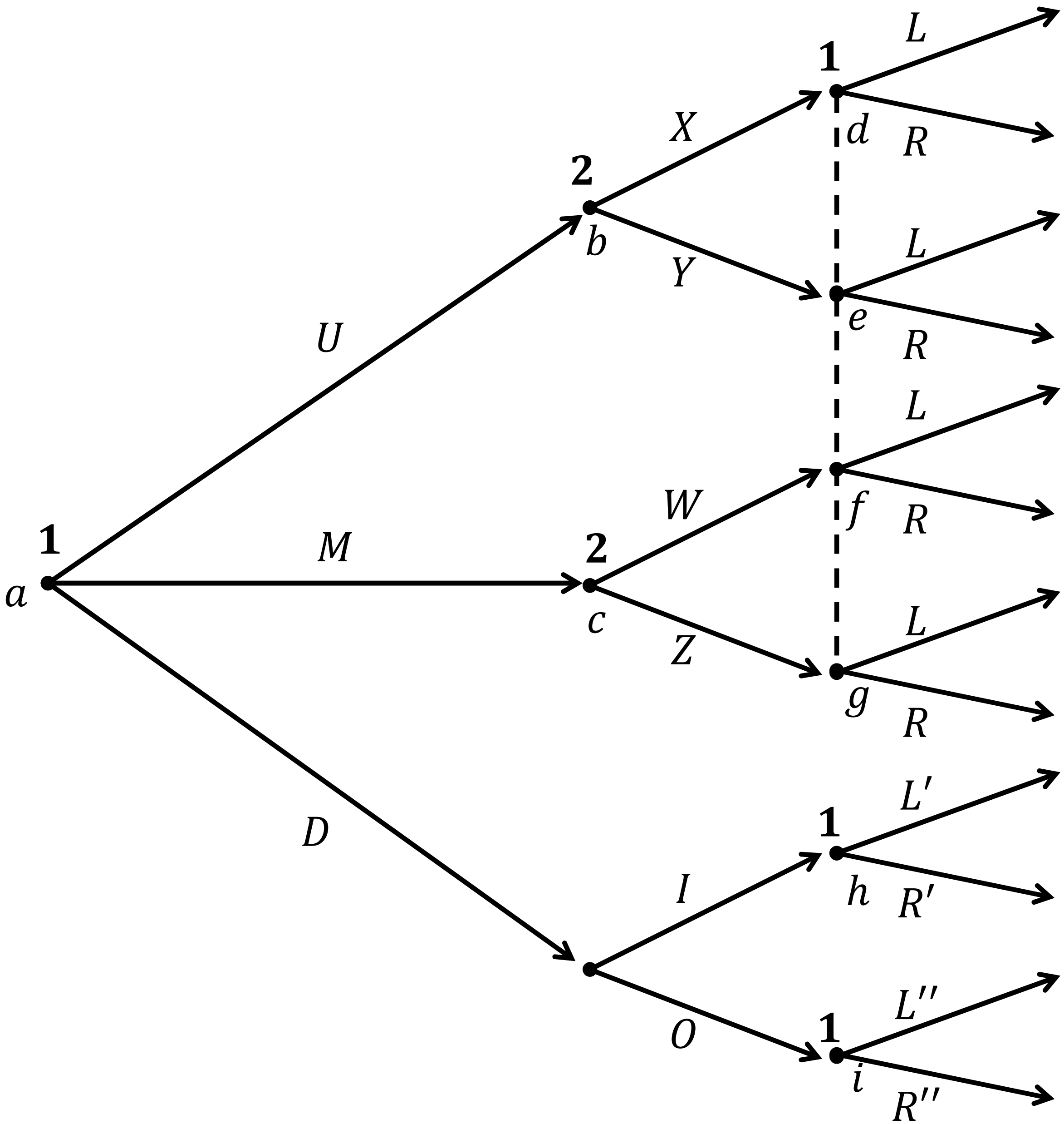


Decision nodes b and c belong in the same information set for Player 2. Node b has three available actions while node c has three. This would immediately reveal to Player 2 which node he is in.

- **Perfect recall:** In the course we will assume that, as the game progresses, players do not forget what they were able to observe in earlier stages of the game.

