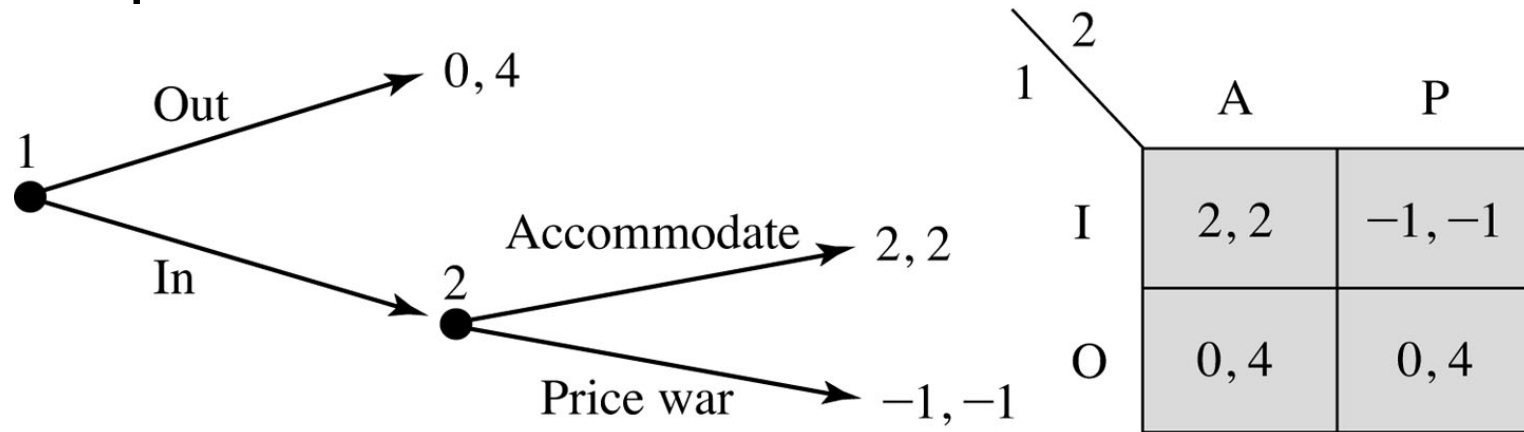


15.- Sequential Rationality and Subgame Perfection

- So far, our analysis of rational behavior and equilibrium behavior **has focused on the normal-form representation of a game.**
- In going from an extensive-form game to its normal-form representation, some key details of the nature of the game are obscured. Specifically:
 - The sequence of moves in the game.
 - The precise nature of the information sets.
- As a result, if we focus exclusively on the normal form of a game, we may fail to observe that some Nash equilibria call for behavior that would make little sense in a game that is truly played sequentially.

- Specifically, we may have equilibrium strategies that look just fine in the normal-form representation, but that call for suboptimal behavior in some branches of the underlying extensive form game.
- **Example:** Consider the following game, describing an industry with a single incumbent firm (a monopolist).
- A potential entrant (Player 1) decides whether to enter or not into an industry. Suppose the incumbent firm (Player 2) observes the decision of Player 1 and then, if the latter decides to enter, then the incumbent decides whether to initiate a price war or accommodate the new entrant.

- Suppose the game in extensive and normal form is represented as follows:



- This game has two Nash equilibria: (I, A) and (O, P) .
- If we focus only on the normal form, then there is nothing more we can say.
- However, if we study the extensive form we can see that the equilibrium (O, P) calls for behavior that is suboptimal in certain regions of the decision tree.

- Specifically, focus on the last stage of the game, where the incumbent has to decide whether to initiate a price war or accommodate the entrant:
- If Player 2 (the incumbent) finds himself in this node, initiating a price war leads to a lower payoff than accommodating the entrant. For this reason, **initiating a price war if Player 1 enters is a non-credible threat by Player 2.**
- The Nash equilibrium (O, P) calls for an action that is suboptimal for Player 2 once he finds himself in the last node of the game.

- Notice the following:
 1. (O, P) is still a well-defined Nash equilibrium because it is a pair of best-responses: If player 2 threatens with initiating a price war, then player 1's best response is to stay out of the market. And if player 1 stays out of the market, then initiating a price war or not becomes irrelevant (both strategies are "best responses" for player 2).
 2. However, if we pay attention to the sequential nature of the underlying extensive form game, we see that initiating a price war is a non-credible threat since it would require player 2 to take a suboptimal action in the last part of the tree.

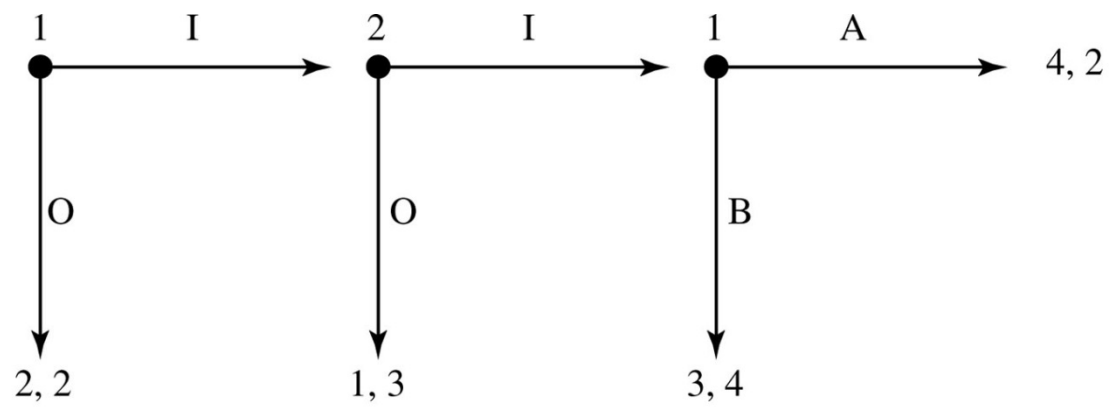
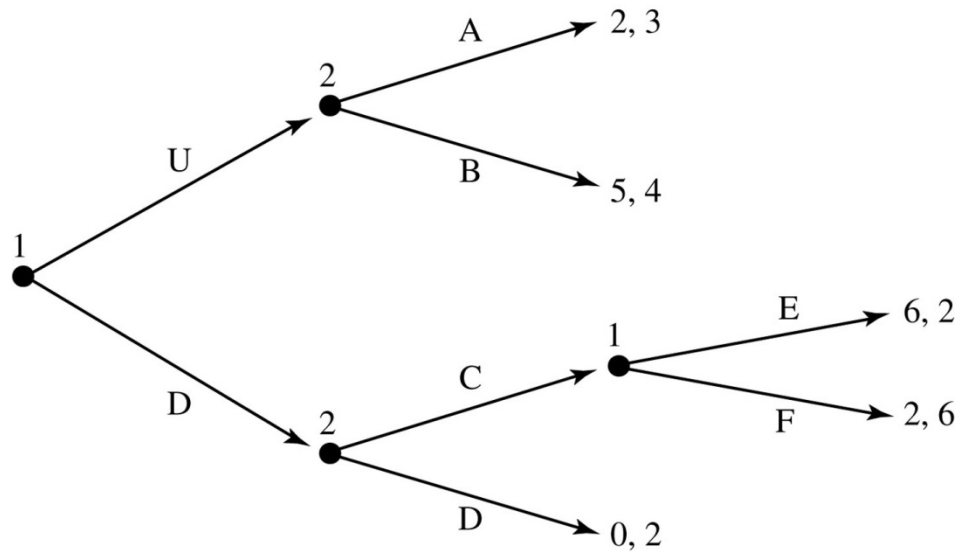
- This example helps motivate the notion of ***sequential rationality*** of players' strategies. Intuitively, a strategy is sequentially rational if it leads to choosing best responses at each decision point in the game.
- More formally, we can define it as follows:
- **Sequential rationality:** Is a strategy for player i that prescribes optimal actions at each one of the information sets of player i .
- In the previous example, the strategy P is not sequentially rational for player 2, since it calls for a suboptimal action in the last decision node of the extensive form game.

- We will discuss three notions of sequential rationality:
 - 1. Backward induction.**
 - 2. Subgame perfect Nash equilibrium.**
 - 3. Conditional dominance and forward induction.**
- We will focus mainly on #1 (backward induction) and #2 (subgame perfect Nash equilibrium) and will describe #3 (conditional dominance and forward induction) only briefly.
- The most important concept in this section will be that of subgame perfect Nash equilibrium. It encompasses backward induction as a special case in games of perfect information.

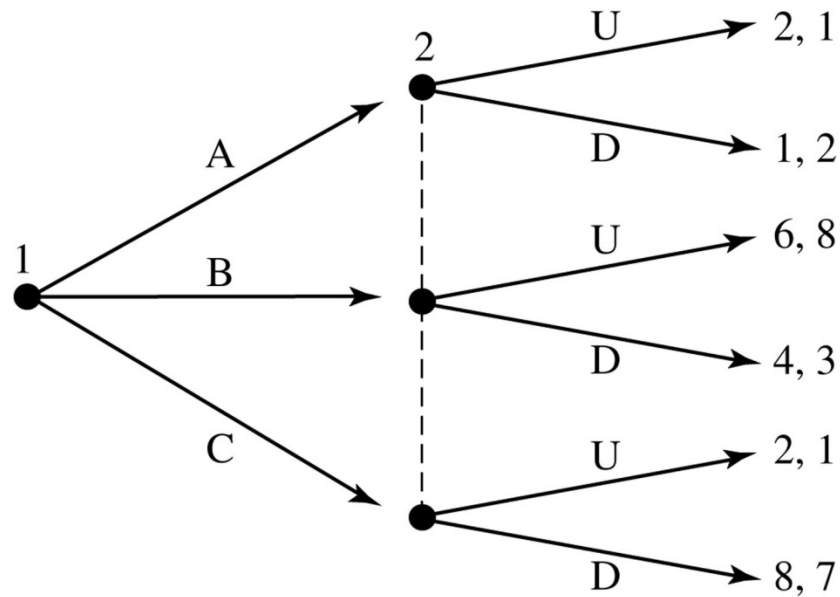
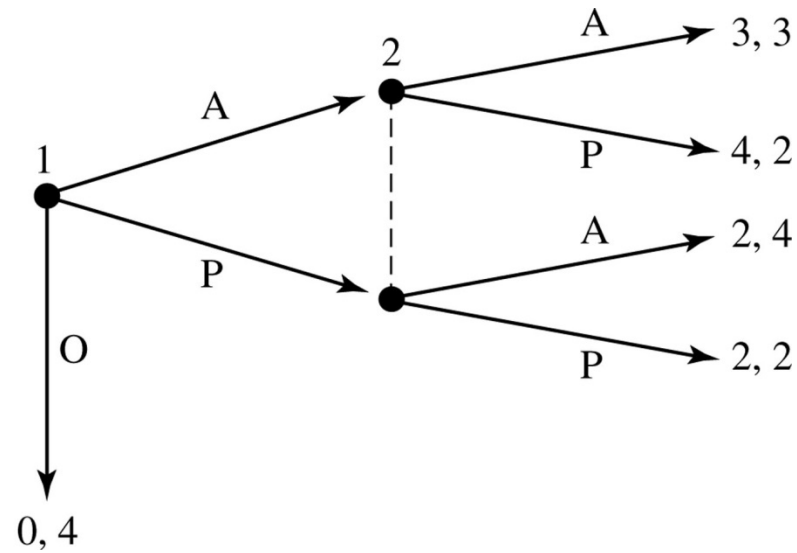
- Note that sequential rationality cannot be determined from the normal-form of a game. We need to study the extensive form to determine if a strategy has this property.
- Suppose we assume sequentially rational behavior for the players. Furthermore, suppose sequential rationality is **common knowledge**.
- This strengthens the notion of common knowledge of rationality to common knowledge of sequential rationality.

- **Backward Induction**
- So, how does this “looking ahead” process look like? We begin by focusing on **games with perfect information** and the notion of **backward induction**.
- **Perfect information games:** These refer to extensive form games where **every information set is a singleton**, so that each player always knows exactly where they are along the game.

- The following games have perfect information:



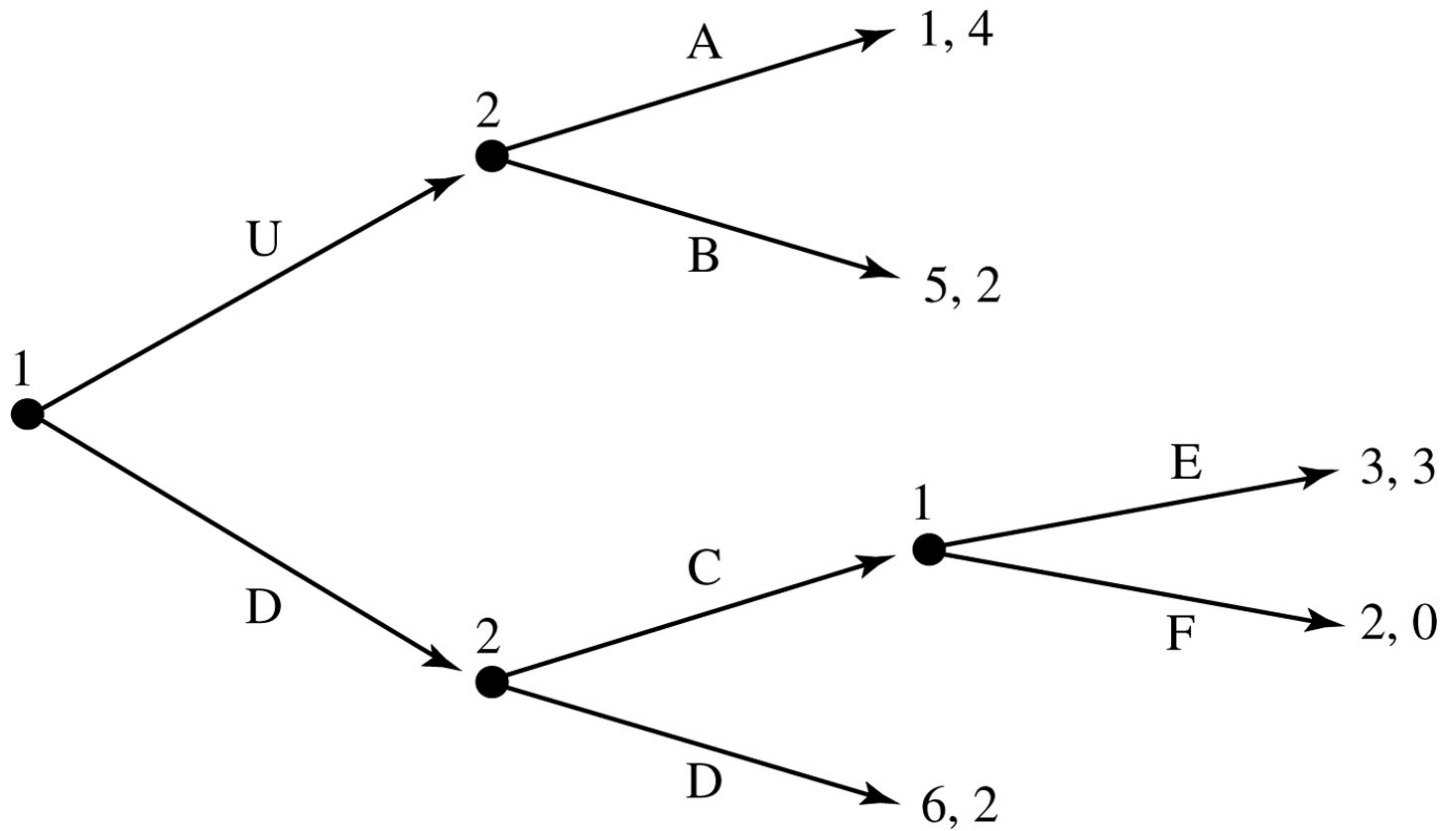
- The following games do NOT have perfect information:



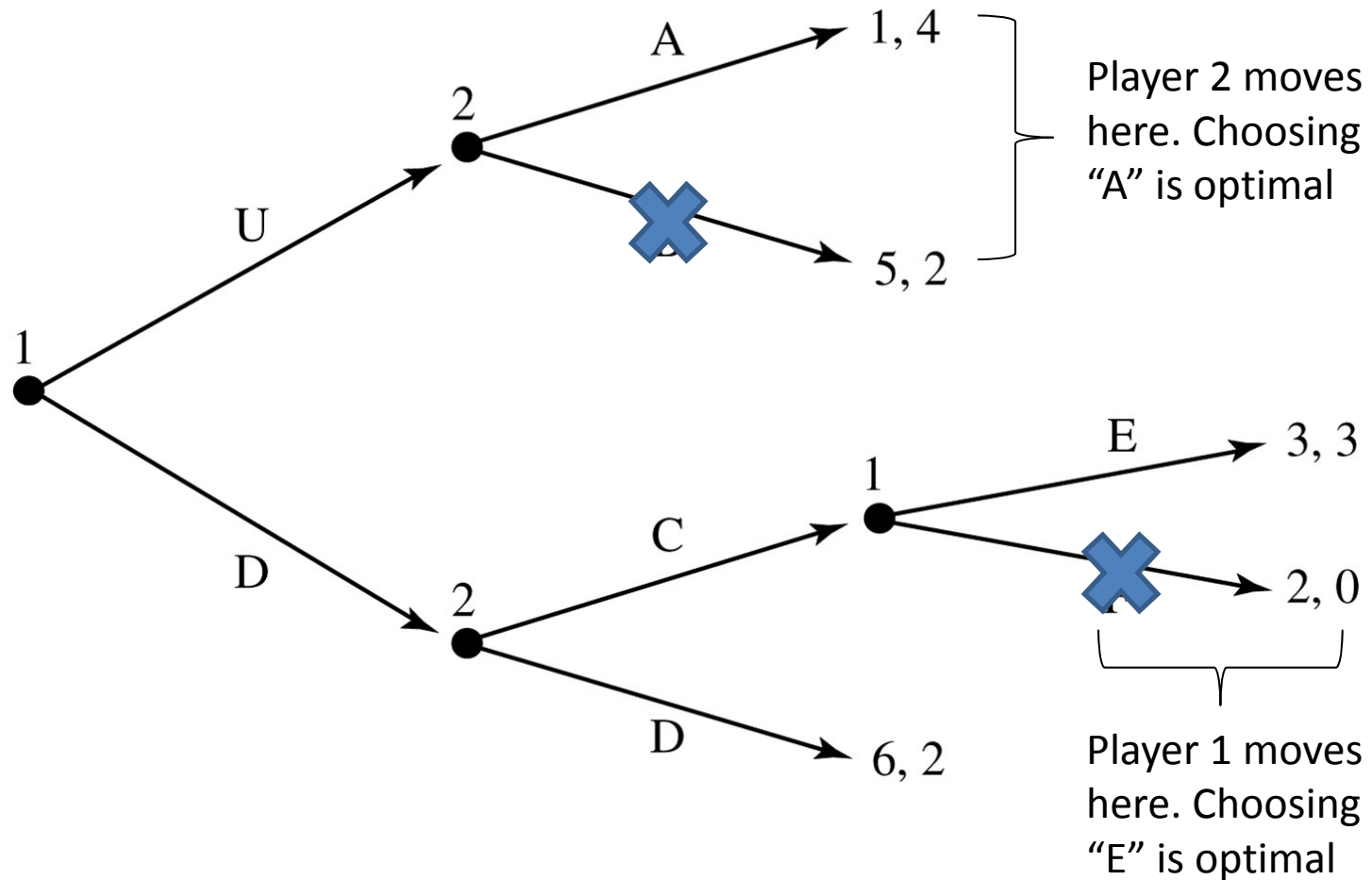
- The process of **backward induction** “solves” an extensive form game by working from the end of the game backwards to the beginning. At each state (starting at the terminal nodes), it identifies the optimal strategies. More formally:
- **Backward induction procedure:** This is the process of analyzing a game from the end to the beginning. At each decision node, we strike out all actions that are dominated. This process starts at the terminal nodes of the game and works backwards to the initial node.
- Because backward induction studies the decisions at **individual nodes**, it can be applied only to games of **perfect information**. We will see in a bit how to extend this notion to games of imperfect information.