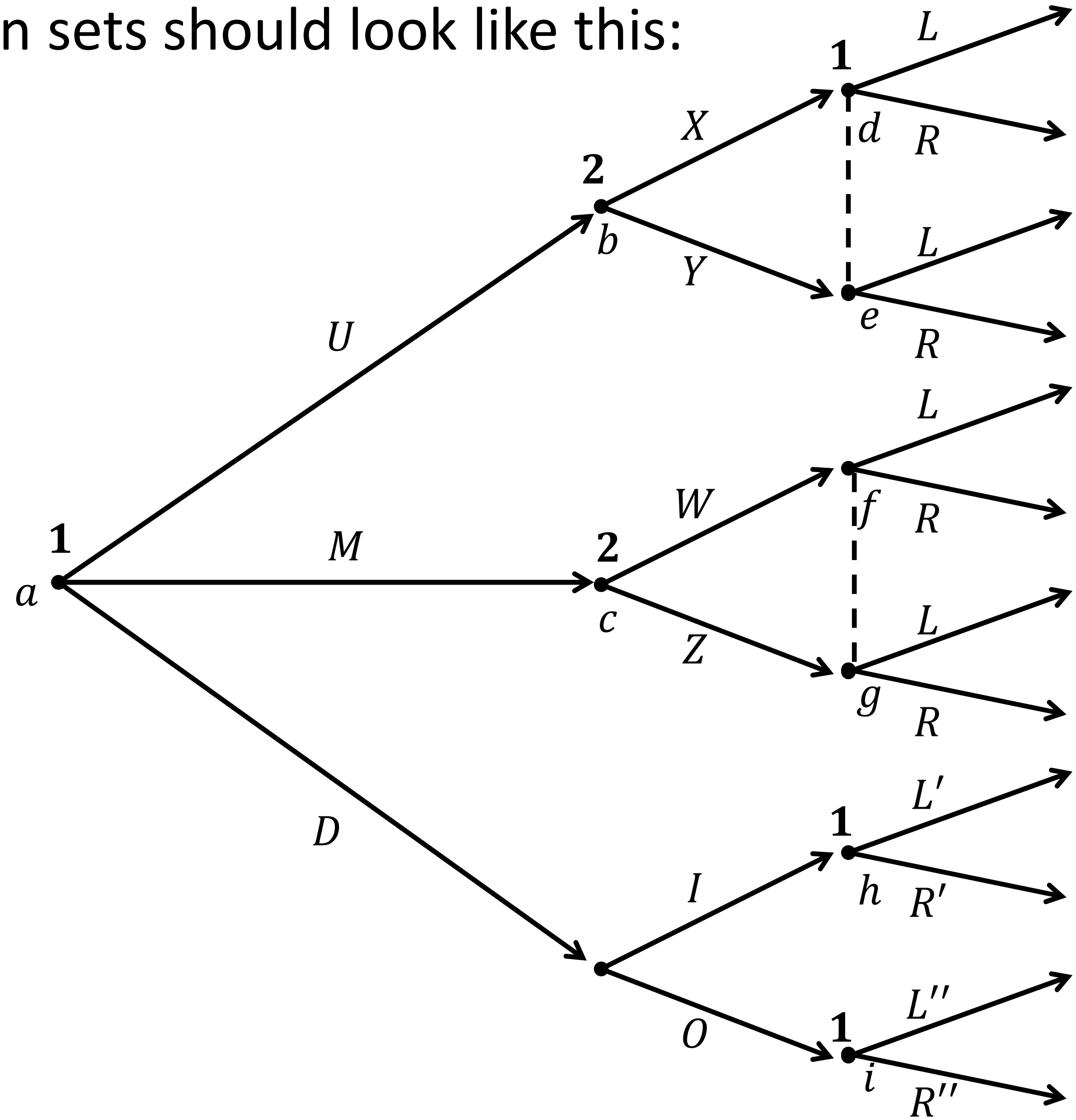
- More precisely: Players do not forget their own past actions, and they do not forget the past actions of others that they were able to observe.

- The following example violates perfect recall:



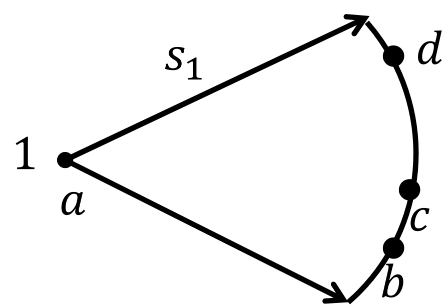
In the information set for player 1 that includes the nodes d, e, f, and g , player 1 has forgotten whether he chose U or M earlier in the game.