- Steps of backward induction:
 1. Begin at the terminal nodes. Identify all the terminal nodes that have the same immediate predecessor node. Strike out all terminal nodes that can be reached only by a branch involving a dominated action.
 2. Move on backwards to the immediate predecessor nodes and repeat this process: Strike out all nodes that can be reached only by a branch involving a dominated action.
 3. Continue all the way back to the initial node.

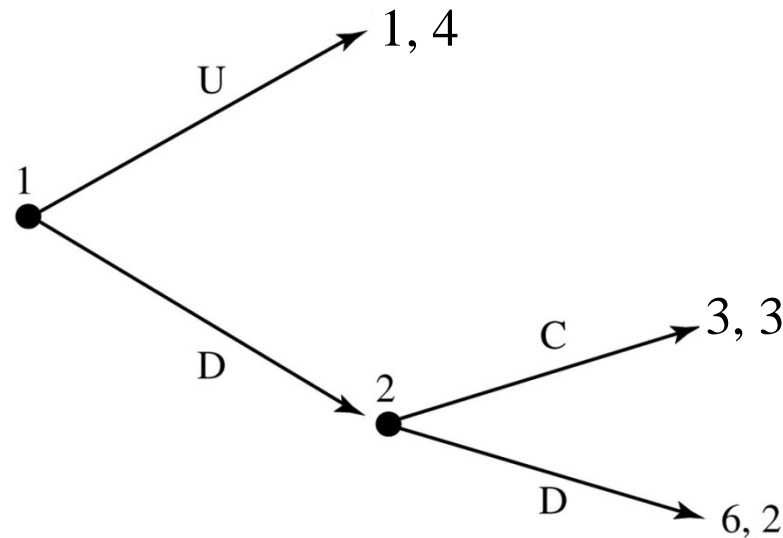
- **Example:** Consider the following perfect information game:



- In the first step of backward induction we strike out the following dominated actions:

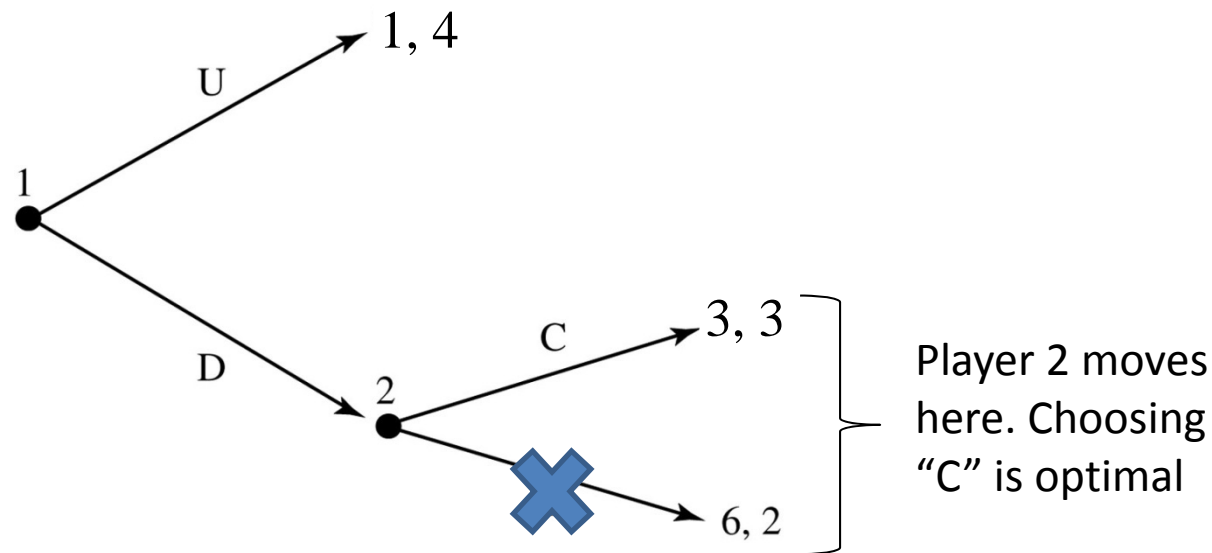


- Once we strike out dominated strategies leading to the terminal nodes, the “continuation payoffs” can be written as:



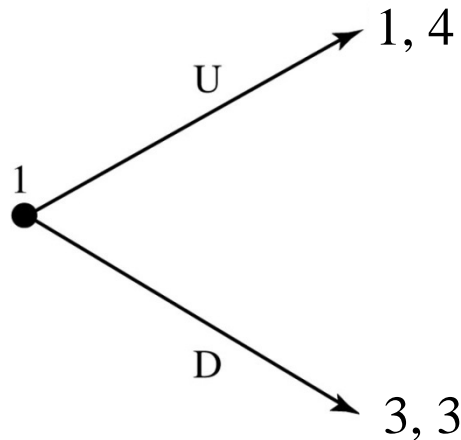
- In the next step, we take these “continuation payoffs” and once again strike out dominated actions

- Striking out the dominated actions we have:



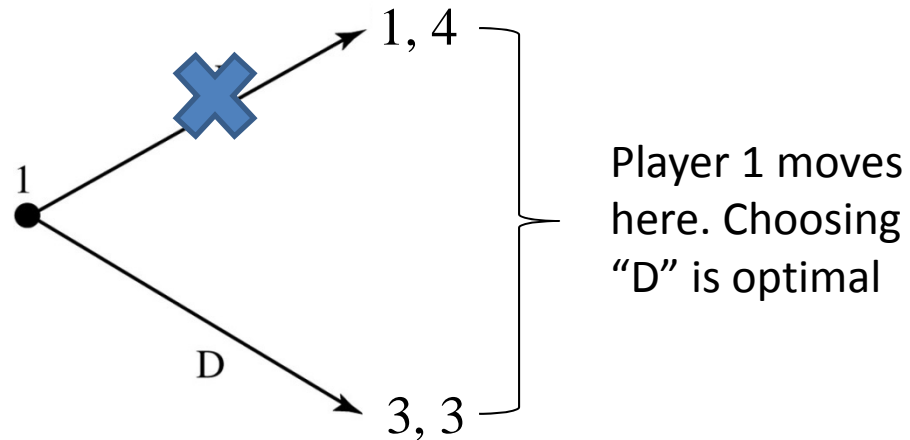
- In the next step, we take these “continuation payoffs” and once again strike out dominated actions

- Once we strike out the dominated actions, the new continuation payoffs look like this:



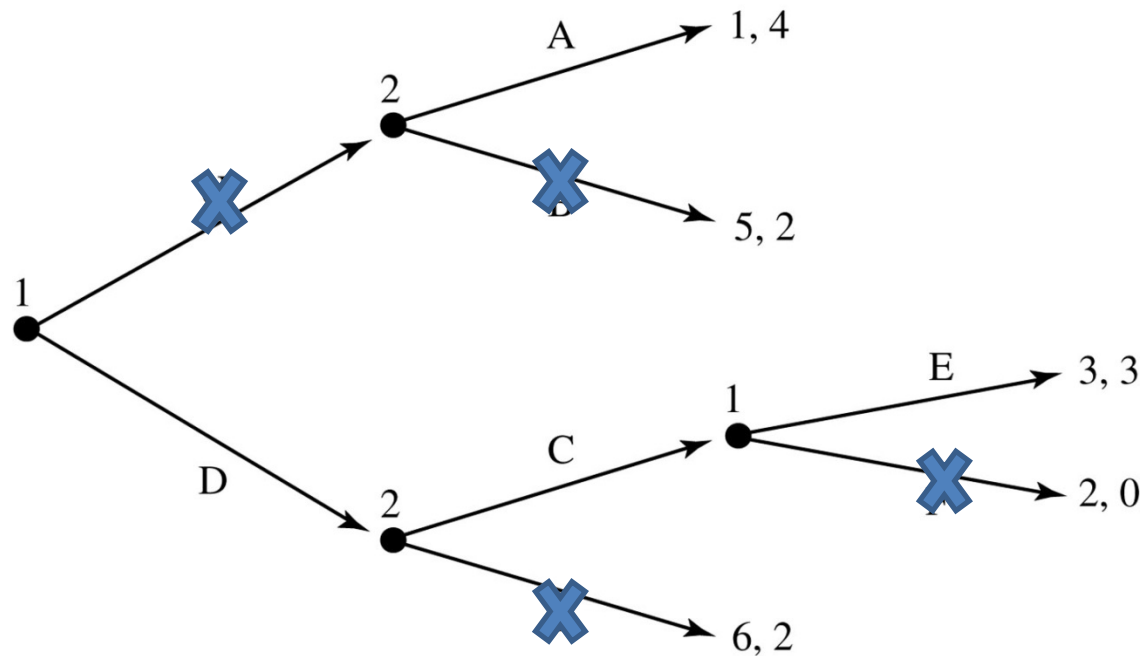
- We have reached the initial node. We only have to strike out one last action...

- We now strike:



- We have solved this game using backward induction. If this sequential rationality is common knowledge, then this is the unique solution players should arrive at.

- The backward induction optimal strategies be recovered by re-tracing all the actions we struck out through the process of backward induction:



- The backward induction optimal strategies are:

$$s_1 = DE \quad \text{and} \quad s_2 = AC$$

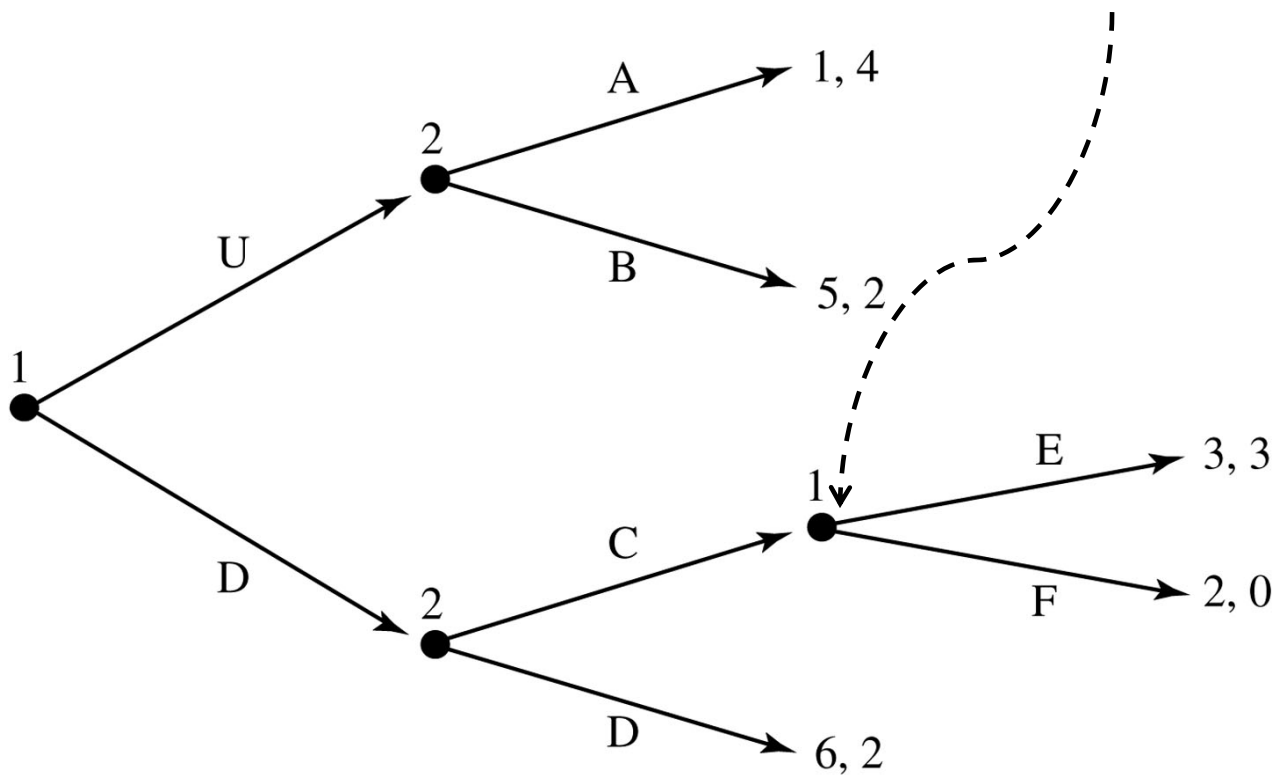
- By construction, since backwards induction always looks for optimal actions at each decision node, every profile of backward induction optimal strategies must be a profile of best responses. Therefore, it must be a pure-strategy Nash equilibrium.
- **Result:** In every perfect-information game, the process of backwards induction always identifies a pure-strategy Nash equilibrium.

- To check this in our previous example, note that the normal form of the game is (Nash equilibria circled):

	AC	AD	BC	BD
UE	1, 4	1, 4	5, 2	5, 2
UF	1, 4	1, 4	5, 2	5, 2
DE	3, 3	6, 2	3, 3	6, 2
DF	2, 0	6, 2	2, 0	6, 2

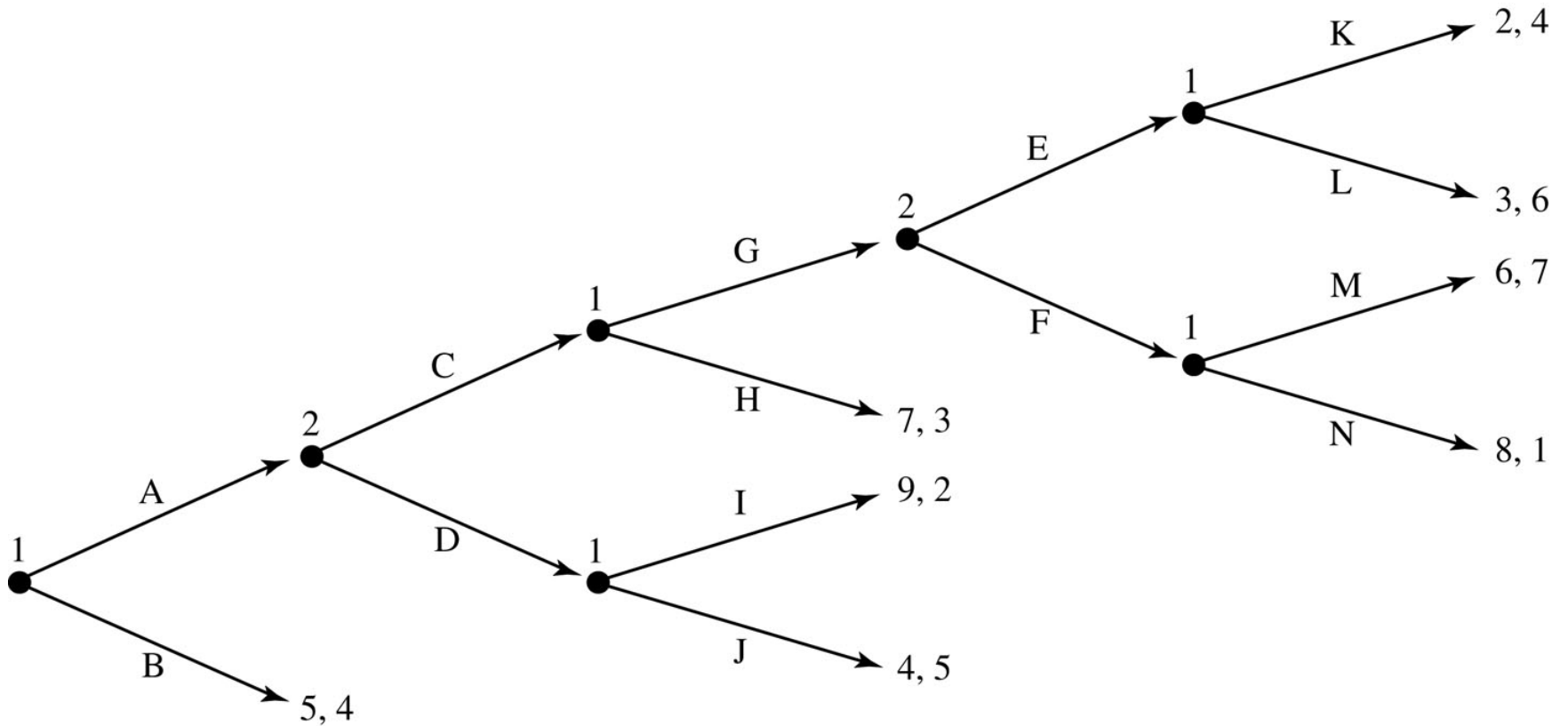
- We identified the profile (DE, AC) as the sole profile consistent with backward induction. We see that it is in fact a Nash equilibrium of the game. However, there are TWO other equilibria: (DF, AD) and (DF, BD) . However, these are NOT consistent with sequential rationality.

- In both of these equilibria, player 1 is supposed to choose F this decision node:



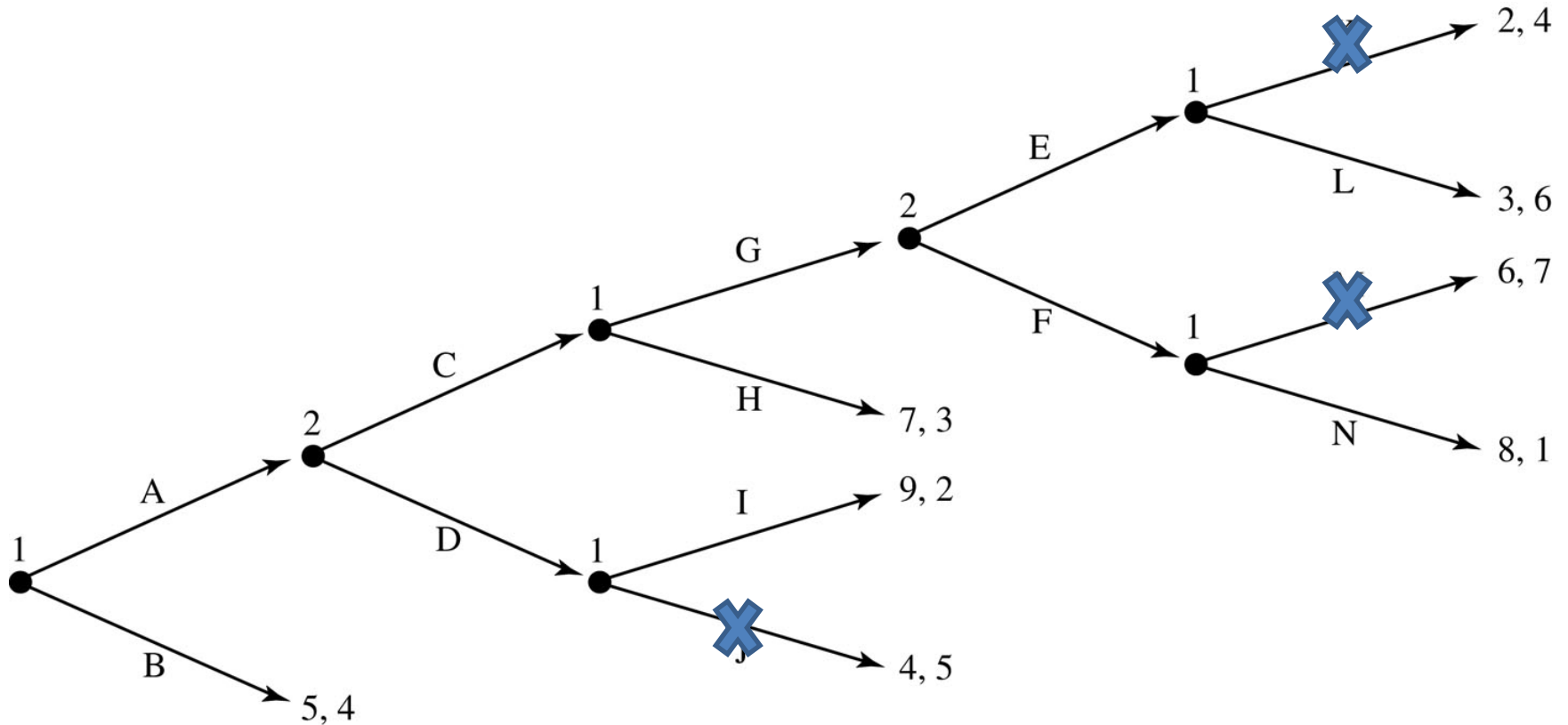
- But if that node is reached, choosing F is dominated by choosing E .

- **Example:** Consider the following perfect-information game:

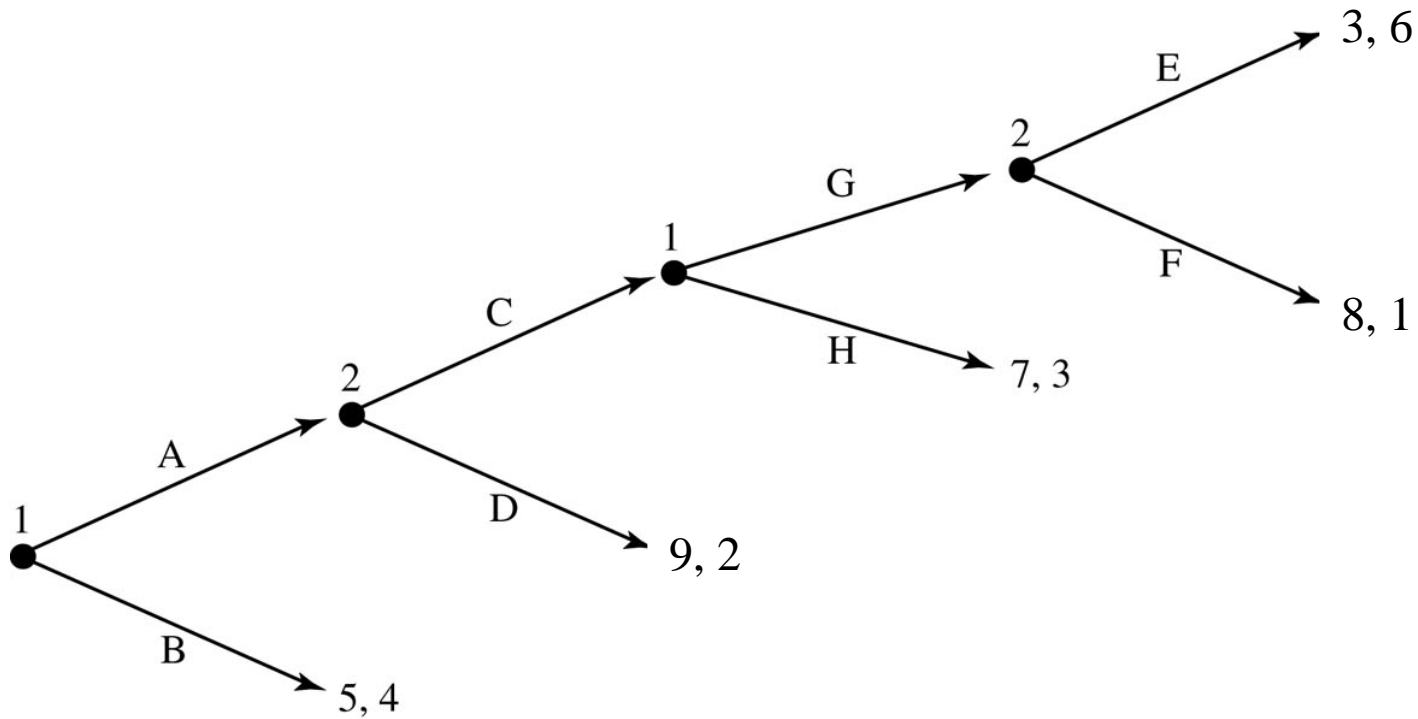


- Player 2 has a 2 decision nodes with 2 actions available in each node. Therefore he has 4 total different strategies.
- Player 1 has 5 decision nodes with 2 actions available in each node. Therefore he has $2^5 = 32$ total different strategies.
- Therefore this game has $4 \times 32 = 128$ total different outcomes in the normal form representation.
- Looking for all Nash equilibria in the normal form would be a tedious exercise in this game. However, looking for the Nash equilibrium that results from backward induction is a relatively simple task.

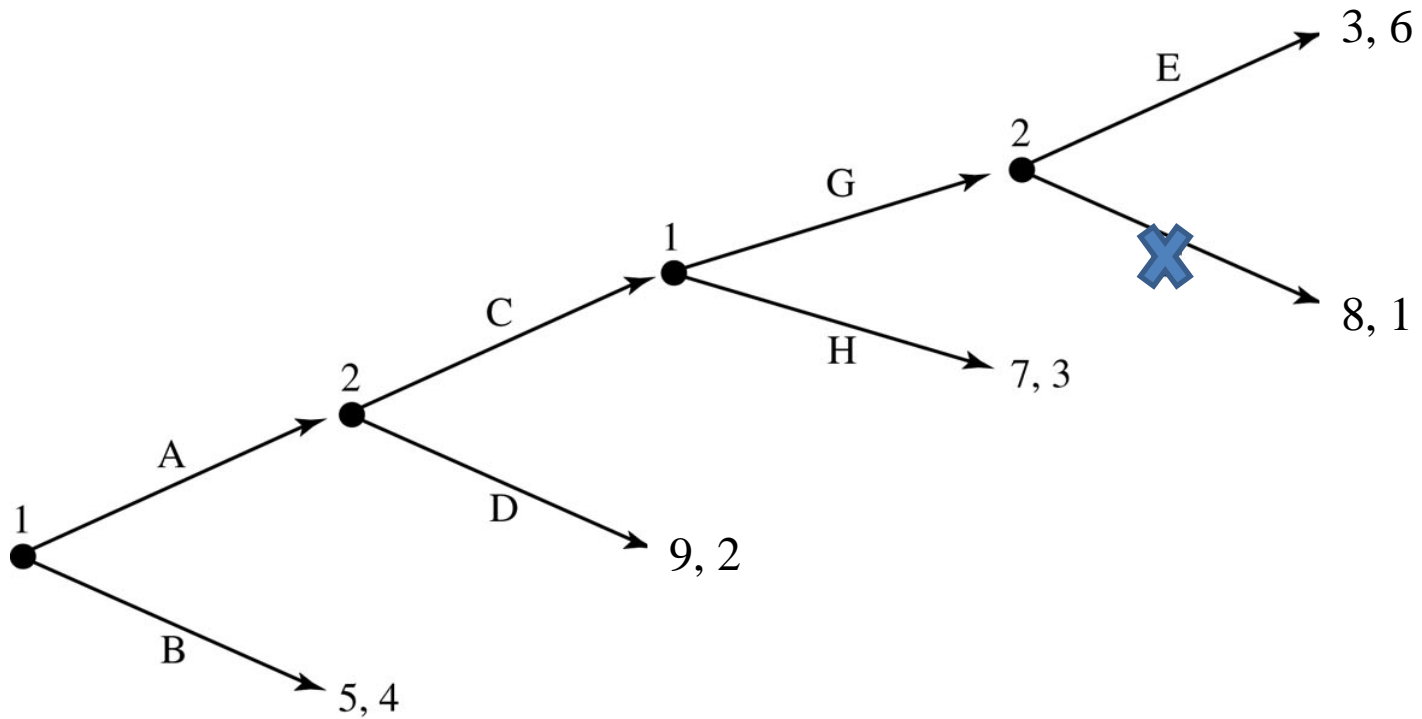
- We start by striking out the dominated actions at the terminal nodes:



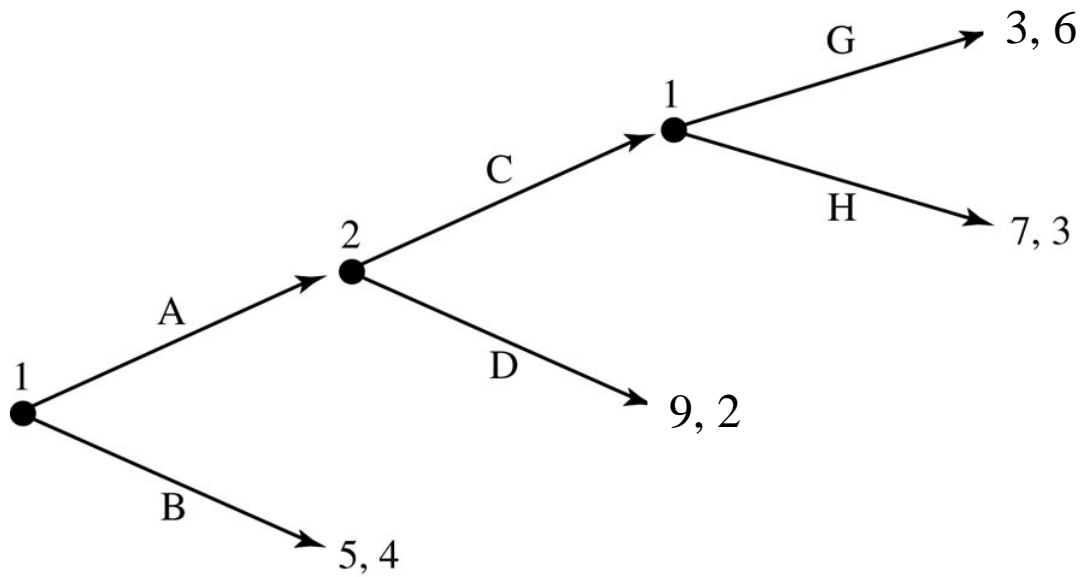
- After this first step, the continuation payoffs look like this:



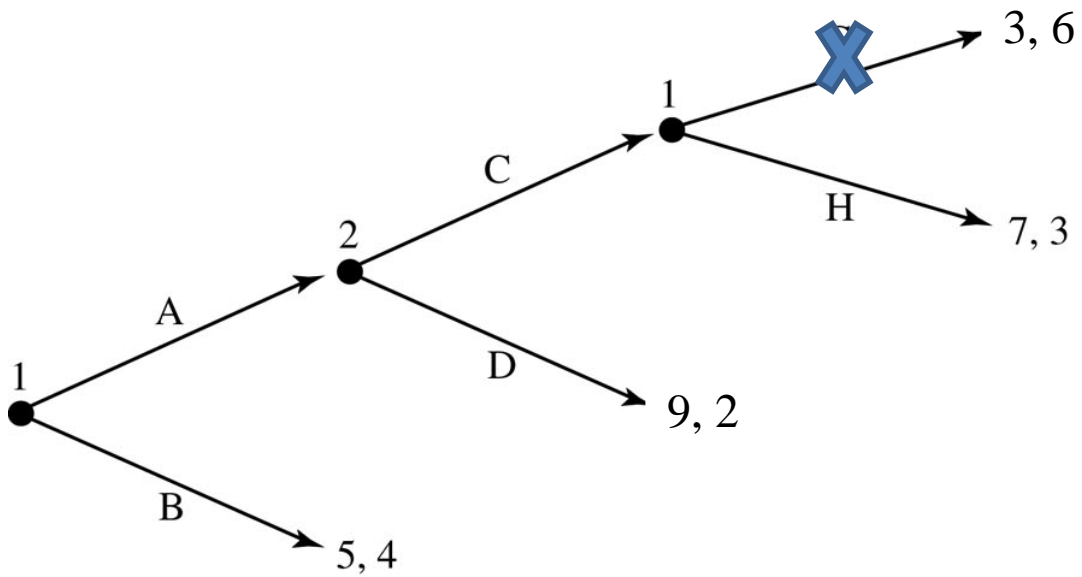
- We now strike out the following strategy:



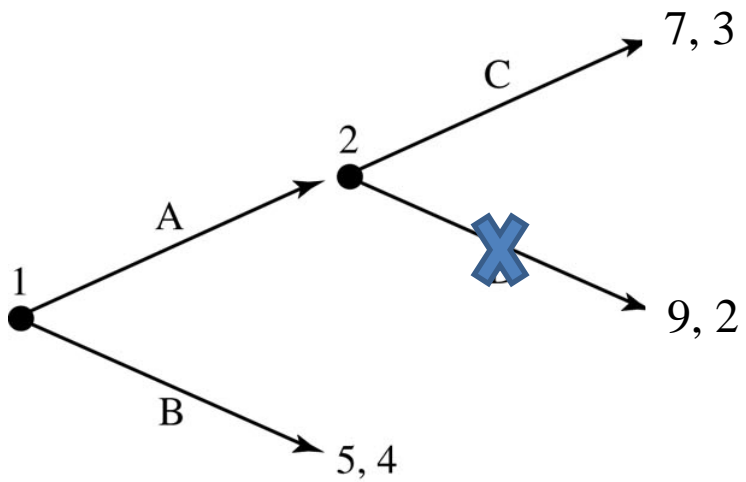
- And the continuation payoffs become:



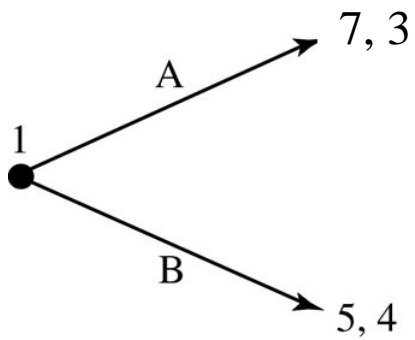
- We now strike out the following strategy:



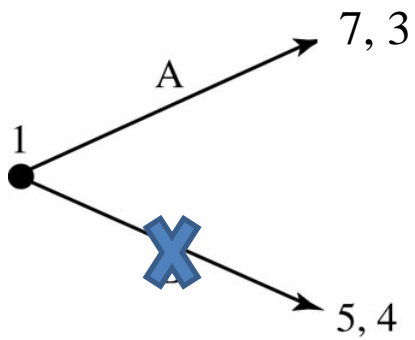
- We now strike the following strategy:



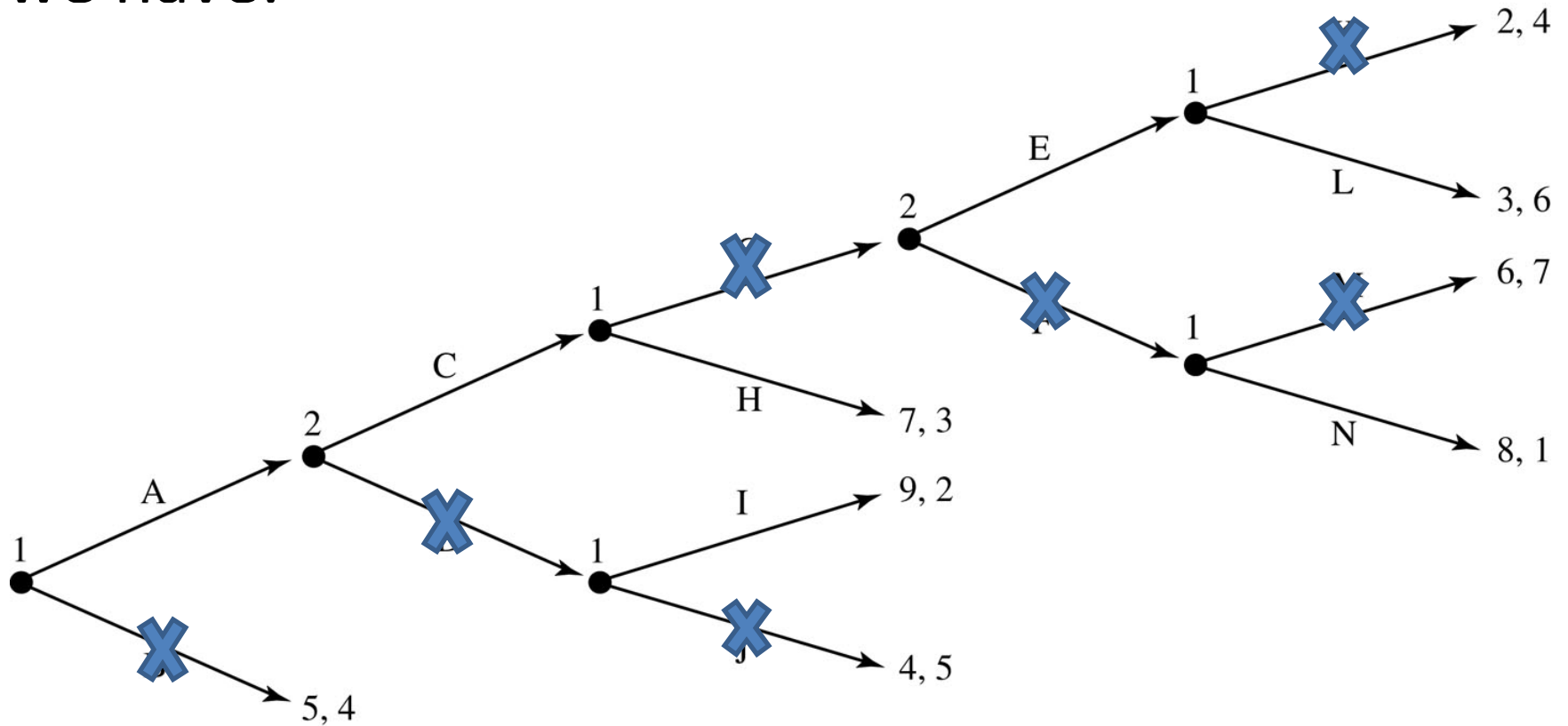
- And the continuation payoffs become:



- In the final step, we strike out:



- Re-tracing all the strategies that were stricken out, we have:

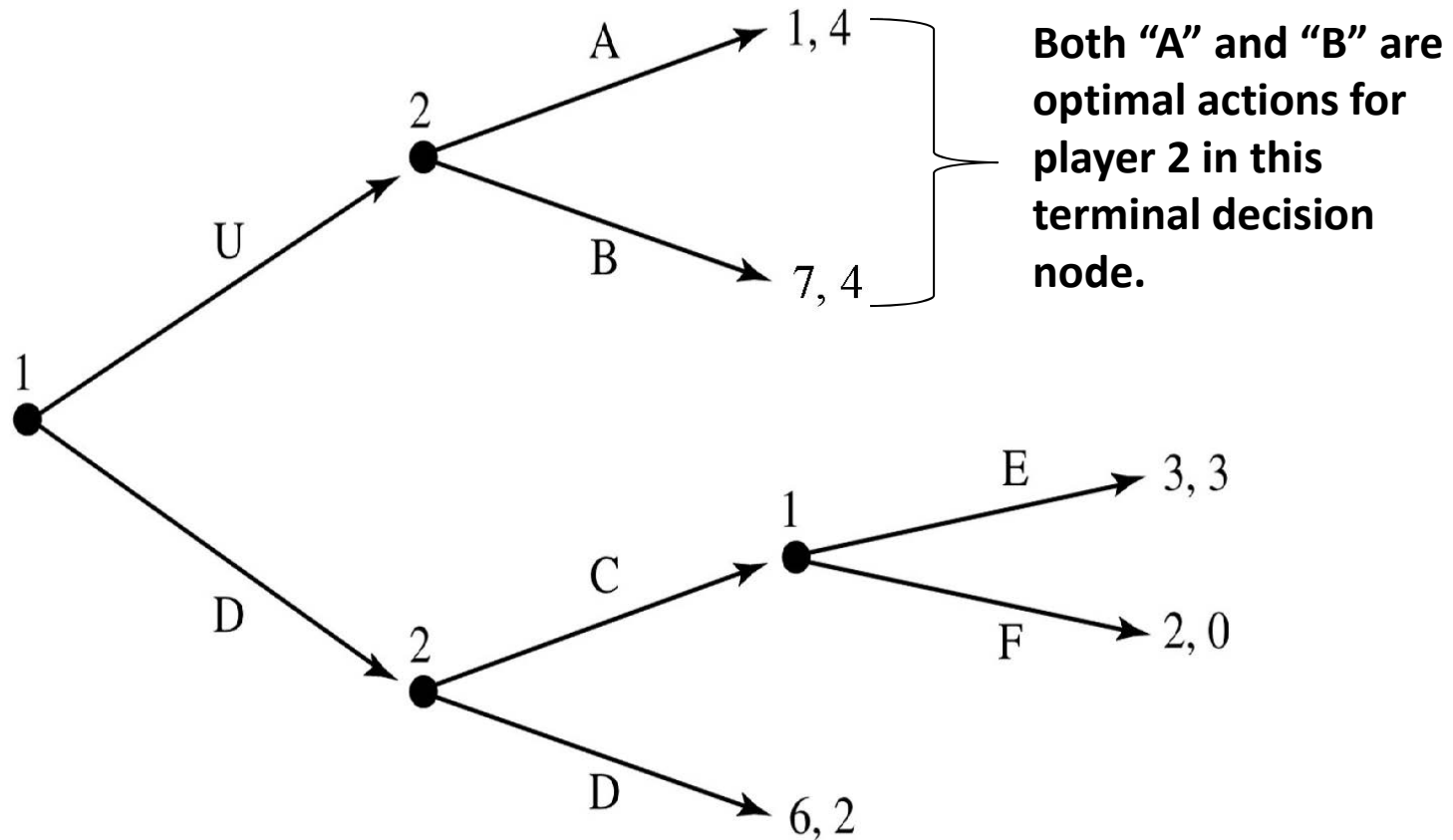


- The backward induction strategies are therefore:

$$s_1 = AHILN \quad \text{and} \quad s_2 = CE$$

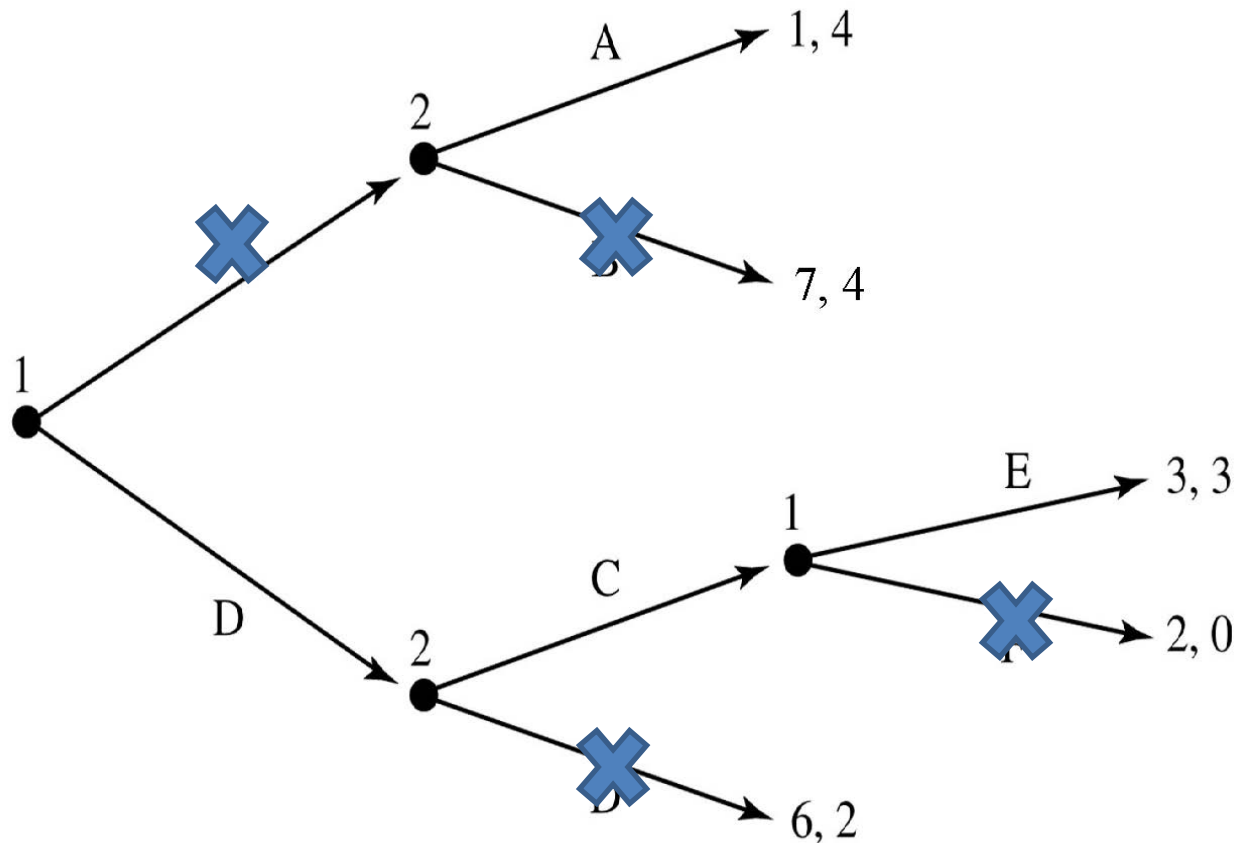
- In the examples we have seen, backward induction produced a unique profile of sequentially rational strategies. This was so because optimal strategies were unique at each step of the backward induction procedure
- We would have multiple optimal strategies if there are **ties** in the continuation payoffs produced by two or more strategies at a given decision node.
- In this case, we would have to do the backward induction exercise separately for each of the optimal strategies identified.

- For example, consider the following game:



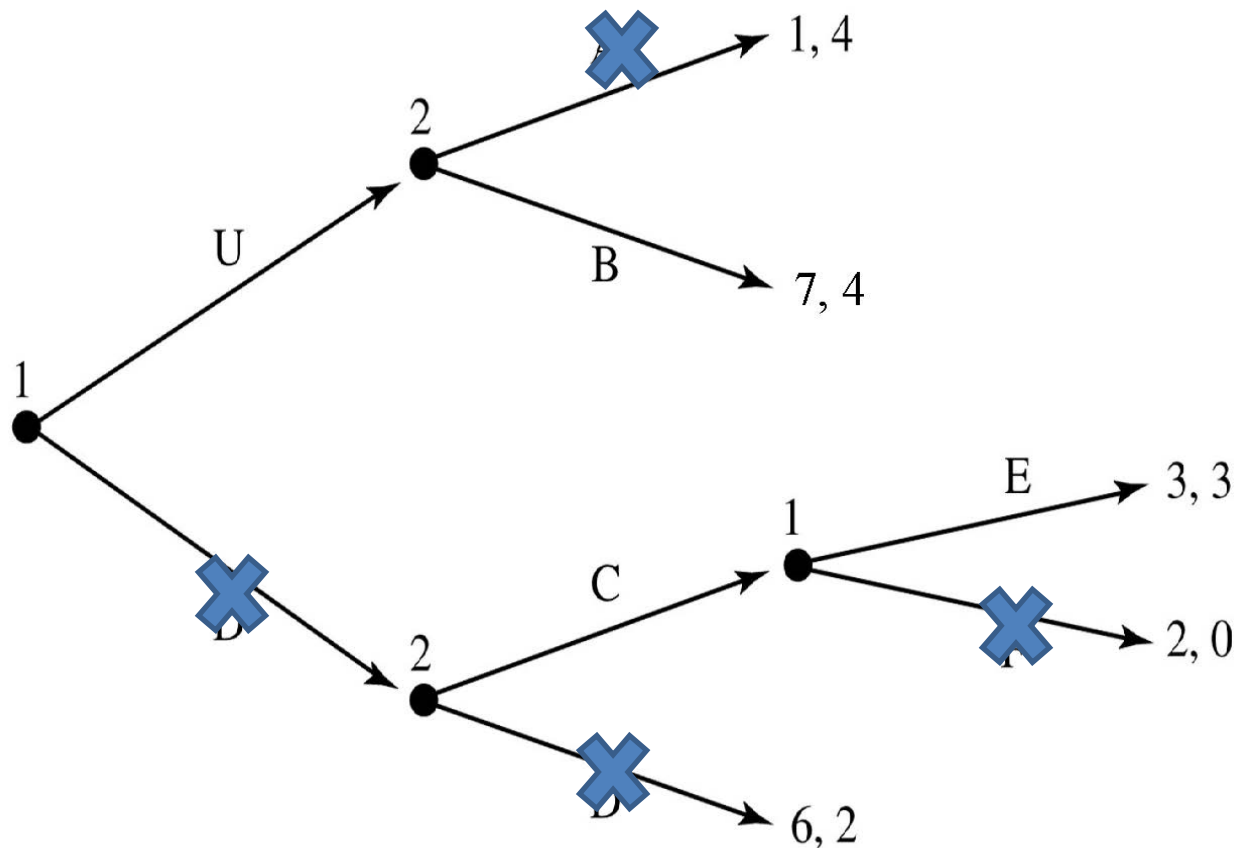
- We do backward induction for the case where player 2 chooses "A" in the final decision node, and then we do it for the case in which he chooses "B".

- Case 1.- Player 2 chooses “A” in the final decision node:



- Backward induction strategies in this case would be: $s_1 = DE, s_2 = AC$

- Case 2.- Player 2 chooses “B” in the final decision node:



- Backward induction strategies in this case would be: $s_1 = UE$, $s_2 = BC$

- We conclude that this game has two strategy profiles consistent with backward induction:

$$(s_1, s_2) = (DE, AC)$$

and

$$(s_1, s_2) = (UE, BC)$$

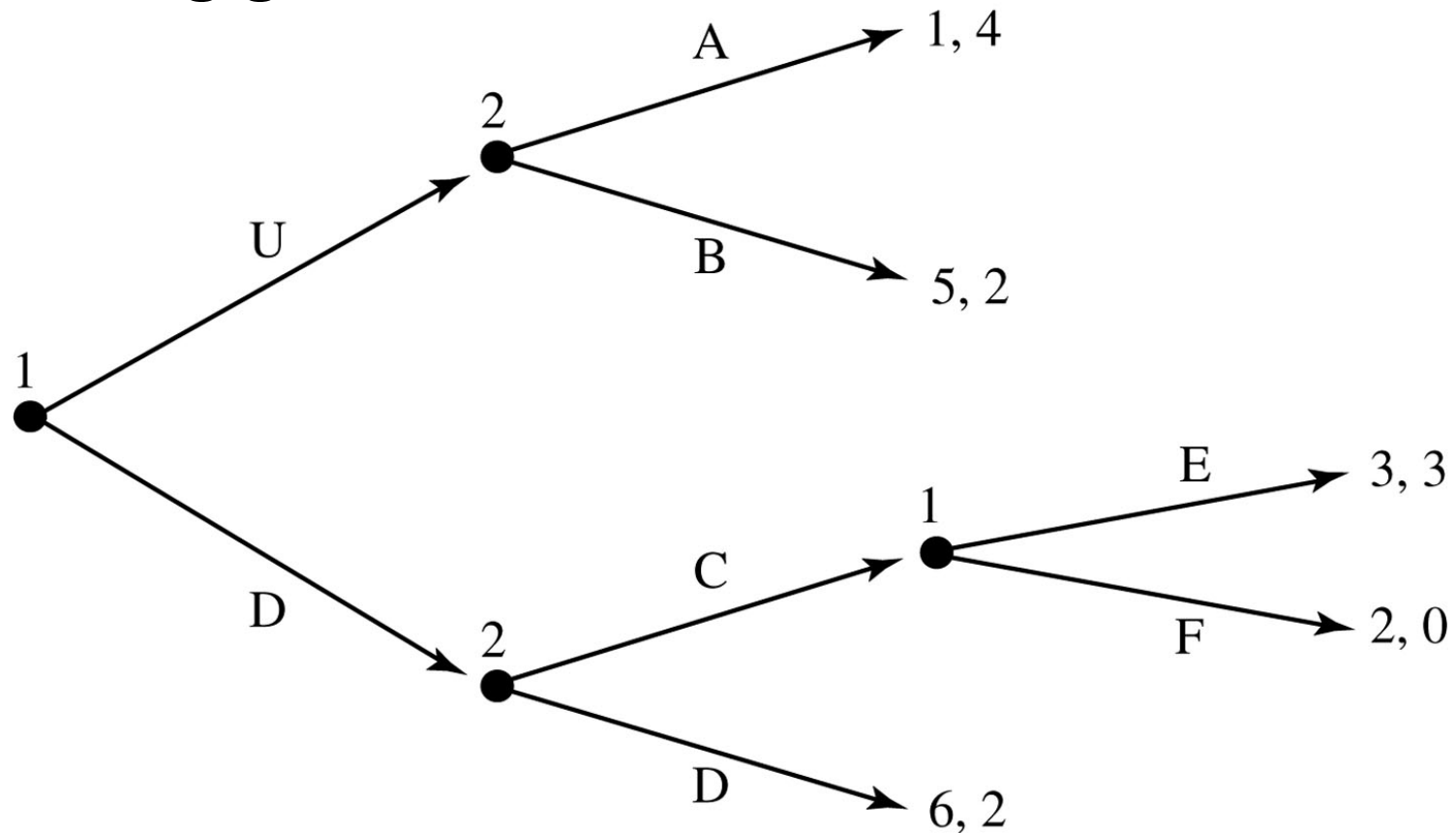
- By our previous results, both of these are also Nash equilibria.

- **Subgame perfect Nash equilibrium**
- Ok, so backward induction helps us find sequentially rational strategies in games of perfect information.
- **How do we look for sequentially rational strategies in games of imperfect information?**
- To show how, we first need to introduce the concept of a **subgame**.
- A **subgame** is a subset of the original game that satisfies certain key properties.
- We define these properties next...

- A **subgame** within a game must satisfy the following:
 1. The initial node x of a subgame must always be a singleton information set.
 2. A subgame must contain all successors of its initial node x .
 3. Every node in the subgame must be a successor of its initial node x .
 4. If a node “ y ” is included in a subgame, then every node in the information set of y must also be included in the subgame.

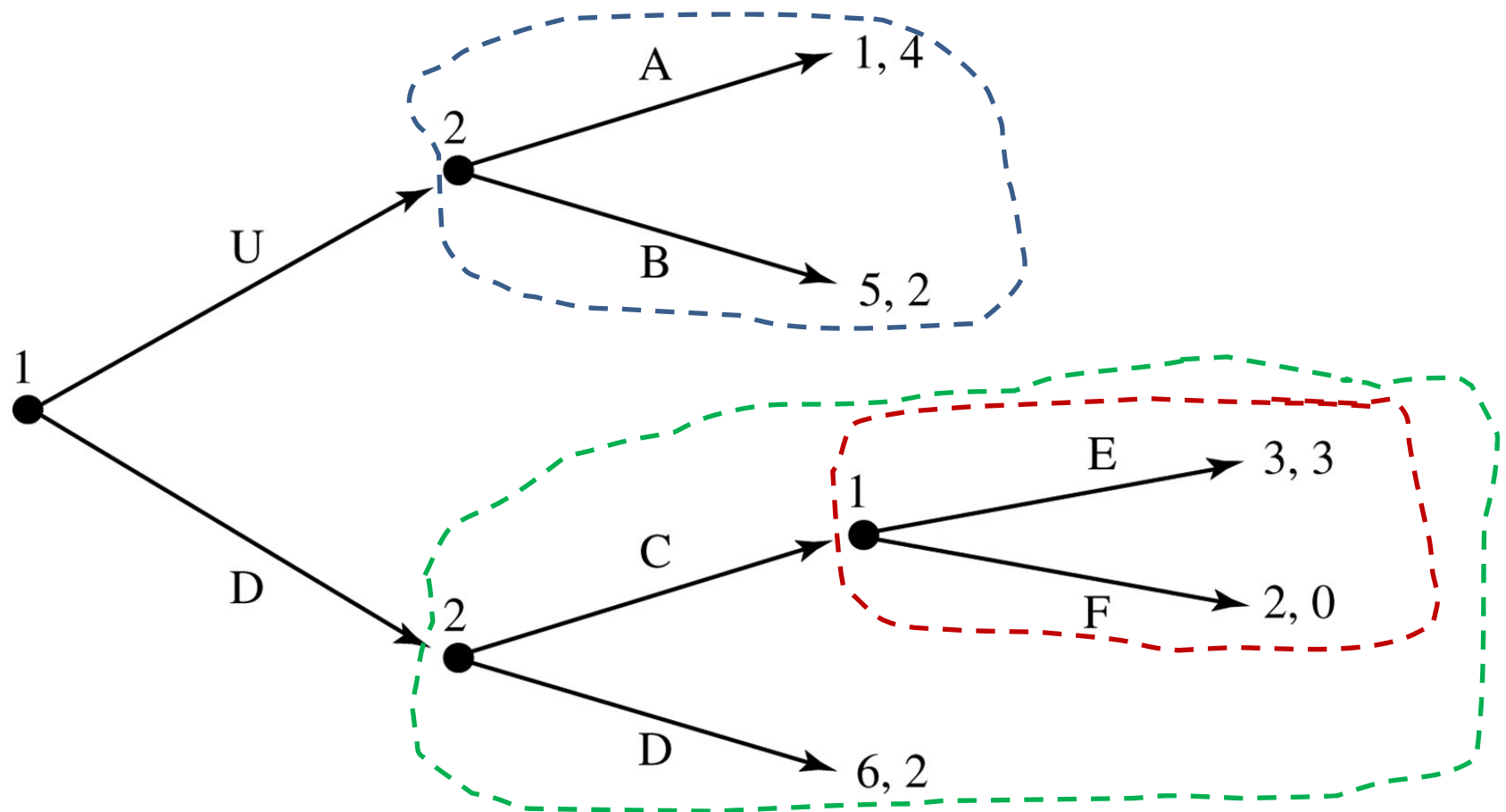
- The following definition of a subgame is given in the textbook. It is exactly equivalent to conditions 1-4 given above:
- A node x is said to *initiate a subgame* if neither x nor any of its successors are in an information set that contains nodes that are not successors of x . If these conditions are satisfied, then the tree structure defined by x and all its successors is a subgame.
- By construction, **the original game is always a subgame of it**. Subgames that start from nodes other than the original game's initial node are called **proper subgames**.

- **Example:** Identify all the subgames in the following game:



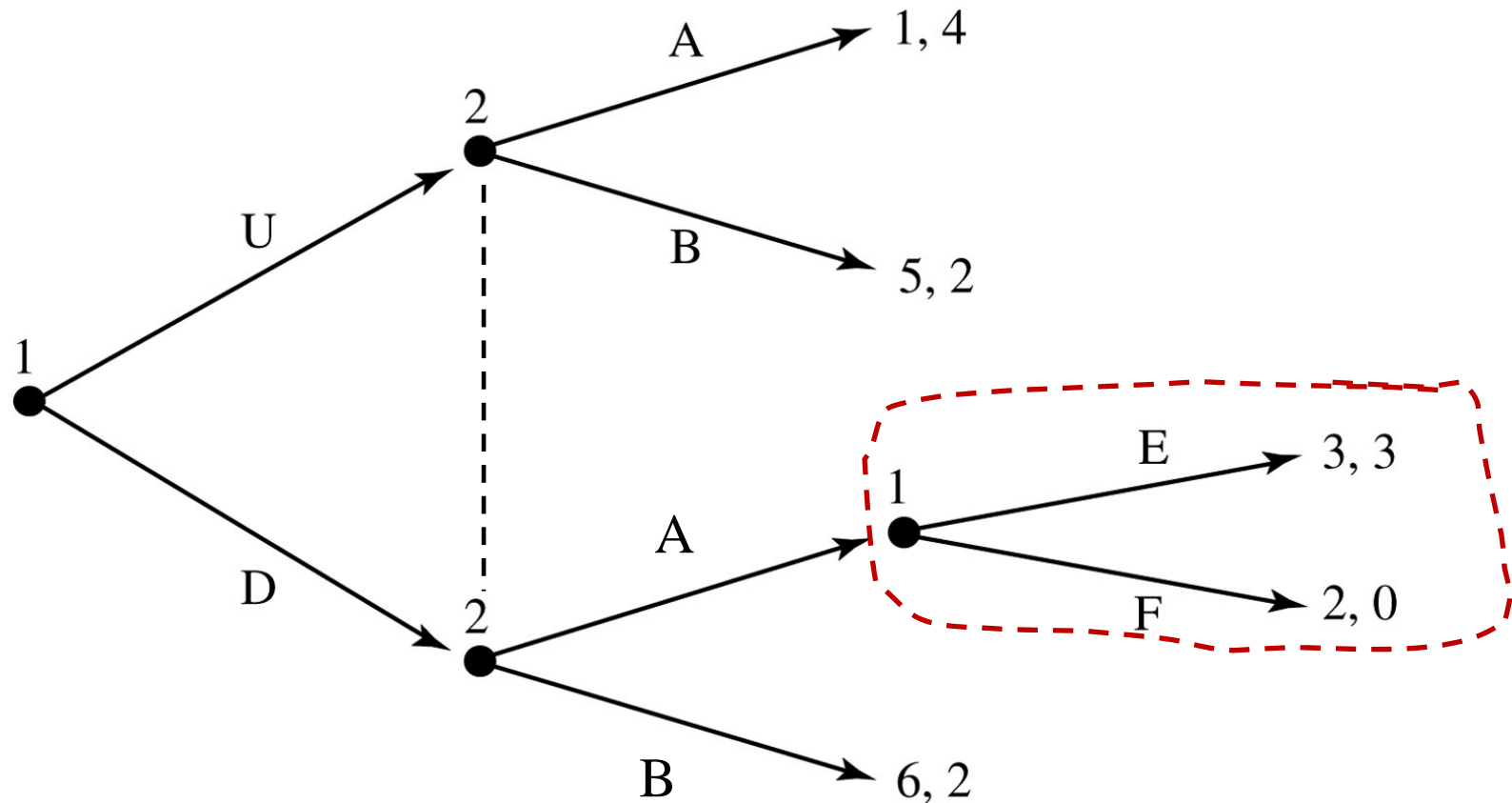
- In addition to the original game, this game has three proper subgames.

- Let us circle out the proper subgames:



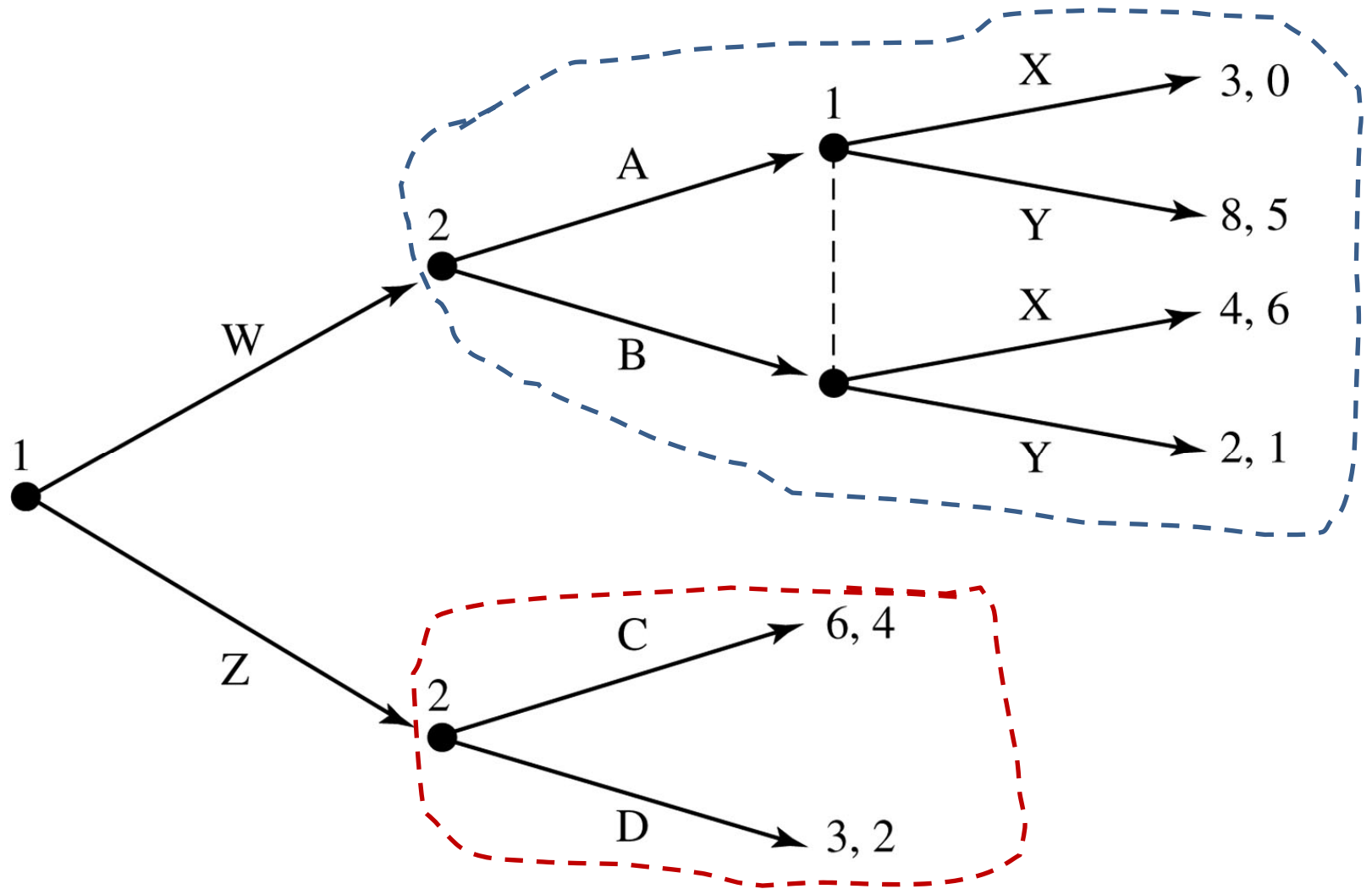
- In games of perfect information, every node initiates a subgame.

- **Example:** Now consider an imperfect-information variation of the previous game:

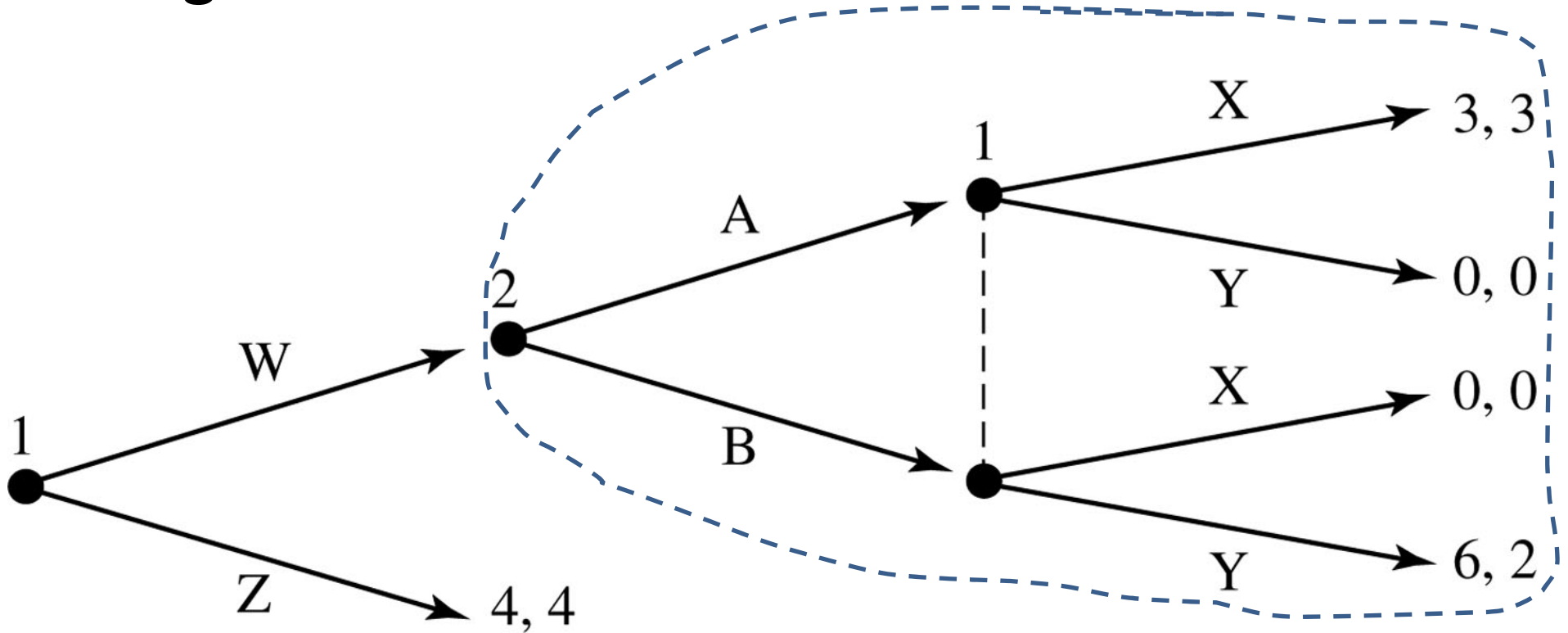


- This game has **only one proper subgame.**

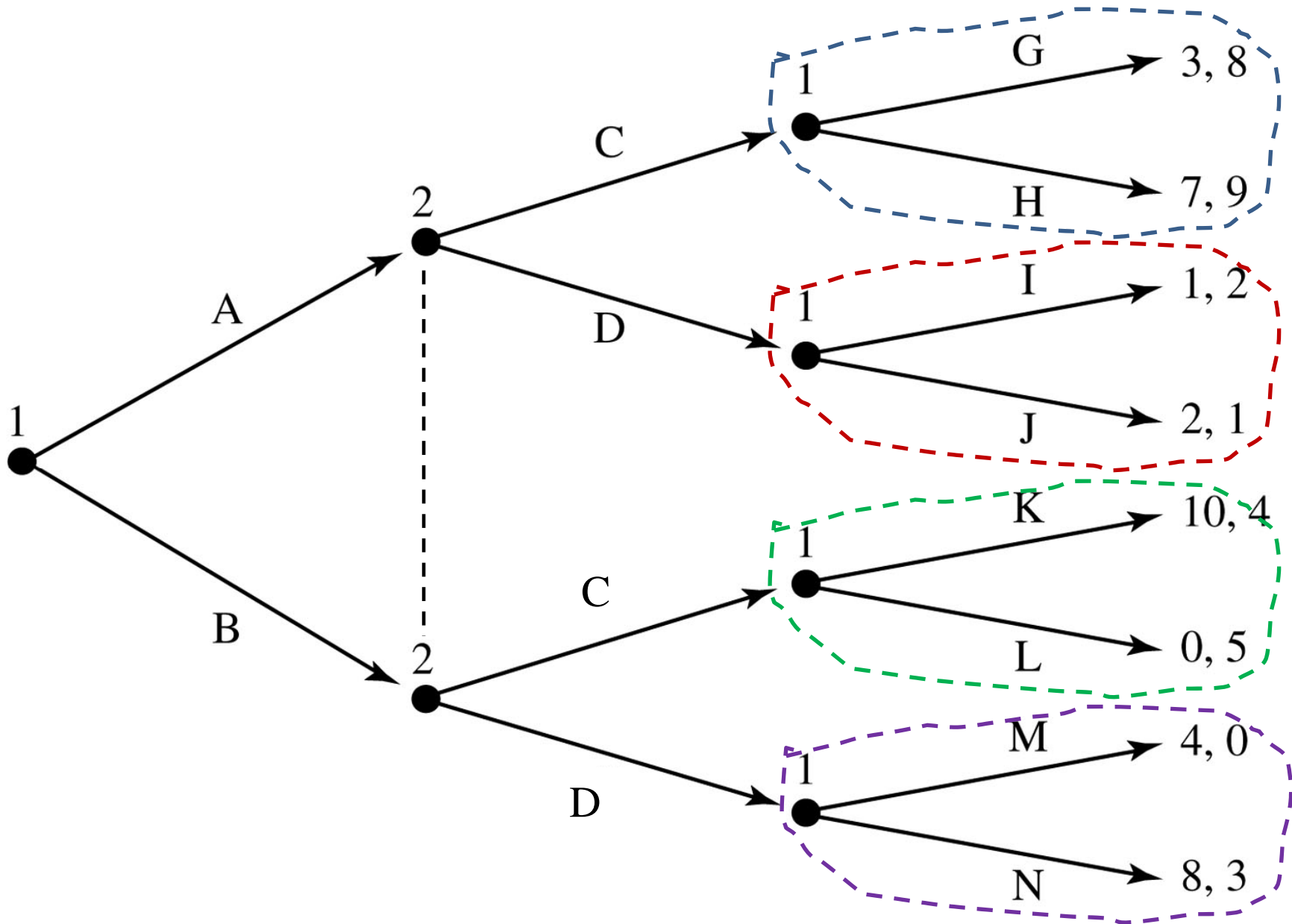
- **Example:** This game has **two proper subgames**:



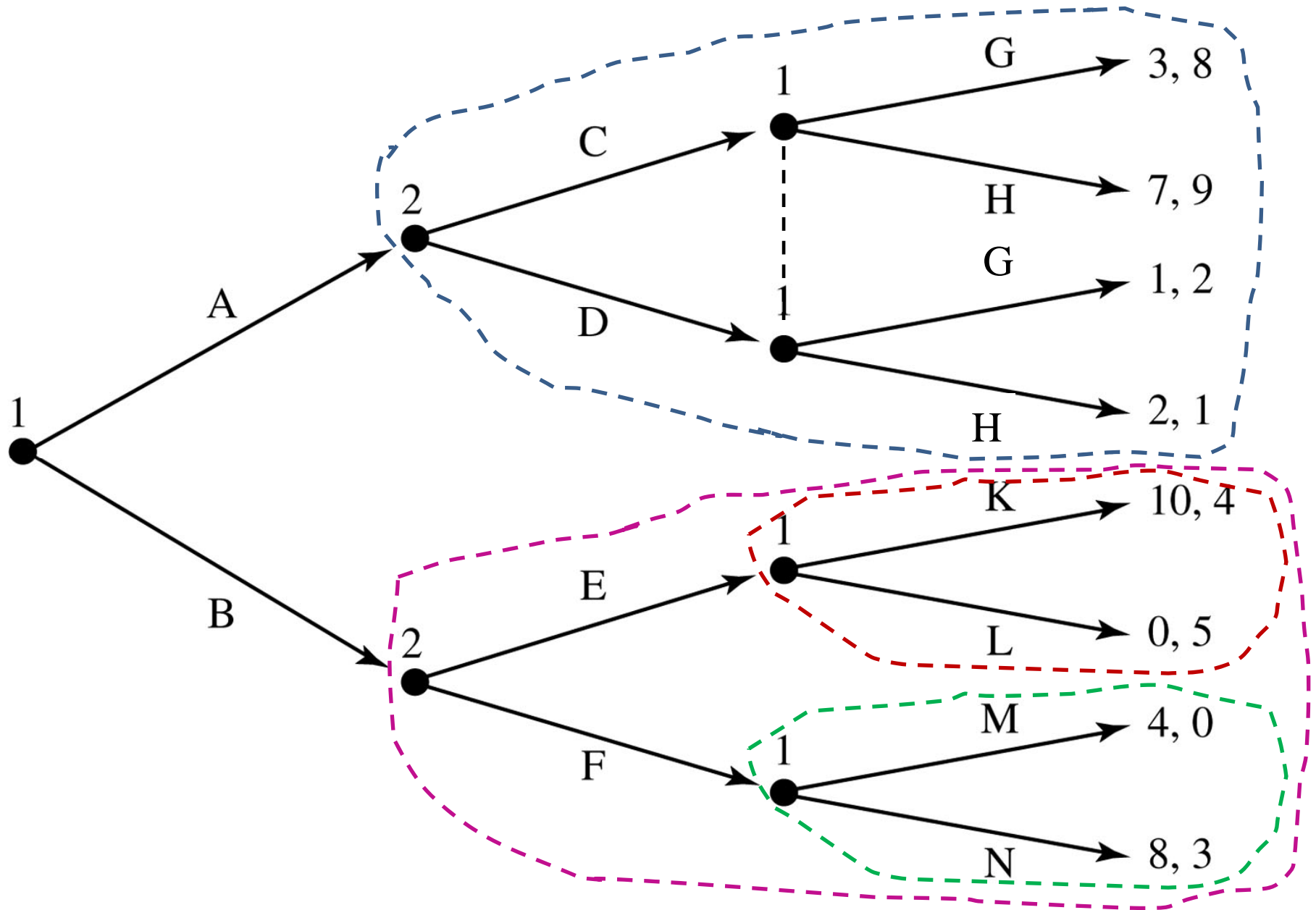
- **Example:** This game has **only one proper subgame:**



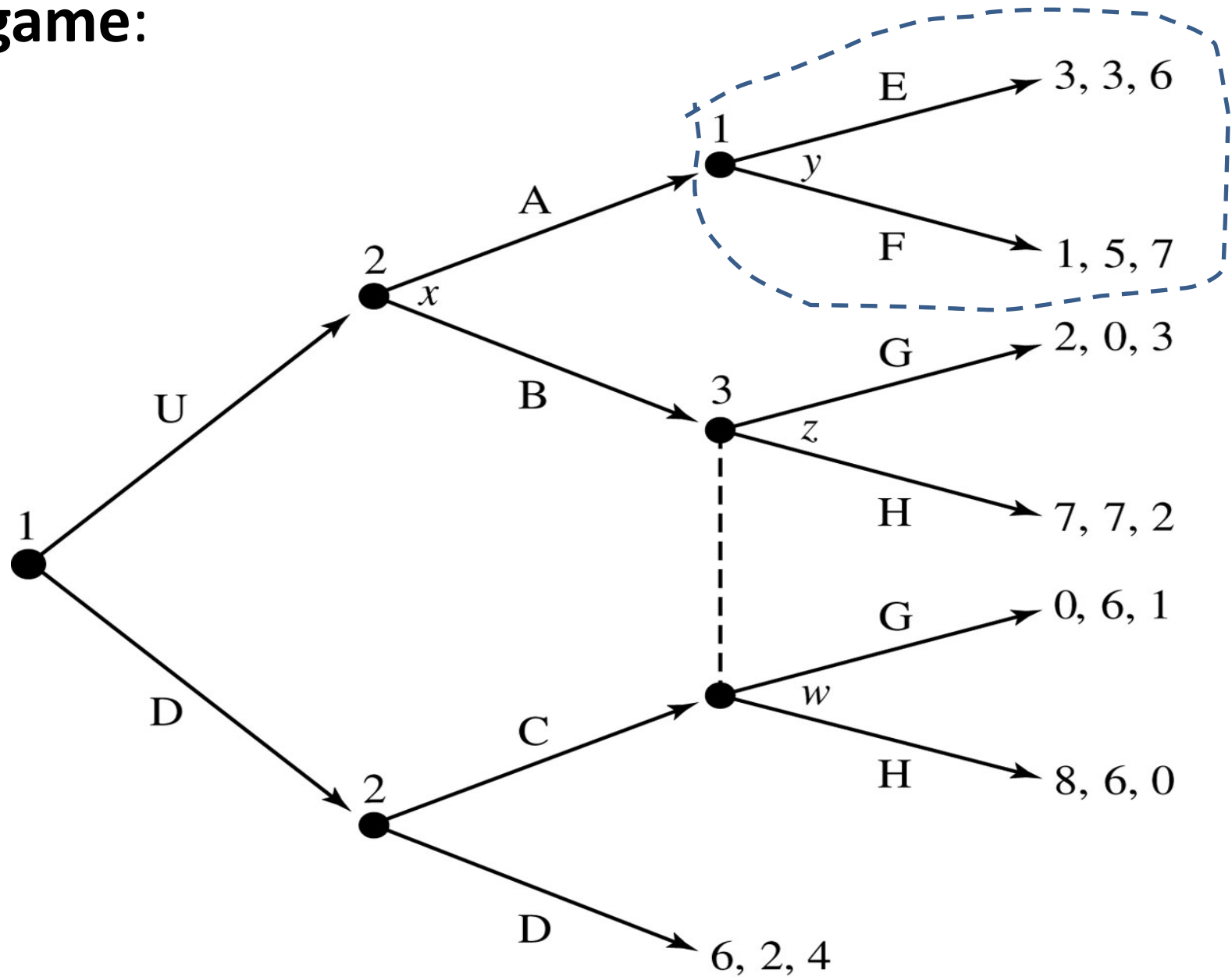
- **Example:** This game has **four proper subgames:**



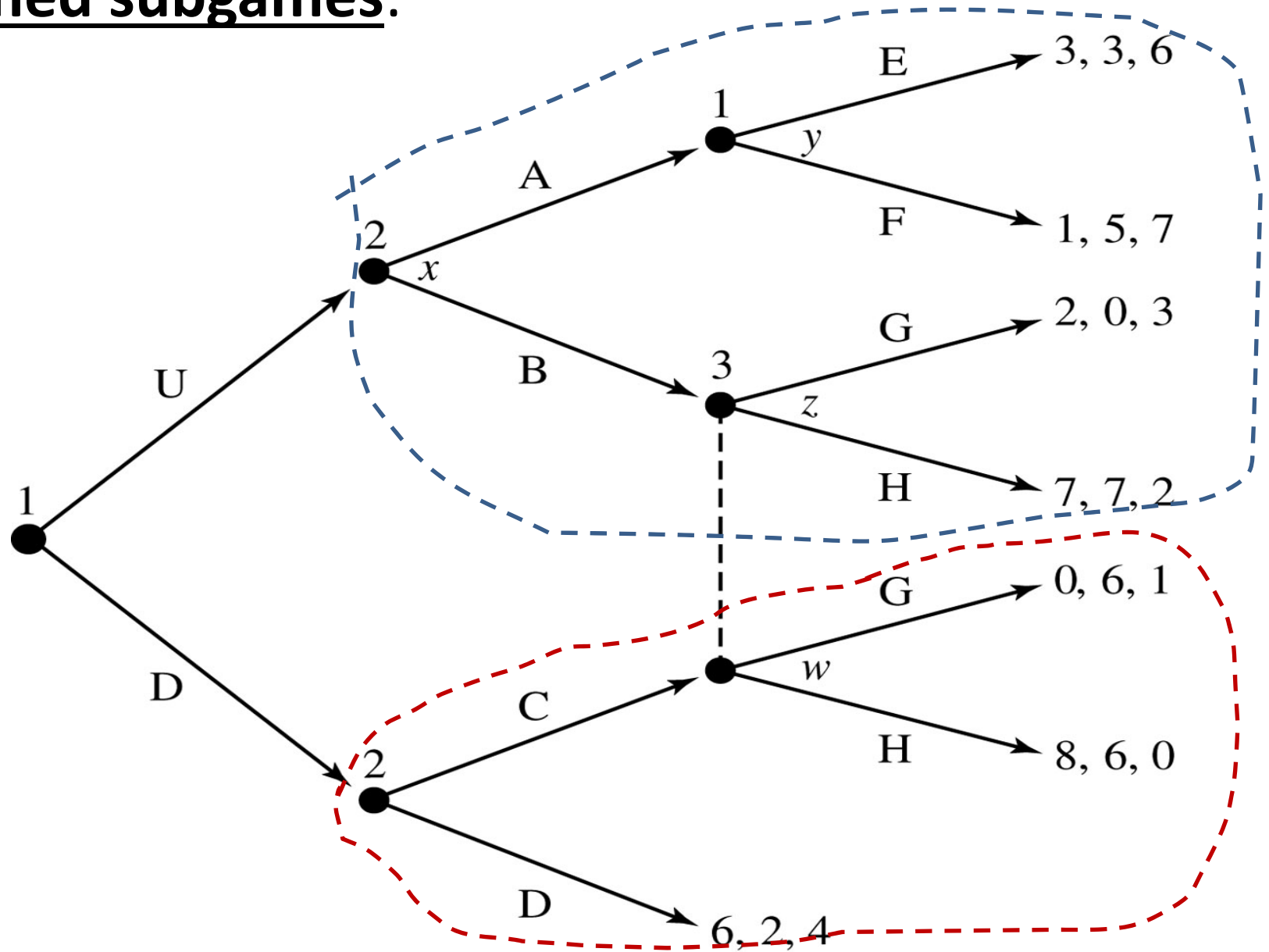
- **Example:** This game has 4 proper subgames:



- **Example:** This game has **only one proper subgame:**

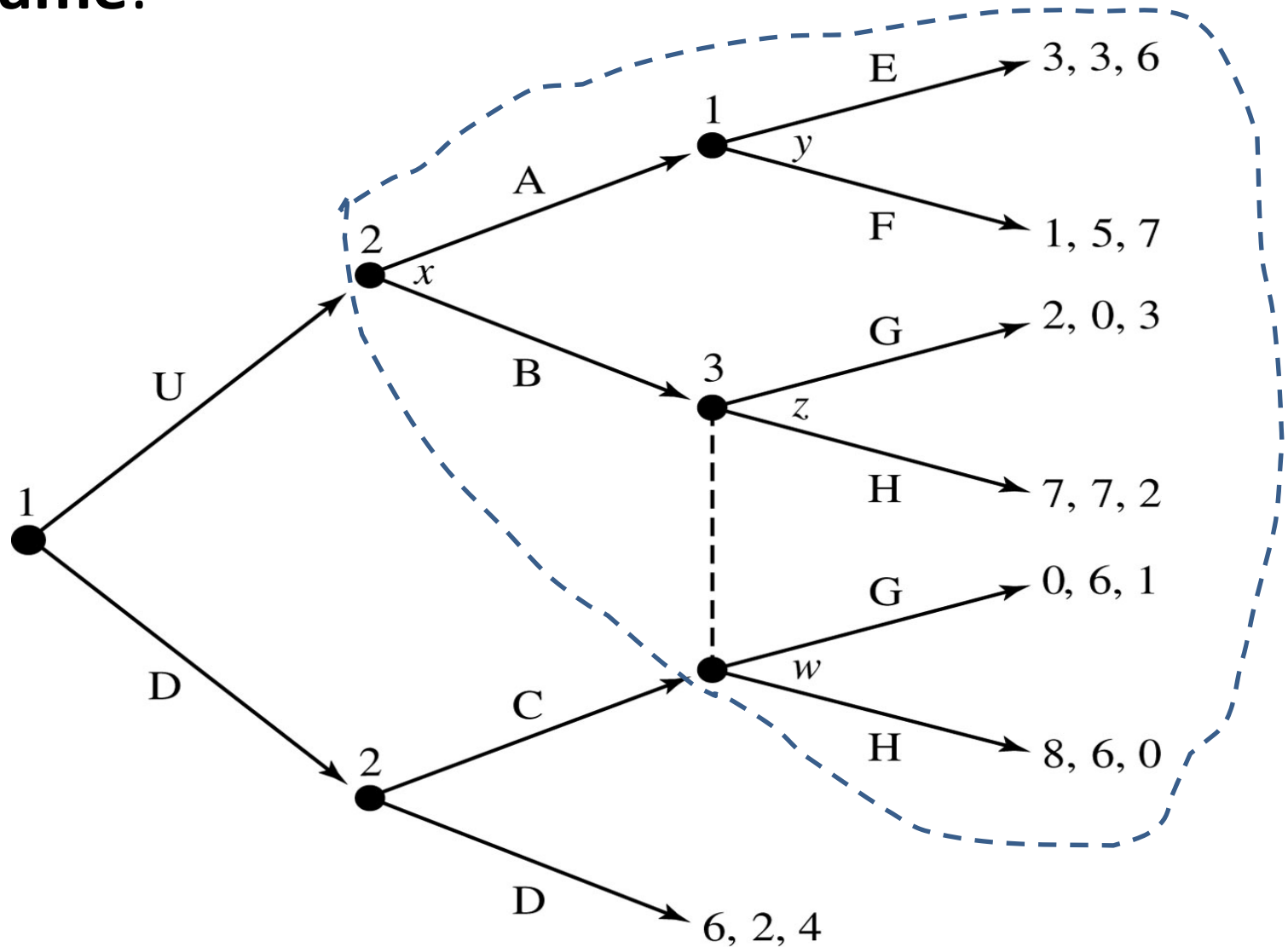


- In particular, note that the following are not well-defined subgames:



- Why are they not subgames? Because subgames are required to include the entire information set of each node included in the subgame. This requirement fails in both cases:
 - It fails for the node "z" in the first case.
 - It fails for the node "w" in the second case.

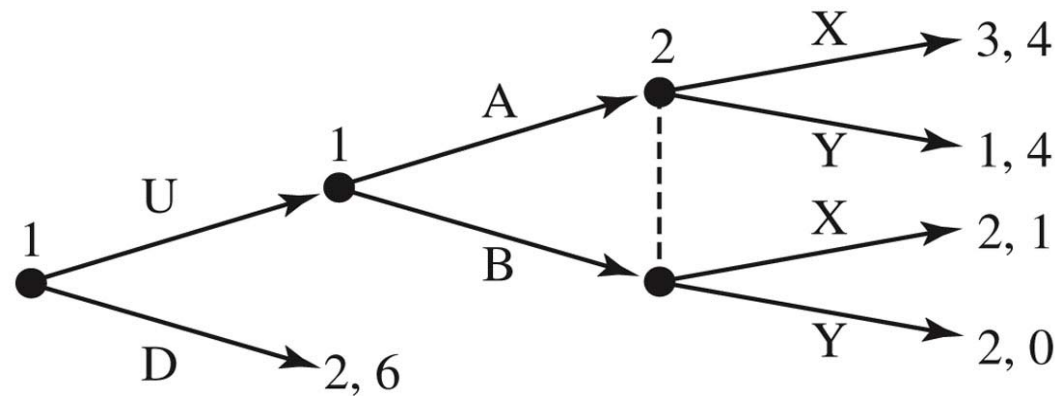
- The following is still **not a well-defined subgame**:



- Why? The initial node of the “candidate subgame” is x , but the node w is not a successor of x .
- **Strategies and subgames:** Since a strategy is a complete contingent plan for each player along his information sets, it follows that a **strategy always specifies instructions of what to do in each subgame.**

- We are now ready to define our main notion sequential rationality, which includes backward induction as a special case.
- This notion is the concept of **subgame perfect Nash equilibrium**.
- **Subgame perfect Nash equilibrium:** A strategy profile is called a subgame perfect Nash equilibrium (SPNE) if it specifies a Nash equilibrium in every subgame of the original game.

- Example:** This game has only two subgames: The original game and one proper subgame. We have identified all (pure-strategy) Nash equilibria for each subgame:



First subgame: (original game).

		2	
		X	Y
1	U	(3, 4)	1, 4
	B	2, 1	2, 0
	D	2, 6	(2, 6)
	B	2, 6	(2, 6)

		2	
		X	Y
1	A	(3, 4)	1, 4
	B	2, 1	2, 0

Second subgame: (Proper subgame)

- The first subgame is the original game. The Nash equilibrium profiles in the original game are:

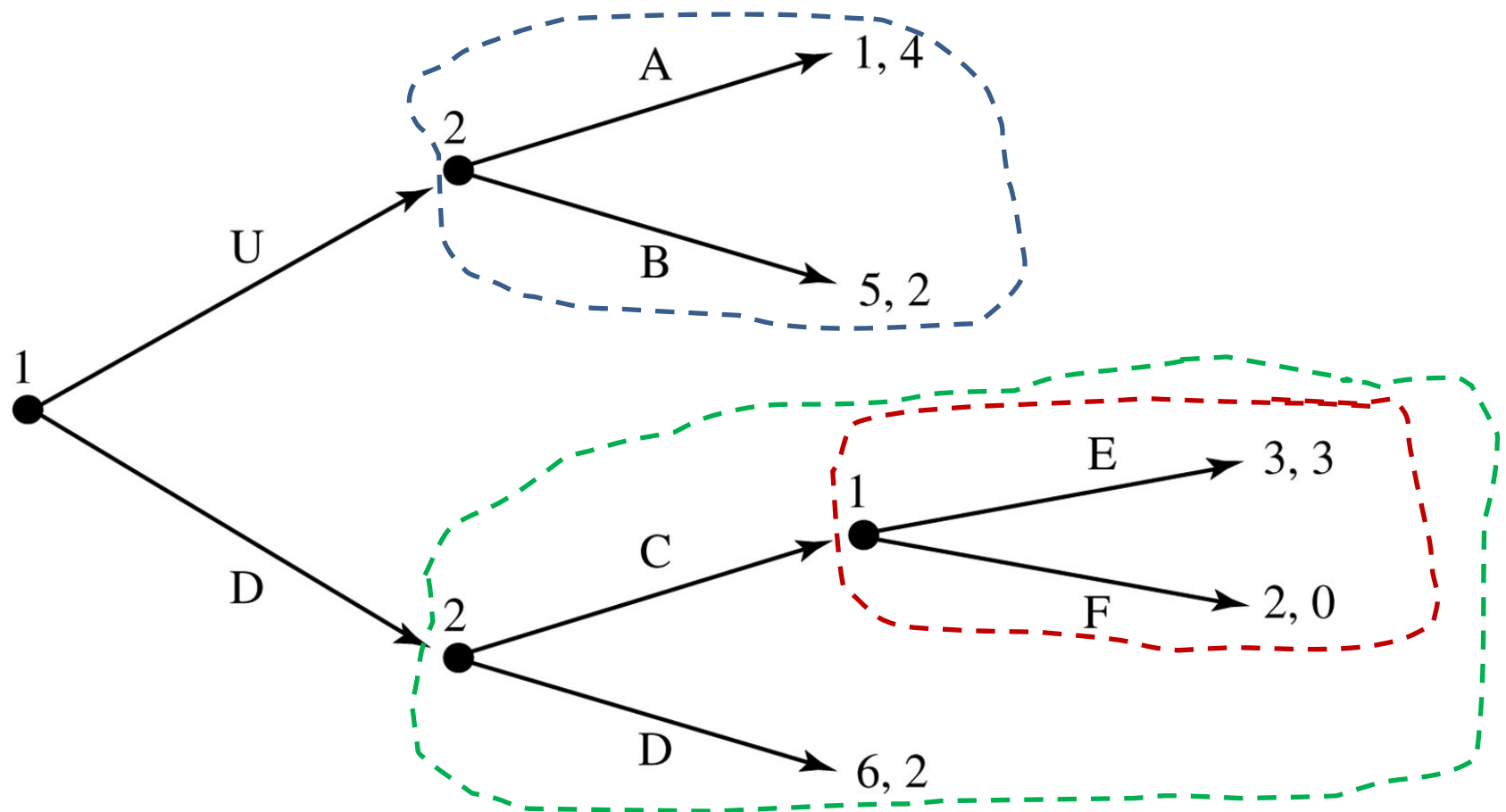
$$(UA, X), (DA, Y), \text{ and } (DB, Y)$$

- The Nash equilibrium profile in the second subgame is only:

$$(A, X)$$

- Therefore, any SPE strategy profile must specify playing “A” for player 1, and playing “X” for player 2 in the second subgame.
- This rules out the profiles (DA, Y) and (DB, Y) as SPE profiles. **The only SPE profile in this game is (UA, X) .**
- (UA, X) is the only profiles that induces a Nash equilibrium in every subgame of the original game.

- **Example:** Consider the following game



- Including the original game, there are four subgames in total.

- The normal forms of each subgame and their Nash equilibria are:

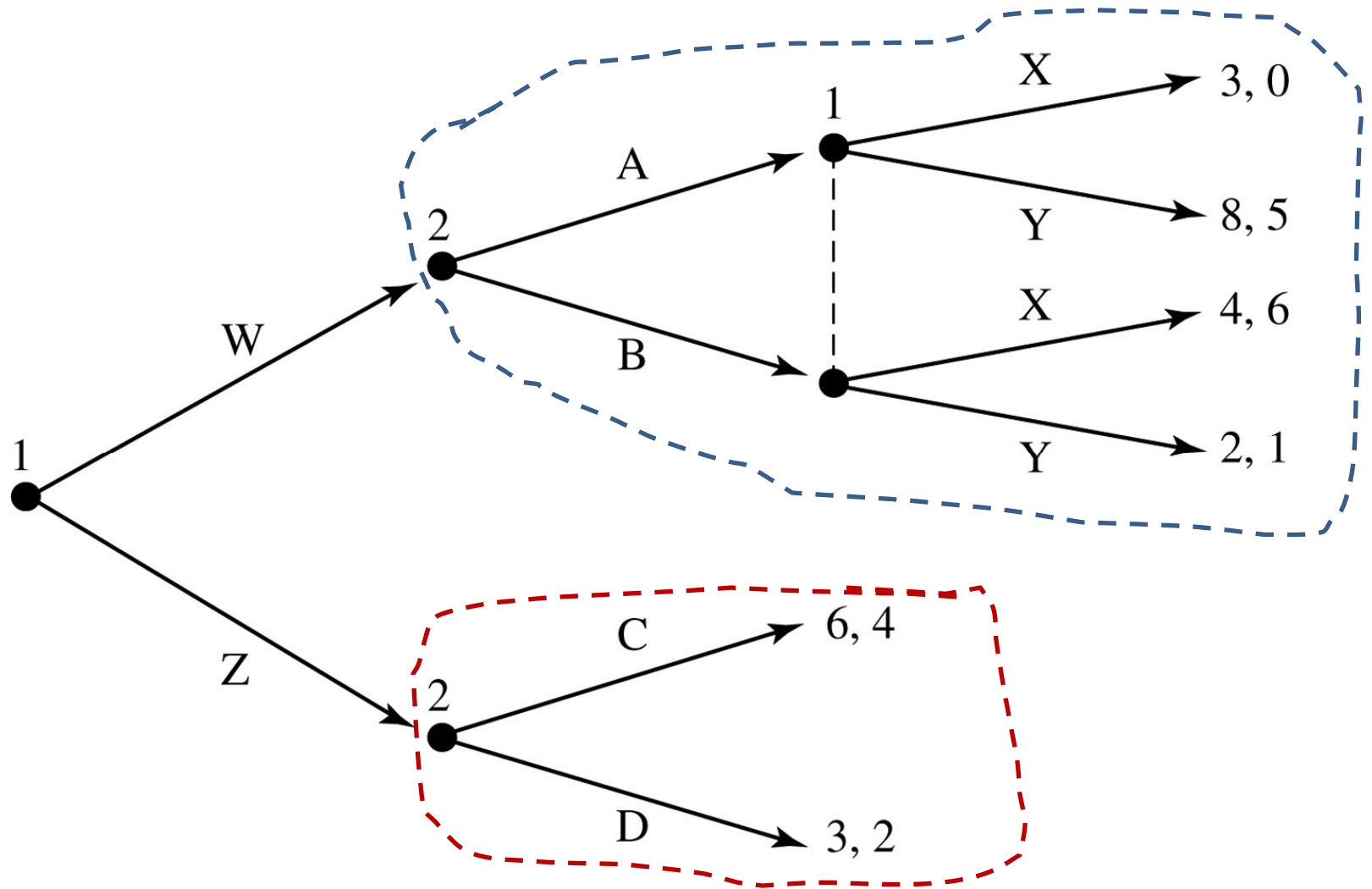
P1 \ P2		AC	AD	BC	BD	
		UE	1, 4	1, 4	5, 2	5, 2
		UF	1, 4	1, 4	5, 2	5, 2
		DE	3, 3	6, 2	3, 3	6, 2
		DF	2, 0	6, 2	2, 0	6, 2

P1 \ P2		C	D				
		E	3, 3	6, 2	P1		
		F	2, 0	6, 2	E	3	
					F	2	
						A	4
						B	2

- From here we see that the only SPE profile of strategies in this game is: ***(DE, AC)***

- The original game has the following Nash equilibria: (DE, AC) , (DF, AD) and (DF, BD) .
- To look for all SPE profiles, we need to check if each of the NE of the original game induces a Nash equilibrium for the other two subgames.
- We do it one at a time:
- Is (DE, AC) an SPE? Yes. It induces a Nash equilibrium in each subgame.
- Is (DF, AD) an SPE? No. It does not induce a Nash equilibrium in the third subgame: Choosing “E” dominates choosing “F” for player 1 in the third subgame.
- For the same reason, (DF, BD) is not an SPE either.
- We conclude that this game has a **unique SPE profile given by: (DE, AC)**

- **Example:** Consider the following game



- The normal forms of each subgame and their Nash equilibria are:

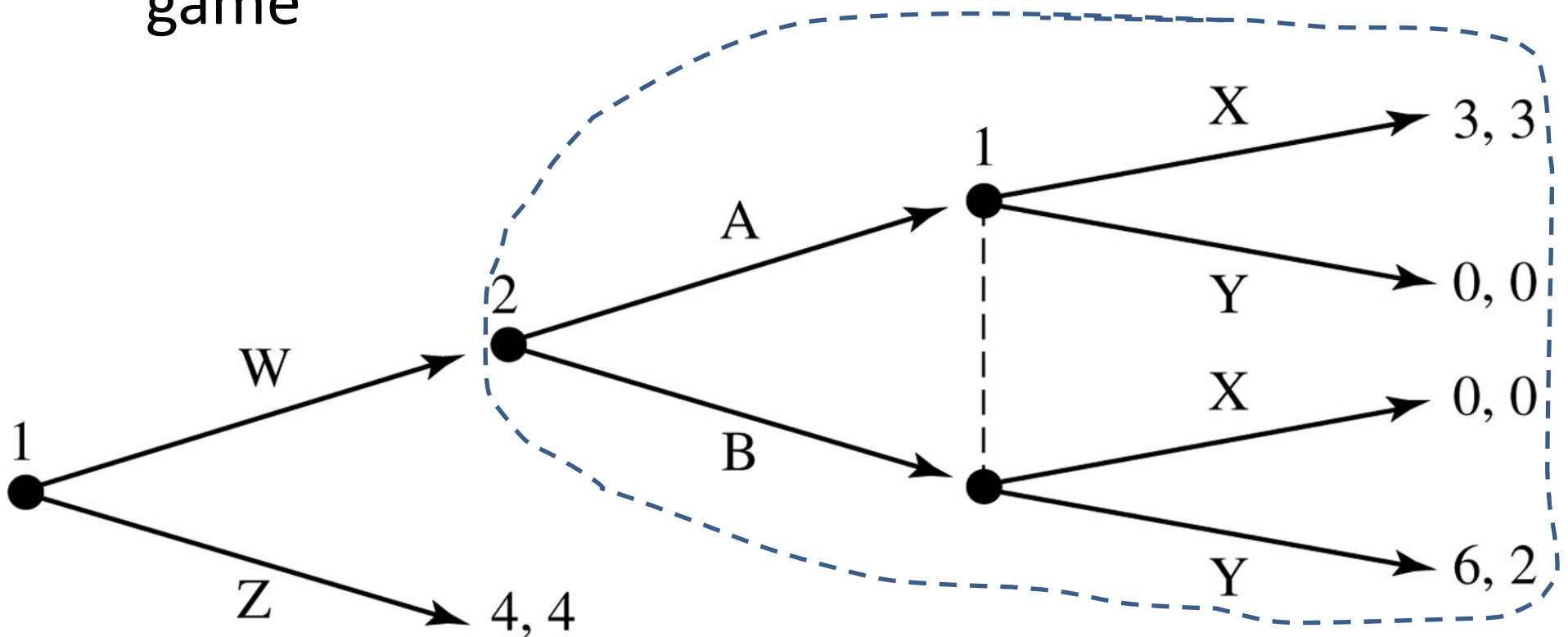
		P2			
		AC	AD	BC	BD
P1	WX	3, 0	3, 0	4, 6	(4, 6)
	WY	(8, 5)	(8, 5)	2, 1	2, 1
	ZX	6, 4	3, 2	(6, 4)	3, 2
	ZY	6, 4	3, 2	(6, 4)	3, 2

		P2			
		A	B		
P1	X	3, 0	(4, 6)	P2	
	Y	(8, 5)	2, 1		
				C	(4)
				D	2

- The Nash equilibrium profiles in the original game are (WX, BD) , (WY, AC) , (WY, AD) , (ZX, BC) and (ZY, BC) .
- To look for all SPE profiles, we need to check if each of the NE of the original game induces a Nash equilibrium for the other two subgames.
- We check each at a time:
- Is (WX, BD) an SPE profile? No because it does not induce a Nash equilibrium in the third subgame (it requires player 2 to choose “D” there, but “C” is the best response).

- Is (WY, AC) an SPE profile? Yes, it induces a Nash equilibrium in each subgame.
- Is (WY, AD) an SPE profile? No because it does not induce a Nash equilibrium in the third subgame (it requires player 2 to choose “D” there, but “C” is the best response).
- Is (ZX, BC) an SPE profile? Yes, it induces a Nash equilibrium in each subgame.
- Is (ZY, BC) an SPE profile? No because it does not induce a Nash equilibrium in the second subgame: The profile (Y, B) is NOT a Nash equilibrium in the second subgame .

- We conclude that the **SPE profiles in this game are: (WY, AC) and (ZX, BC) .**
- **Example:** Look for the SPE profiles in this game



- The normal forms of each subgame and their Nash equilibria are:

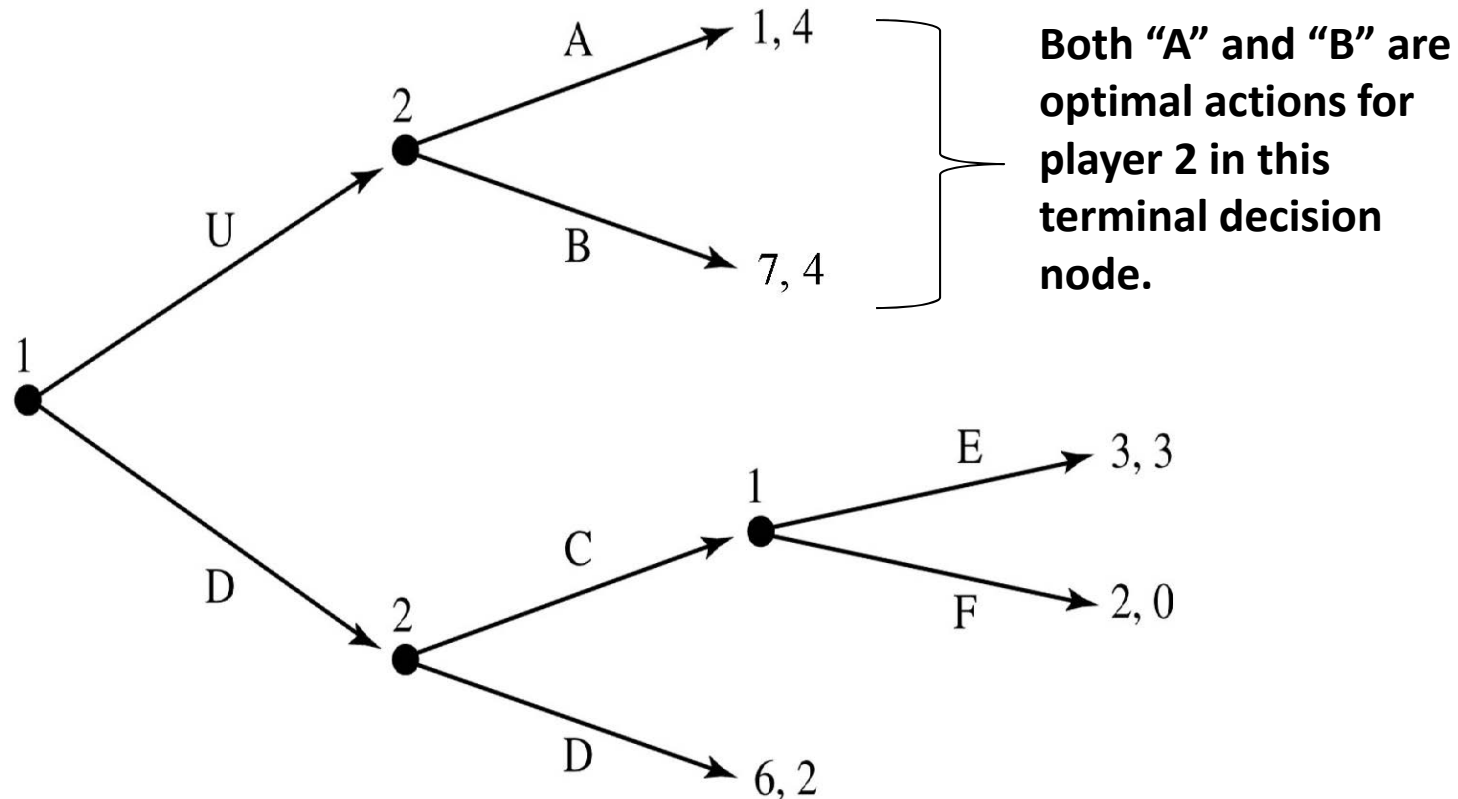
		P2	
		A	B
P1	WX	3, 3	0, 0
	WY	0, 0	(6, 2)
	ZX	(4, 4)	4, 4
	ZY	(4, 4)	4, 4

		P2	
		A	B
P1	X	(3, 3)	0, 0
	Y	0, 0	(6, 2)

- The original game has three Nash equilibrium profiles: (ZX, A) , (WY, B) and (ZY, A) .
- Is (ZX, A) an SPE? Yes, it induces a Nash equilibrium in each subgame.
- Is (WY, B) an SPE? Yes, it induces a Nash equilibrium in each subgame.

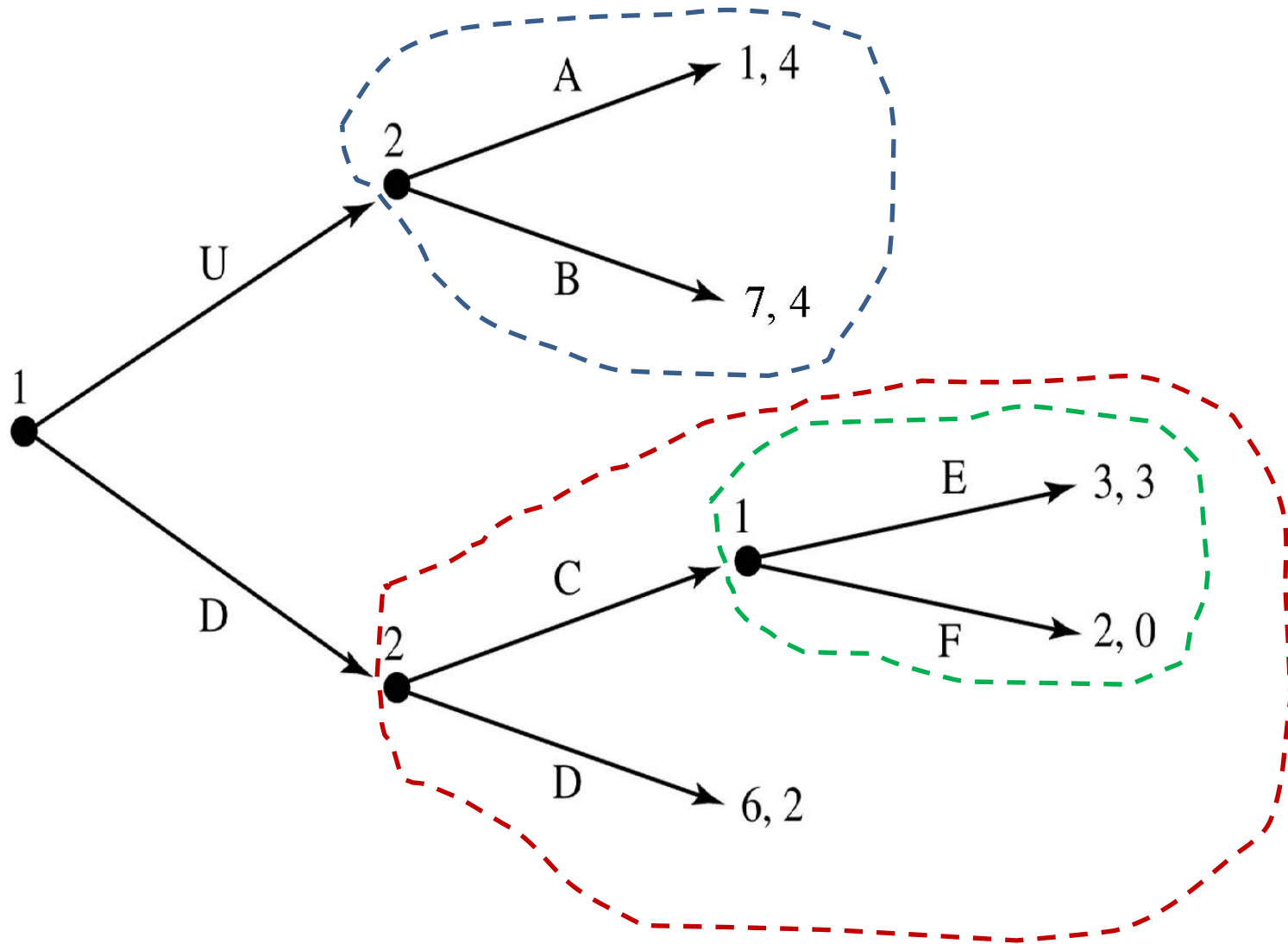
- Is (ZY, A) an SPE? No. It does not induce a Nash equilibrium in the second subgame: Note that (Y, A) is not a Nash equilibrium in the second subgame.
- Therefore, **the SPE profiles in this game are: (ZX, A) and (WY, B) .**
- In games of perfect information, **backward induction is a special case of SPE.**
- Thus, in games with perfect information we can either do backward induction or we can write down the subgames and compute the Nash equilibrium in each one. The latter is more efficient in cases where there are **ties**.

- For example, let us go back to the example:



- Above, we did backward induction for the case where player 2 chooses "A" in the final decision node, and then we did it for the case in which he chooses "B". We can save time and do it in one step if we write down the subgames.

- The proper subgames are indicated below:



- The normal forms of each subgame and their Nash equilibria are:

		P2			
		AC	AD	BC	BD
P1	UE	1, 4	1, 4	(7, 4)	(7, 4)
	UF	1, 4	1, 4	(7, 4)	(7, 4)
	DE	(3, 3)	6, 2	3, 3	6, 2
	DF	2, 0	(6, 2)	2, 0	6, 2

		P2	
		C	D
P1	E	(3, 3)	6, 2
	F	2, 0	6, 2

		P1
		E
E	(3)	
F	2	

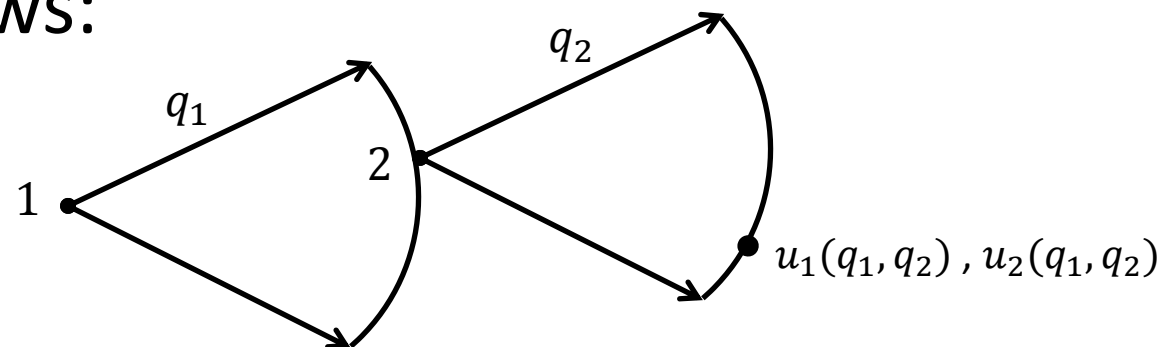
		P2
		A
A	(4)	
B	(4)	

- The original game has six Nash equilibrium profiles: (UE, BC) , (UE, BD) , (UF, BC) , (UF, BD) , (DE, AC) and (DF, AD)

- Is (UE, BC) an SPE? Yes. It induces a Nash equilibrium in each subgame.
- Is (UE, BD) an SPE? No. It fails to induce a Nash equilibrium in the second subgame: The profile (E, D) is NOT a Nash equilibrium in the second subgame.
- Is (UF, BC) an SPE? No. It fails to induce a Nash equilibrium in the third subgame: Choosing “ E ” dominates choosing “ F ” in the third subgame for player 1.
- For the same reason, the profiles (UF, BD) and (DF, AD) are not SPEs either.
- Is (DE, AC) an SPE? Yes. It induces a Nash equilibrium in every subgame.

- Therefore, the SPE profiles in this game are: (UE, BC) and (DE, AC) .
- We had already determined that these were the profiles that resulted from backward induction.
- In games of perfect information, backward induction is equal to subgame perfection.
- With ties (especially if there are multiple ties throughout the game), it may be more efficient (time-wise) to write down the subgames instead of doing backward induction multiple times.

- **SPE in games with continuous actions.**
- **Example: Stackelberg duopoly model.**- This is a sequential version of the Cournot duopoly game. The sequence of moves is as follows:
 1. In the first period, firm 1 chooses its quantity produced, q_1 .
 2. Firm 2 observes q_1 and then chooses the quantity to produce, q_2 .
- The extensive form representation of this game looks as follows:



- Suppose specifically the following:
 - Market price is given by the equation:

$$p = 12 - q_1 - q_2$$

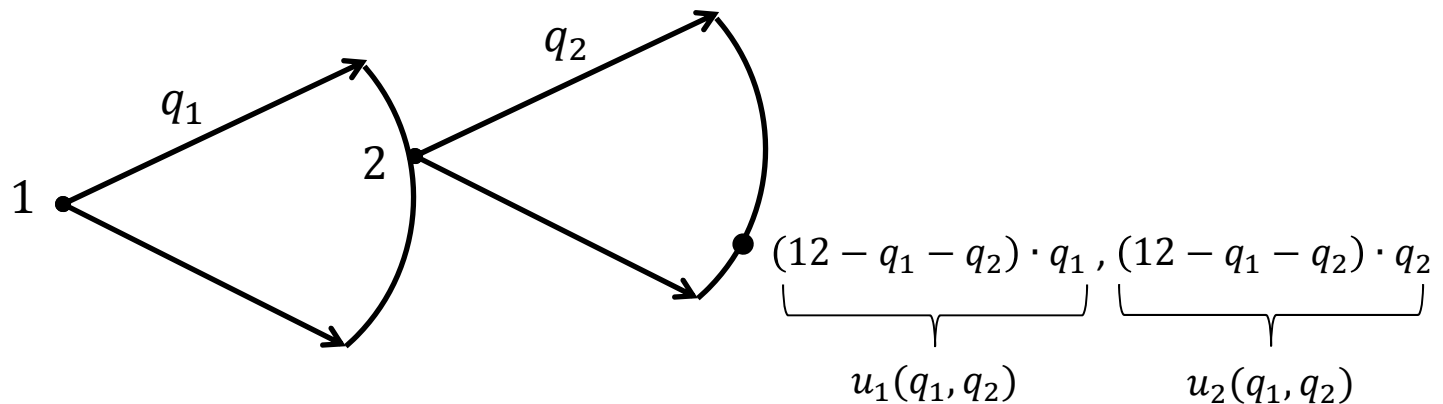
- The production cost is zero for both firms.

- This means that profits are given by:

$$u_1(q_1, q_2) = (12 - q_1 - q_2) \cdot q_1$$

$$u_2(q_1, q_2) = (12 - q_1 - q_2) \cdot q_2$$

- The extensive game looks like:



- How do strategies look like in the Stackelberg game?
 - Firm 1 moves first without observing the choice of Firm 2. Therefore, the strategy of Firm 1 is simply q_1 , the quantity it chooses to produce.
 - Firm 2 observes q_1 before making its choice. Since a strategy form Firm 2 is defined as a complete contingent plan. Therefore, it must provide a full description of what to do for each value of q_1 . Therefore, a strategy for Firm 2 must be a function of q_1 . We write it as $s_2(q_1)$.
 - Therefore, a **strategy profile in this game is of the form: $(q_1, s_2(q_1))$**

- We described this game in Chapter 14, where we noted that strategy profiles here must be of the form:

$$(q_1, s_2(q_1))$$

- Where $s_2(\cdot)$ (the strategy of player 2) is a function that tells player 2 how much to produce given the production level q_1 of player 1.
- **Question: Find the SPE strategy profile.**
- How do we proceed? **First note that this is a game of perfect information, so we can use backward induction to find the SPE.** As we do in backward induction, we need to look at player 2's optimal decision in the second stage of the game.

- In the second stage of the game, player 2 has observed q_1 . Given this production level, player 2 will then choose the production level that maximizes his profits. That is, he will choose the value of q_2 that maximizes:

$$u_2(q_1, q_2) = (12 - q_1 - q_2) \cdot q_2$$

- This is done by solving the first order conditions

$$\frac{\partial u_2(q_1, q_2)}{\partial q_2} = 0$$

with respect to q_2 .

- We have

$$\frac{\partial u_2(q_1, q_2)}{\partial q_2} = 12 - q_1 - 2 \cdot q_2$$

- Therefore, the best response of player 2 in the second stage is:

$$BR_2(q_1) = \frac{12 - q_1}{2}$$

- Since SPE requires optimal behavior in all stages of the game, **in any SPE, player 2 must follow the strategy:**

$$s_2(q_1) = \frac{12 - q_1}{2}$$

- Now let us go back to stage 1 of the game. In stage 1, player 1 looks ahead and realizes that if player 2 is rational, his strategy must be

$$s_2(q_1) = \frac{12 - q_1}{2}$$

- Using this knowledge, player 1 can **compute the continuation payoffs of choosing q_1 in stage 1**. How? Recall that the payoff to player 1 is:

$$u_1(q_1, q_2) = (12 - q_1 - q_2) \cdot q_1$$

- For any q_1 , the continuation payoffs are computed by replacing q_2 with $s_2(q_1) = \frac{12 - q_1}{2}$

- This yields:

$$\begin{aligned} u_1(q_1, s_2(q_1)) &= \left(12 - q_1 - \frac{12 - q_1}{2} \right) \cdot q_1 \\ &= \left(\frac{12 - q_1}{2} \right) \cdot q_1 \end{aligned}$$

- If player 2 behaves rationally in the second stage, then the continuation payoffs of choosing q_1 in stage 1 are therefore:

$$u_1(q_1, s_2(q_1)) = \left(\frac{12 - q_1}{2} \right) \cdot q_1$$

- Sequential rationality then requires that player 1 **choose the value of q_1 that maximizes his continuation payoffs.**

- The value of q_1 that maximizes his continuation payoffs is the one that solves the first order conditions:

$$\frac{\partial u_1(q_1, s_2(q_1))}{\partial q_1} = 0$$

- We have:

$$\frac{\partial u_1(q_1, s_2(q_1))}{\partial q_1} = 6 - q_1$$

- Therefore, using his continuation payoffs, the optimal strategy for player 1 is:

$$q_1^* = 6$$

- And the strategy in stage 2 for player 2 is:

$$s_2(q_1^*) = \frac{12 - q_1^*}{2} = 3$$

- This game has a unique **SPE profile**, given by:

$$(q_1^*, s_2^*(q_1)) = \left(6, \frac{12 - q_1}{2} \right)$$

- The **outcome of the game for the SPE profile** is:

$$(q_1, q_2) = (6, 3)$$

- The corresponding payoffs are:

$$(u_1, u_2) = (18, 9)$$

- Note that in an SPE, there is a **first mover advantage for player 1**: Being able to anticipate the behavior of player 2 in the second stage allows player 1 to obtain a higher payoff (in other settings, being a first mover can be a disadvantage).

- **Stackelberg vs. Cournot:** Suppose both players move simultaneously. Then we are back in the Cournot model where players' strategies are simply (q_1, q_2) .

- It is easy to see that the Cournot equilibrium would be:

$$(q_1^*, q_2^*) = (4, 4)$$

- And players' payoffs in equilibrium would be:

$$(u_1, u_2) = (16, 16)$$

- So, if we were to go from simultaneous to sequential moves, there is clearly an **incentive to be the first mover in this particular game**: Payoffs go from 16 to 18 to the first mover, and they fall from 16 to 9 to the other player. **In other sequential games there can be a disadvantage to being the first mover.**

- **Are there Nash equilibria in the Stackelberg game that are not SPE? YES.** As it turns out, this game has an infinite number of Nash equilibria, but a unique SPE.
- For example, let $x \in [0, 12]$ be any production level for player 1 between 0 and 12. And consider the following strategy \bar{s}_2 for player 2:

$$\bar{s}_2(q_1) = \begin{cases} \frac{12-x}{2} & \text{if } q_1 = x \\ 12 - q_1 & \text{if } q_1 \neq x \end{cases}$$

- This strategy amounts to the following statement from player 2 to player 1:

“If you produce $q_1 = x$, I will set the production level q_2 that corresponds to my best response to x . But if you deviate from producing $q_1 = x$, I will retaliate with a production level q_2 that drives the market price (and both of our profits) to zero”.

- It is easy to verify that the profile of strategies $(q_1, s_2) = (x, \bar{s}_2)$ is a **Nash equilibrium for any $x \in [0, 12]$.**

- To prove this, we need to show that $q_1 = x$ is a best response to $s_2 = \bar{s}_2$ and viceversa.
- Let's begin with player 1. Suppose $s_2 = \bar{s}_2$. Then:

a) If $q_1 = x$:

$$u_1(q_1, \bar{s}_2) = \left(12 - x - \frac{12 - x}{2} \right) \cdot x = \left(\frac{12 - x}{2} \right) \cdot x$$

b) If $q_1 \neq x$:

$$u_1(q_1, \bar{s}_2) = (12 - q_1 - (12 - q_1)) \cdot q_1 = 0$$

- Therefore, **$q_1 = x$ is a best response to \bar{s}_2 for any $x \in [0, 12]$.**

- For player 2, note that from our previous analysis we already know that, if $q_1 = x$, then choosing $q_2 = \frac{12-x}{2}$ is the best response by player 2.
- Therefore, if $q_1 = x$, then the strategy \bar{s}_2 yields the best response to player 2.
- However, **these Nash equilibria are not SPE because they entail a non-credible threat by player 2:** If player 1 deviates from choosing $q_1 = x$ in the first stage, it will not be in player 2's advantage to drive the price down to zero in stage 2. Doing this would be dominated by responding with $q_2 = \frac{12-q_1}{2}$, which is player 2's best response to q_1 .

- **Remark:** SPE and backward induction (a special case of SPE) are **refinements of the Nash equilibrium concept** since they focus on the subset equilibria that satisfy subgame perfection. By doing this, we can reduce the range of predicted behavior in a game.

- **Conditional dominance and forward induction**
- Just like SPE is a refinement of Nash equilibrium resulting from sequential rationality, conditional dominance is the refinement of dominance that results from sequential rationality.
- We say that a strategy s_i is **conditionally dominated** if there exists some information set h that can be reached under strategy s_i , and s_i is dominated if we restrict attention to the subset of strategies that also reach h .
- Proceeding iteratively, we can extend the concept of iterated dominance in a normal-form game, to **iterated conditional dominance** in an extensive form game.

- Iterated conditional dominance is a refinement of iterated dominance that results from sequential rationality.
- Intuitively, conditional dominance is based on the following reasoning made by players: *“OK, the game has reached information set h . Given all the different ways in which information set h could have been reached, and given that I believe that players are rational, what is my expectation about other players’ behavior going forward in the game?”*
- Conditional dominance is a special form of what is called **forward induction**, which assumes that past play was rational and uses this to predict future behavior in the game.

- We will not get into further details about conditional dominance and forward induction because it is a bit more complicated than backward induction and SPE.
- We will restrict attention to the details of backward induction and SPE.