- Even if player 1 cannot observe whether player 2 chooses X, Y, W or Z, he should remember his own earlier action. The information sets should look like this:



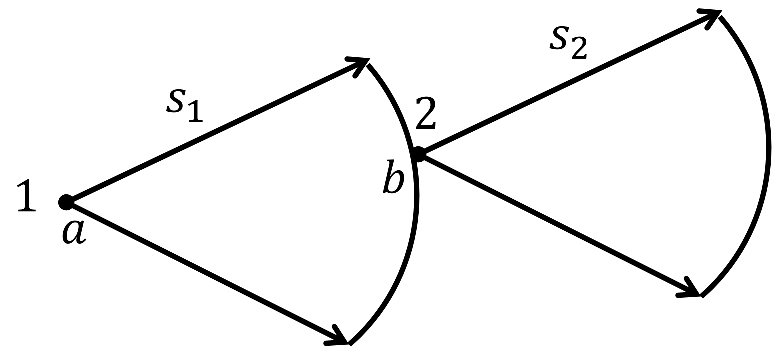
- **Extensive form representation of games with continuous actions.-** Drawing a game tree with a finite number of branches and outcomes is possible only in discrete games.
- However, we can still sketch out the extensive form representation of games with continuous strategies, even though the tree itself will be only a summary of the game, it will not describe the game completely as it is the case in discrete games.
- Note that games with continuous actions have infinitely many branches and outcomes.

- Since it is impossible to draw all the possible branches of a continuous game, we can use a continuous arch to describe the fact that there is a continuum of possible branches stemming from a node.
- A successor node would be indicated generically as a point along the arch of "continuous branches":

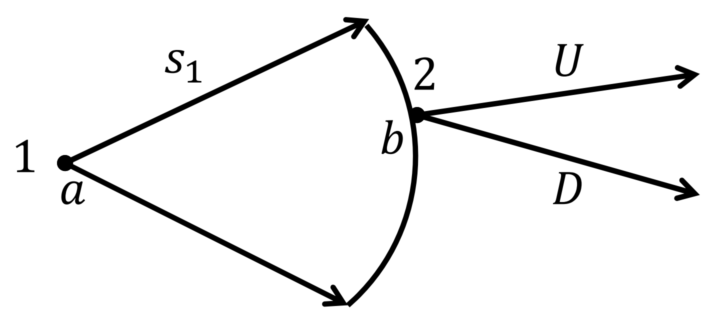


- ➢ There is a continuum of immediate successor nodes to a . Each one represents a point along the arch.
- ➢ Here we have depicted three in particular: b, c and d.

- Sequential continuous strategies can be represented like this:

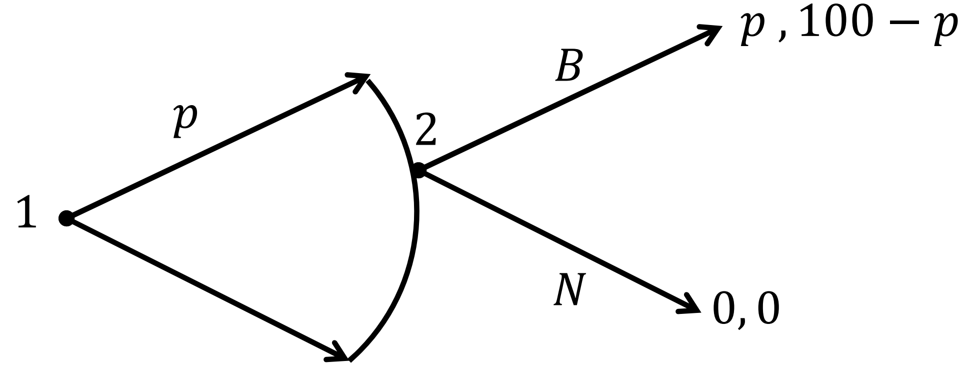


- We can also have a combination of continuous and discrete strategies. For example:



- **Example: Ultimatum game.-** This is a special case of a **bargaining game** in which player 1 makes player 2 a "take it or leave it" offer. Once player 2 observes the offer, he decides whether to accept (A) or reject (R) the offer.
- Specifically, consider the following setting:
  - Player 1 is selling a good which is worth zero for player 2 and $100 for player 2.
  - Stage 1: Player 1 announces a selling price $p \in [0, 100]$ to player 2.
  - Stage 2: Observing p, player 2 decides whether to buy (B) or not buy (N) the good.

- The extensive form depiction of this game is the following:

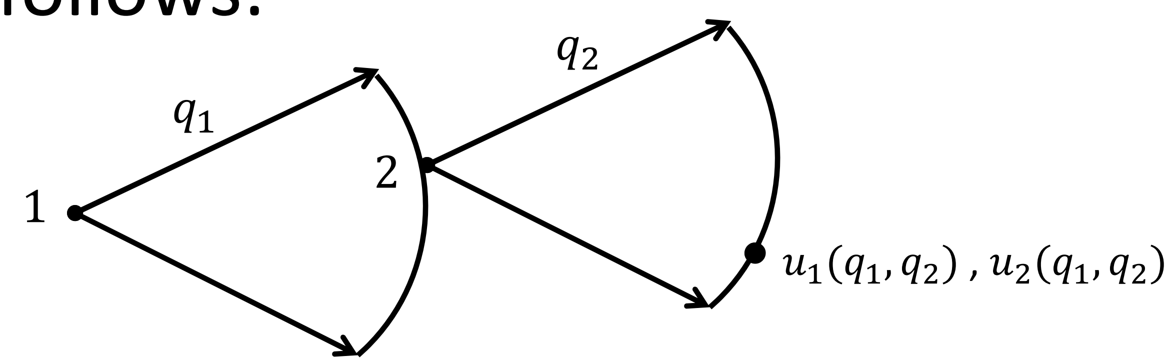


- As before, the terminal nodes describe the payoffs to the players, in the order in which they make their moves along the game.

- **Example: Stackelberg duopoly model.-** This is a sequential version of the Cournot duopoly game. The sequence of moves is as follows:

1. In the first period, firm 1 chooses its quantity produced, q_1.
2. Firm 2 observes q_1 and then chooses the quantity to produce, q_2.

- The extensive form representation of this game looks as follows:



- **Strategies in games with continuous actions when players move sequentially.-** A strategy is a complete contingent plan. In discrete, extensive form games, a strategy is simply a description of what action to take at each information set.
- In games with continuous actions there are infinitely many information sets for some players.
- Therefore, strategies must be described as functions that depends on the information that players have in the game.

- **In the bargaining game example**:
  - Player 1 must propose a price without observing the action (accept or reject) of player 2. Therefore, the strategy for player 1 is simply given by "p“, the price he proposes.
  - Player 2 observes p and then makes a decision. The proposed price p summarizes all the information player 2 possesses. Therefore, a strategy for player 2 must be a function of p. It must be of the form $s_2(p)$. For every value of p, the strategy $s_2(p)$ must tell the player whether to “buy” or “not buy” the good.
  - Therefore, **a (pure) strategy profile in this game must be of the form $(\boldsymbol{p}, \boldsymbol{s_2}(\boldsymbol{p}))$**

- Since we can find infinitely many different functions $s_2(p)$, there are infinitely many different possible strategies for player 2.
- One particular example of a strategy could be the following:

$$s_2(p) = \begin{cases} "B" \text{ if } p \leq 75 \\ \\ "N" \text{ if } p > 75 \end{cases}$$

- Another strategy could be:

$$s_2(p) = \begin{cases} "N" \text{ if } p \in [0,10) \\ "B" \text{ if } p \in [10,70] \\ "N" \text{ if } p > 70 \end{cases}$$

- **In the Stackelberg duopoly example**:
  - Player 1 must decide how much to produce, q_1 before observing the action of player 2. Therefore, the strategy for player 1 is simply given by q_1
  - Player 2 observes q_1 and then makes a decision. The quantity produced q_1 summarizes all the information player 2 possesses. Therefore, a strategy for player 2 must be a function of q_1. It must be of the form $s_2(q_1)$. For every value of q_1, the strategy $s_2(q_1)$ must tell player 2 how much to produce.
  - Therefore, **a (pure) strategy profile in this game must be of the form $(\boldsymbol{q_1}, \boldsymbol{s_2}(\boldsymbol{q_1}))$**

- **Perfect Information Games:** We say that a game has perfect information if **every information set is a singleton**. That is, a game is of perfect information if each player knows exactly where they are along the tree.
- Otherwise, if there are information sets that consist of two or more decision nodes, we say that the game has **imperfect information**.
- In a game of perfect information, players move sequentially and observe exactly other players' past choices.

- We already know that in discrete games with imperfect information all we have to do is to connect the decision nodes in a given information sets with dotted lines.
- In the case of games of imperfect information with continuous actions we can also use dotted lines to indicate information sets. For example: