2.- Extensive Form Representation of Games

- As discussed earlier, a game is a complete mathematical summary of a strategic interaction setting.
- Any representation of a game must include the following elements:
  1. The list of **players**.
  2. The **actions** available to each player at each possible point in the game.
  3. The description of players' available **information** at each possible point in the game.
  4. A specification of all possible **outcomes** of the game and how each outcome arises from players' actions.
  5. A specification of players' preferences over each outcome, or players' **payoffs** for each outcome.

- There are two general representations of a (noncooperative) game:
    - **Extensive form** representation.
    - **Normal (or strategic) form** representation.
- The **extensive form** relies on a graphical representation of the sequence of actions in the game. It depicts the decision tree of all players in the game.

- The decision tree used in an extensive form representation summarizes all five components of the game:
- The **nodes** of the tree represent stages of the game where a given player has to make a move.
- The **branches** coming out of each decision node represent the actions available to the player who has to make a decision at the node in question.
- The **final branches** of the tree correspond to all the possible **outcomes** of the game. Players' **payoffs** can be summarized **numerically** there.
- The **information** available to players at each stage of the game can be graphically represented by **connecting decision nodes through dotted lines**.

- **Example from Chapter 2**: Describes a real-world scenario involving two players:
  - Jeffrey Katzenberg (K)
  - Michael Eisner (E)

- From 1984-1994, (K) worked for (E) as head of Disney studios. (E) was CEO of the Walt Disney Company. During his tenure, (K) revived the studios, especially their animation division.

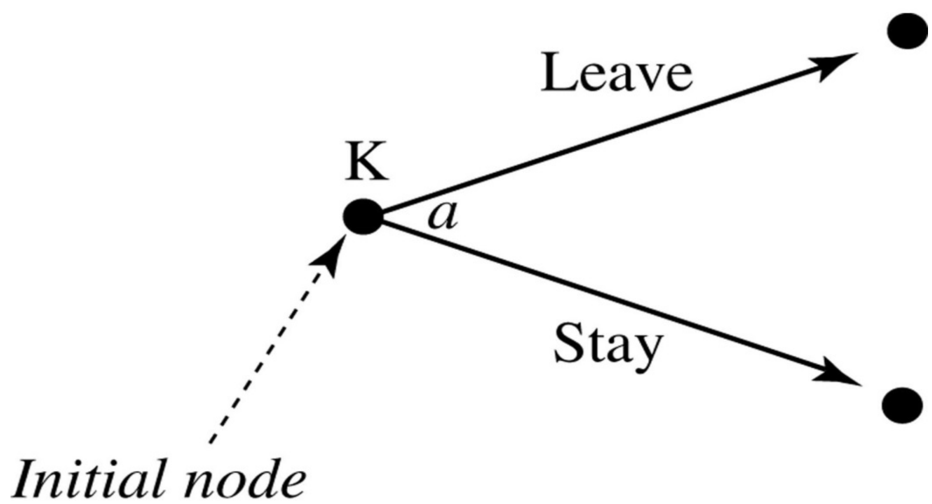
- In 1994, (E) refused to promote (K) to the position of president of the company, leading to a falling out between them and (K)'s departure from Disney in 1994.

- Following his departure, (K) co-founded Dreamworks SKG with Steven Spielberg and David Geffen.

- Shortly afterwards, SKG produced the animated movie *Antz*, coinciding almost simultaneously with Disney's release of *A Bug's Life*.

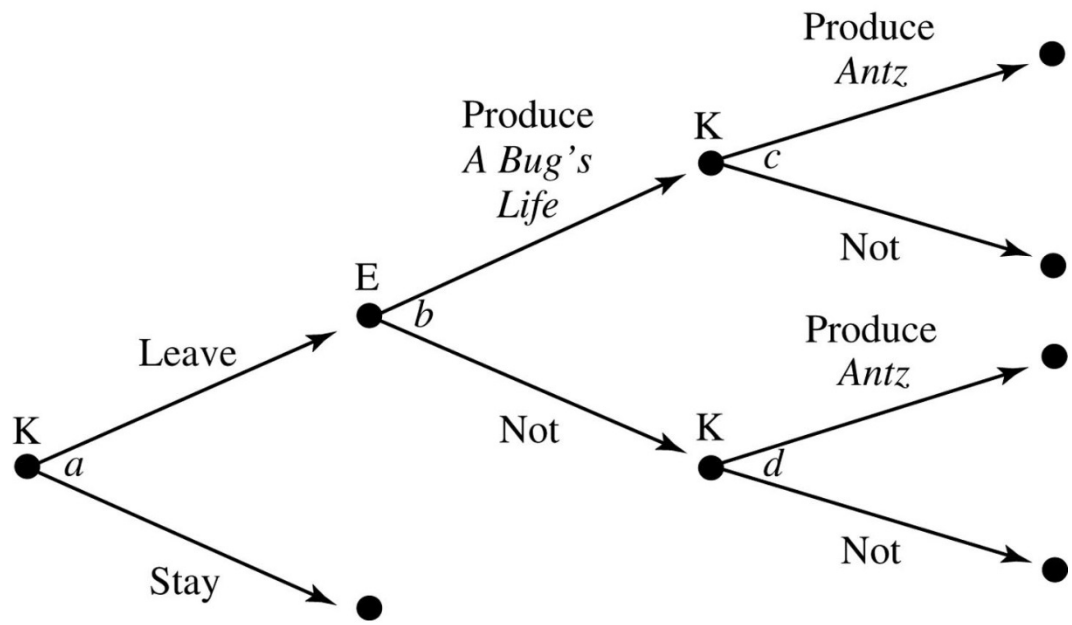
- Chapter 2 represents this real-life event as an extensive form game, where (K) first decides whether to leave or not, and then both (K) and (E) must **simultaneously** (**i.e, without exact knowledge of what the other one will do**) must decide whether to release the aforementioned movies.

- To construct the extensive form game, we take it step-by-step. First we represent the initial stage of the game:



- If (K) decides to stay, we assume that the game "ends" (at least as it pertains to this specific situation). If (K) decides to leave, then the subsequent strategic interaction takes place.

- So, the next step is to sketch the decision tree that follows if (K) leaves:



- The next step is to represent the **information available to the players** in the stage of the game that follows (K) leaving.

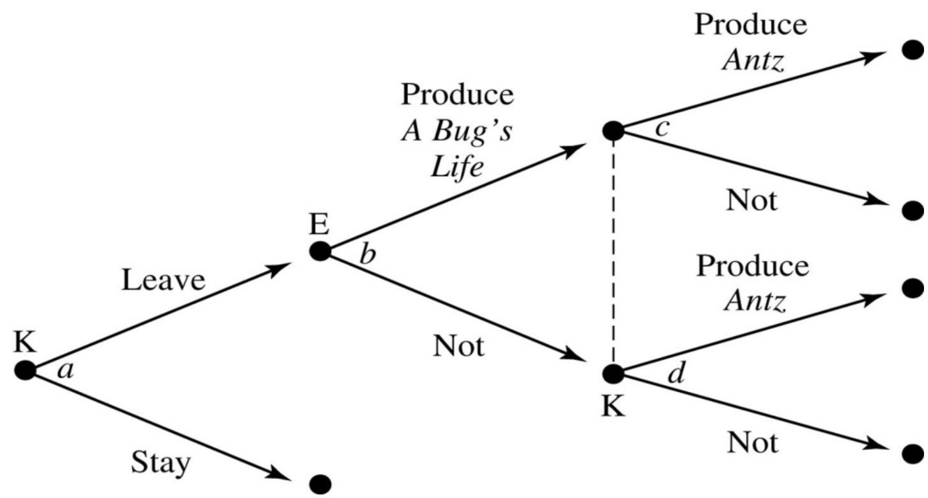
- The following is an adequate description of the information available to (K) and (E):

1. (E) is able to observe if (K) decides to leave or stay.
2. Once (K) leaves, each player must decide **individually** and “secretly” whether they will release the animated movie, but neither of them knows exactly what the other one will do. They must **commit to a decision before knowing exactly what the other one decided**.

- How can this be represented graphically in the tree?

- The information available to each player at each stage of the game is represented through **information sets. These are decision nodes connected to each other through dotted lines.**
- Players always know in which information set they are at each stage of the game. However, **players do not know which node within the information set they are at**.
- If an information set is a **singleton**, this means that the player in question knows exactly which node he is at. If an information set includes two or more nodes, then the player does not know which of these nodes he is at.

- Given the information described, the tree should be represented as:



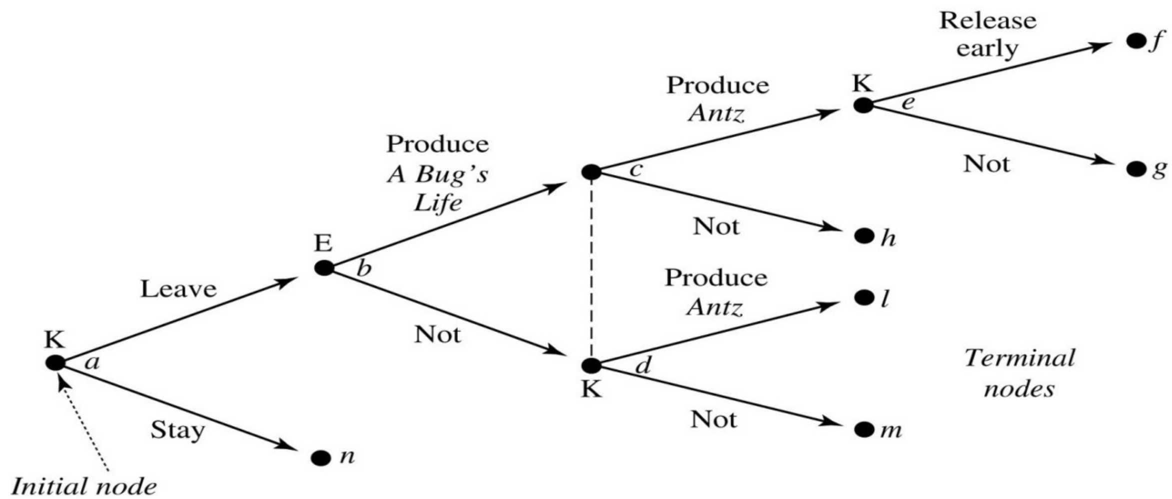
- The information set represented through the dotted line depicts (K)'s lack of knowledge of what (E) will do. The fact that (E) must make his decision before (K) (sequentially in the tree) means that (E) also does not know what (K) will decide before making his own choice.

- To make the game a more realistic description of the real-world situation behind it, we can add a final decision node.

- Suppose once (K) and (E) decide simultaneously whether to make their movies, (K) is able to observe (E)'s decision and (K) now has the choice to **release his movie early or not**.

- This can be represented through a final decision node for (K), as follows.

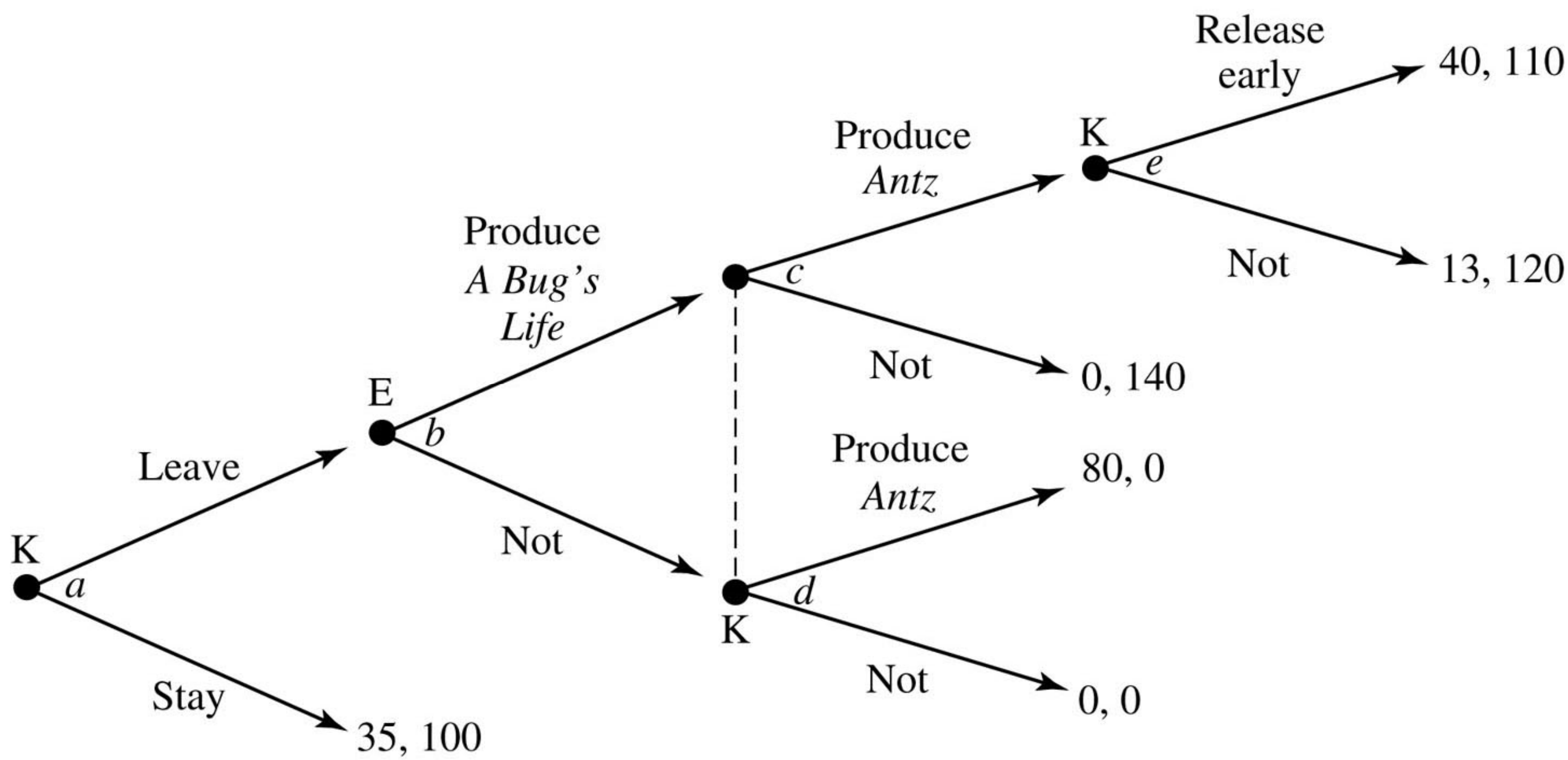
- Full tree looks like this:



- The **terminal nodes** represent all possible **outcomes** of the game.
- Only remaining component are players' **payoffs**, summarizing their preferences over the outcomes of the game.

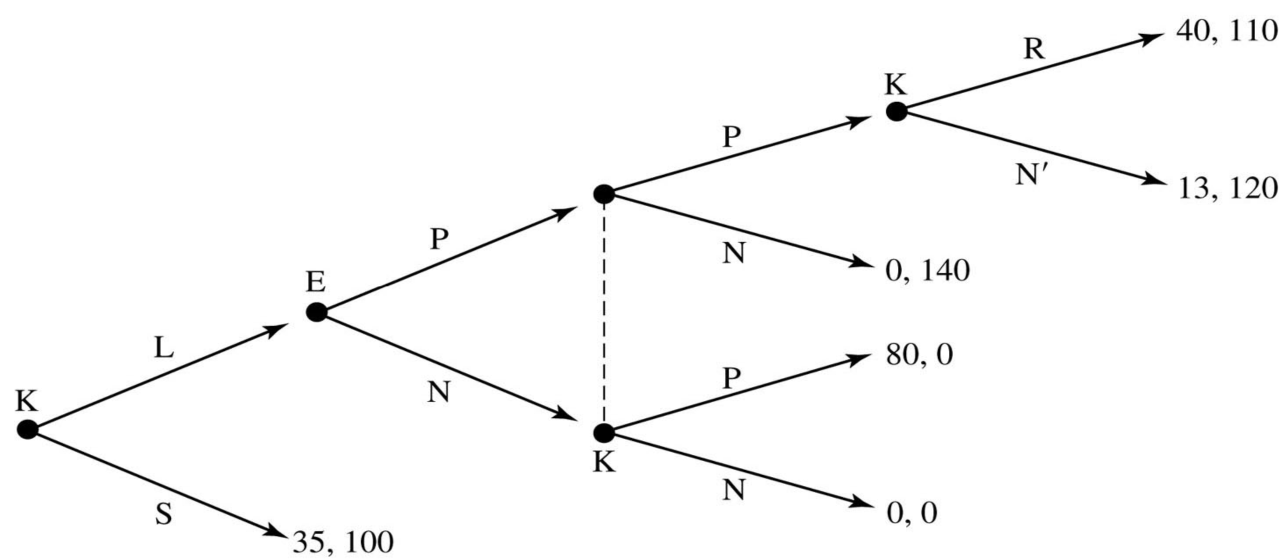
- Assigning exact monetary payoffs in this case would require precise information of the profits in each scenario.
- However, in this simple game we can use payoffs that represent how each player ranks their preferred outcomes. That is, payoffs that represent only the **ordinal preferences** of each player.
- Using arbitrary payoffs that represent ordinal preferences is ok in simple games like this one, where strategies are not random. In more complicated scenarios with random strategies, the actual numerical values used for payoffs (and not just their rankings) are important.

- To assign payoffs, suppose we assume the following about players' preferences:
  - Each player is better off if they are the only one making the movie.
  - If both decide to make the movie, (K) is better off releasing it earlier.
  - The best possible outcome for (K) is leaving and being the only one releasing the movie.
  - The best possible outcome for (E) is if (K) leaves, does not make the movie and (E) is the only one making the movie.
- Given this, suppose we assign payoffs according to the following rules:
  1. If either player decides NOT to produce their movie, that player's payoff is normalized to zero.
  2. If (K) decides to stay, he gets $35M and (E) gets $100M
  3. If (K) decides to leave and produce the movie, he makes $80M if (E) opts NOT to produce the movie.
  4. If BOTH decide to produce their movies, (K) makes $40M if he releases earlier and $13M otherwise.
  5. If BOTH decide to produce their movies, (E) makes $110M if (K) releases early and $120M otherwise.



K
a
Leave
Stay
35, 100
E
b
Produce
A Bug's
Life
Not
c
Produce
Antz
Not
0, 140
K
e
Release
early
40, 110
Not
13, 120
K
d
Produce
Antz
80, 0
Not
0, 0

- For simplicity, actions along the decision tree of an extensive form representation are **abbreviated**. The previous example can be represented as:



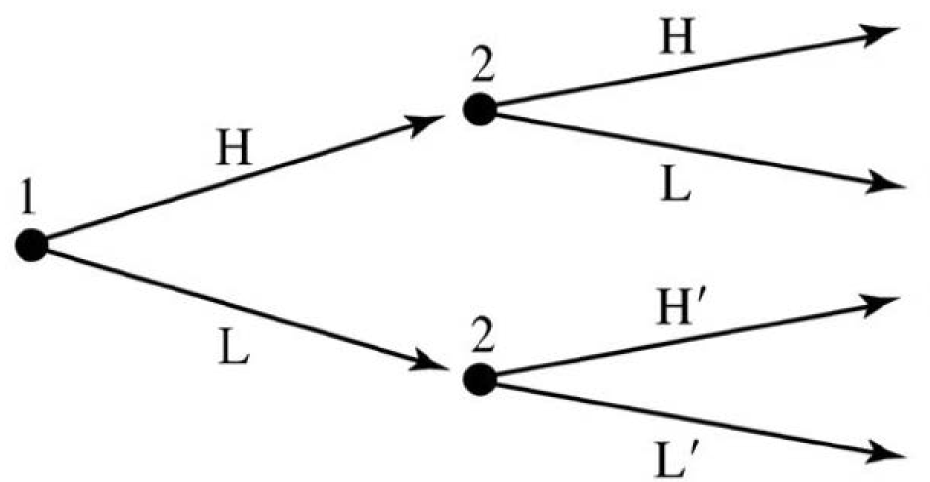
- Different informational structures can be easily represented in extensive form games by depicting the information sets appropriately.

- When we label actions, we should avoid using the same label for different places where decisions are made.

- For example, consider a setting with two players (labeled ‘1’ and ‘2’), each of which has two actions: “High” or “Low”.

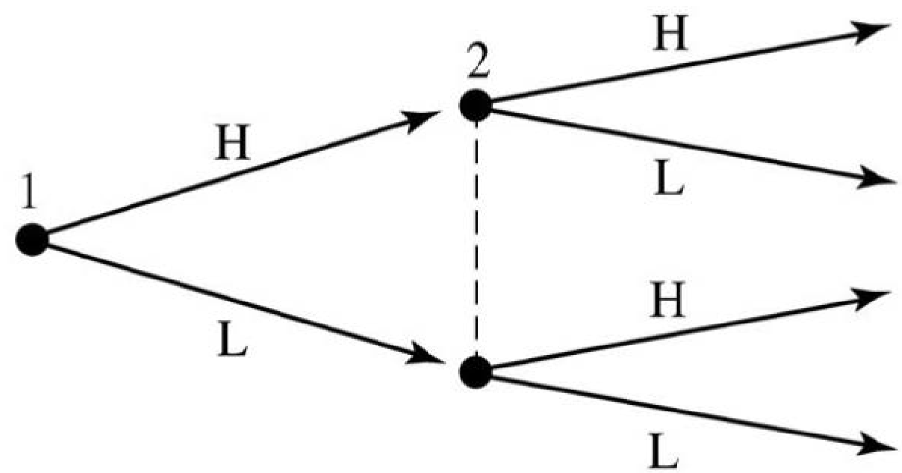
- Consider two alternative information structures:
  a) Player 1 moves first, then player 2 observes 1’s choice and makes his own decision.
  b) Both players move simultaneously.

- In (a), the actions "High" and "Low" for player 2 are labeled H and L if player 1 chose H, and H' and L' if player 1 chose L, to emphasize that player 2 knows exactly which node he is at.



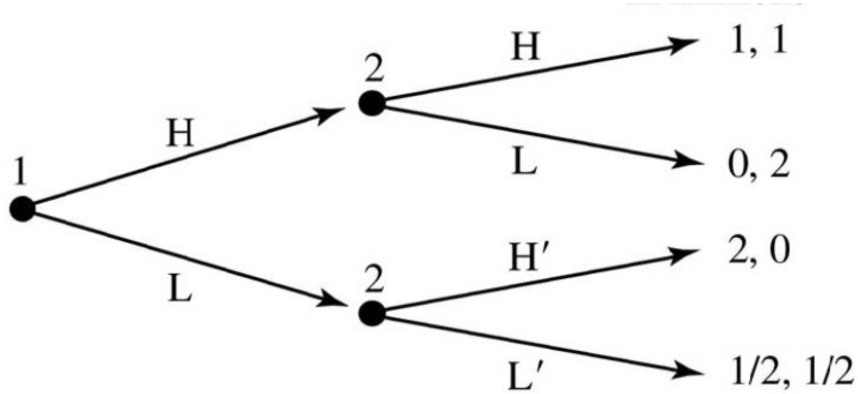
(a)

- In (b), the actions “High” and “Low” for player 2 are labeled H and L in all cases, to emphasize that player 2 cannot distinguish which node he is at.

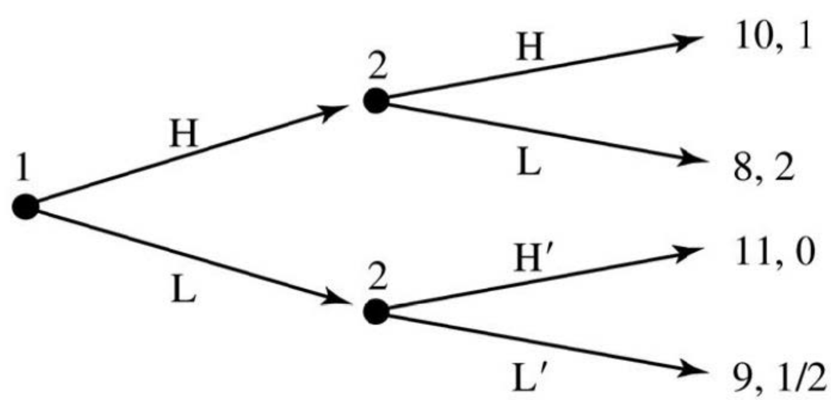


(b)

- **In games where the strategies are not random** (we will clarify this precisely later on in the course), numerical **payoffs** only need to represent the **preference-ranking** of outcomes for each player. For example, the following two trees would be equivalent in terms of the ranking of payoffs for each player:



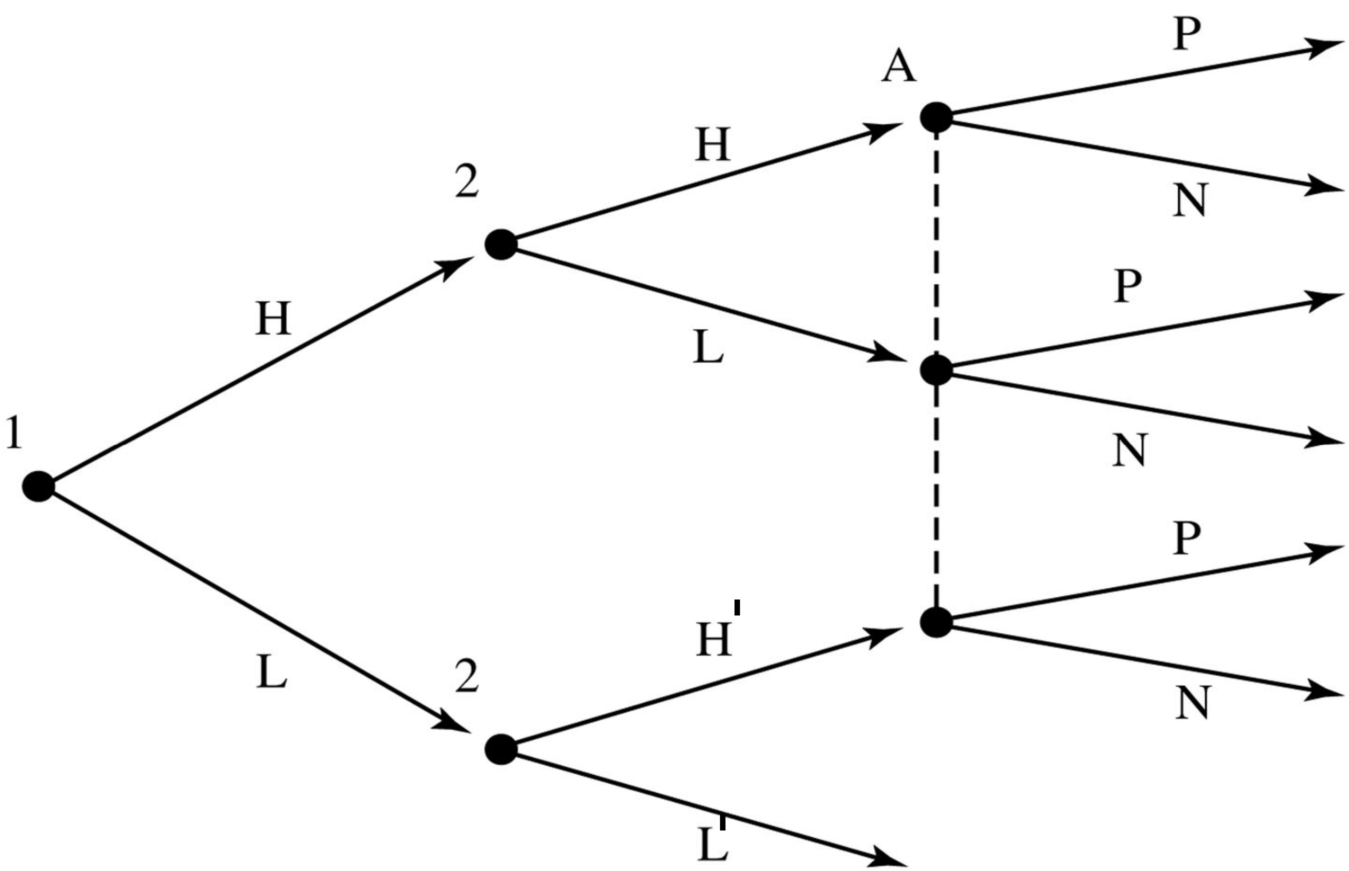
(a)



(c)

- Extensive form representations can be used in more complicated settings. An immediate extension could involve games with three or more players.
- **Example 2.3:** Consider an industry with two competitors: Players 1 and 2.
- Player 1 first decides whether to set a high (H) or low (L) price.
- Player 2 observes the choice of 1 and decides also whether to set a high (H) or low (L) price.
- If both decide (L,L) the game ends.
- If at least one of them chooses H, the attorney general decides whether to prosecute (P) or not (N) for anticompetitive behavior. However, the attorney general **does not observe exactly** which of the two firms set the High (H) price.

- Letting “A” denote the Attorney General, the extensive form (without payoffs) of this game looks as follows:



- Extensive form representations can take more "unconventional" graphical forms.
- **Example: "Let's Make a Deal" game**
- This represents a real-life three-door guessing game made famous on the TV game show.
- There are two players: The host (Player 1) and the contestant (Player 2).
- There are three doors, labeled "a", "b" and "c".
- First, the host places a prize behind one of the three doors.
- Without observing the host's choice, the contestant must choose one of the three doors.

- After this, the host opens one of the doors, but it cannot be either the one chosen by the contestant, nor the one containing the prize.
- Focusing on the two remaining doors, the contestant is then given the choice to "switch" (S) or "don't switch" (D) the door she initially selected.
- The remaining doors are opened and the contestant wins the prize if it is behind the door she chose.
- Suppose the contestant obtains a payoff of 1 if she wins and zero if she loses. Suppose the host is indifferent between all of the outcomes (and normalize his payoff to zero).

- Player 1 has complete information throughout.

- Therefore, the key to getting the extensive form representation correct is to **accurately describe the information sets for Player 2**.

- To do this, we have to use the rules of the game to realize exactly what Player 2 observes at each stage of the game.

- Prior to her first move, Player 2 does not know which door contains the prize.

- Prior to her last move, Player 2 knows the door she chose first AND the door opened by Player 1, but she still does not know which door contains the prize.

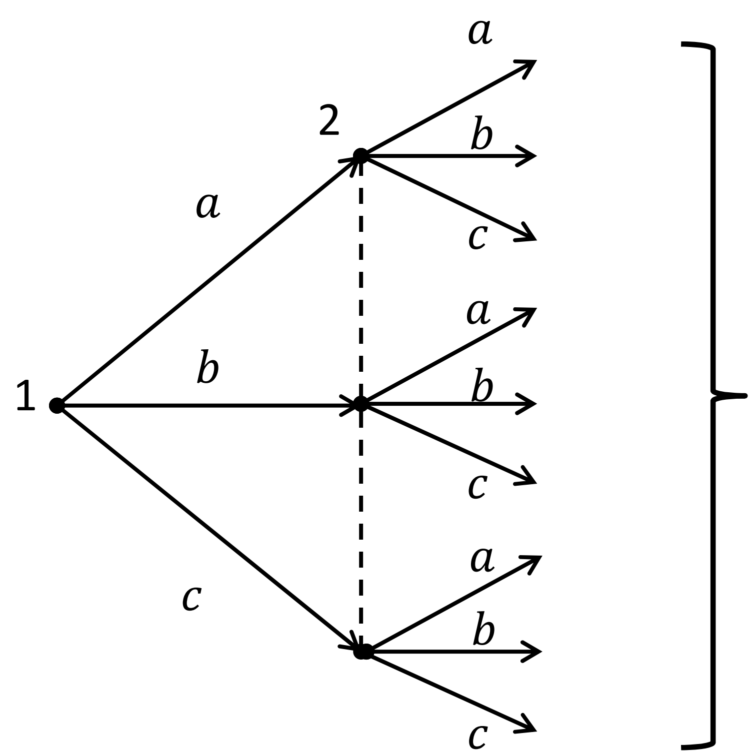
- These two facts fully characterize the information sets for Player 2 at each of the two stages where she makes a move.

- A feature of this game can help us simplify the extensive form representation.

- This is the fact that if, in the first stage, Player 2 chooses a door different from the one where Player 1 placed the prize, then Player 1 does not have a choice regarding which door to open in the second stage.

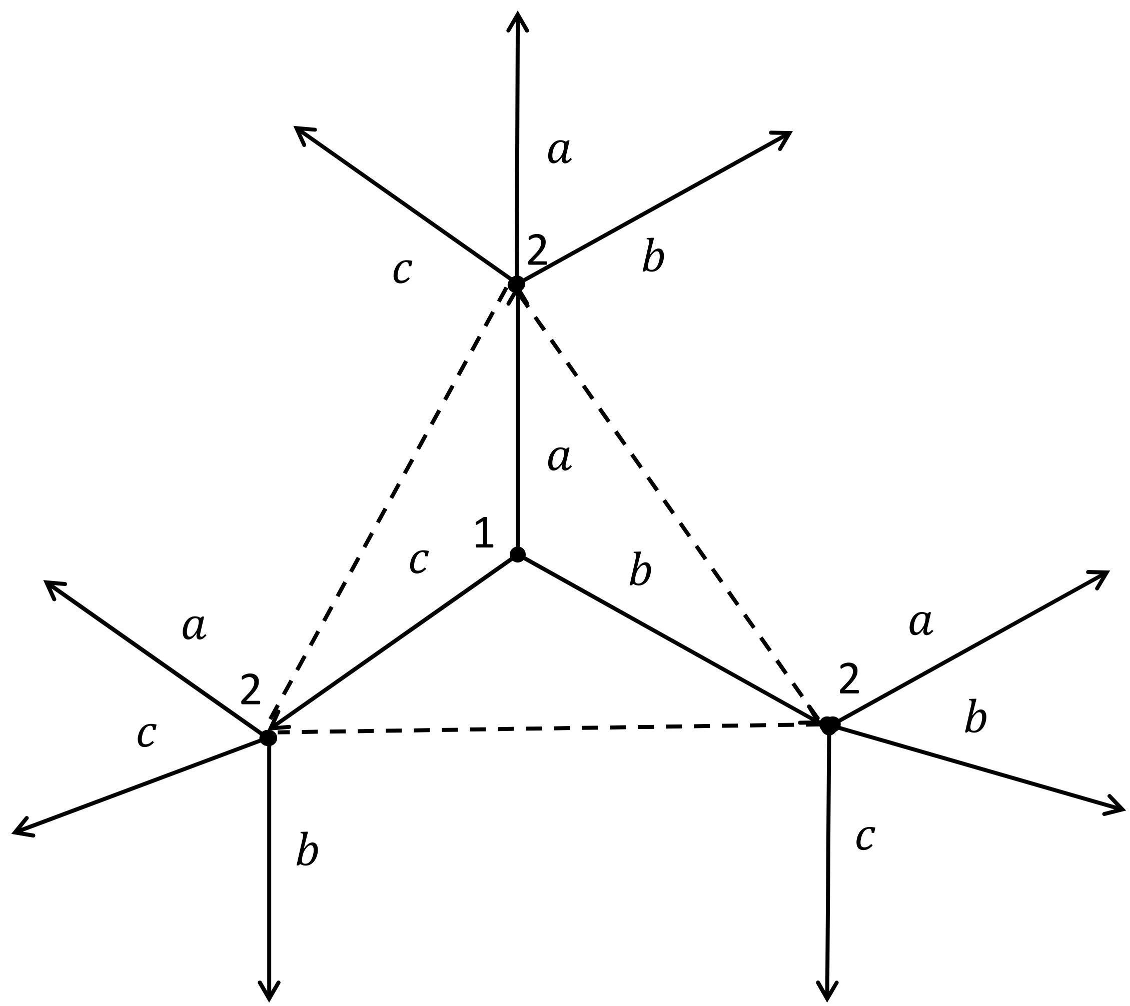
- For example, suppose the prize is in door “a”. Suppose Player 2 chooses door “b”. Then Player 1 MUST open door “c”.

- First two stages of the game can be represented as:



The information set of player 2 illustrates that he does not know where the prize is when he has to choose between doors “a, “b” and “c”.

- However, I think the following alternative tree will yield a “less messy” representation in the end:



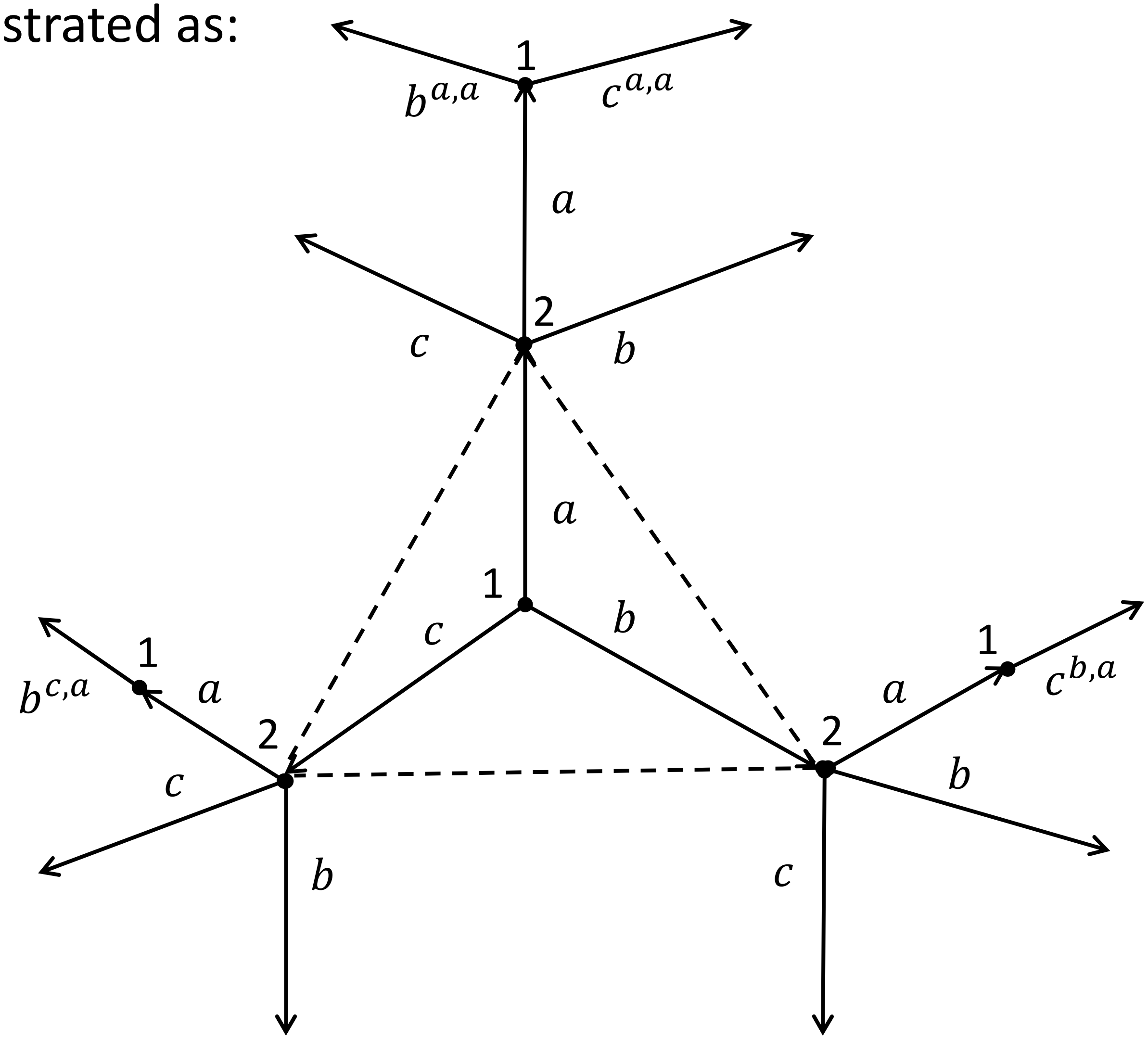
- In the next stage of the game, player 1 opens one of two doors, neither of which can be the door 2 chose, or the door that contains the prize.
- To draw the tree for the next stage, let us think about the rules of the game.
- <u>Suppose player 2 chooses door "a"</u>. Then:

i. If 1 put the prize in "a", then 1 has the choice to open either "b" or "c".

ii. If 1 put the prize in "b", then 1 <u>must</u> open door "c".

iii. If 1 put the prize in "c", then 1 <u>must</u> open door "b".

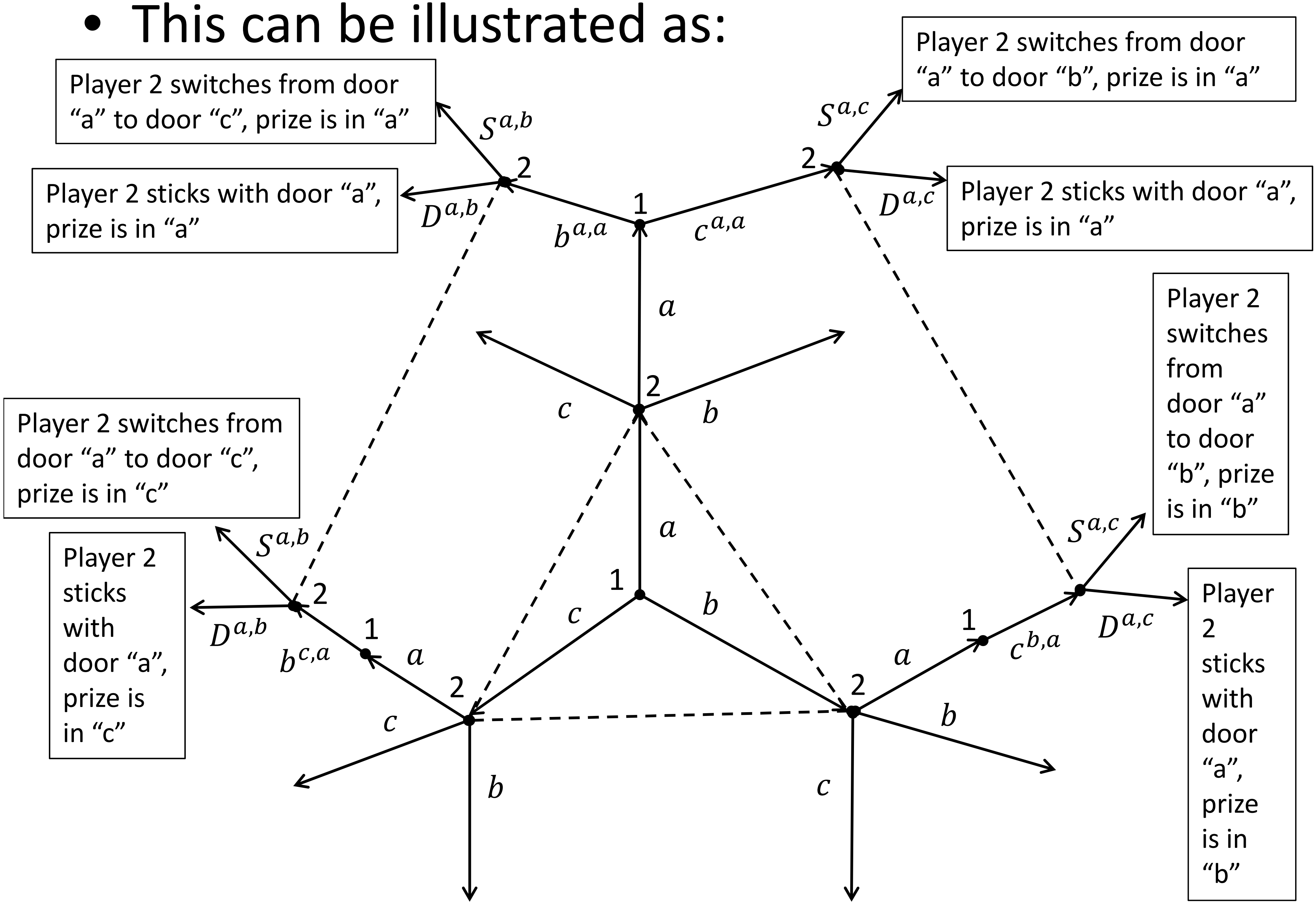
- This can be illustrated as:



- Here, $c^{a,a}$ refers to player 1 choosing to open "c" given that the prize was placed in "a" and player 2 opened door "a". $b^{c,a}$ denotes 1 choosing "b" given that the prize was placed in "c" and 2 opened "a". Analogously for $c^{b,a}$ and $b^{a,a}$.

- In the final stage of the game, player 2 must decide whether to switch (S) to the door that remained unopened, or don’t switch (D) and stay with the door that he chose originally.
- The key here is to **get the information sets for player 2 correct**.
- Let’s keep focusing on the case where player 2 choose door “a” in his first choice. There are TWO **information sets in this case**:
  i. Player 2 chooses “a” and then Player 1 opens “b”.
  ii. Player 2 chooses “a” and then Player 1 opens “c”.
- We need to connect the nodes that correspond to these two possible histories of play with dotted lines.

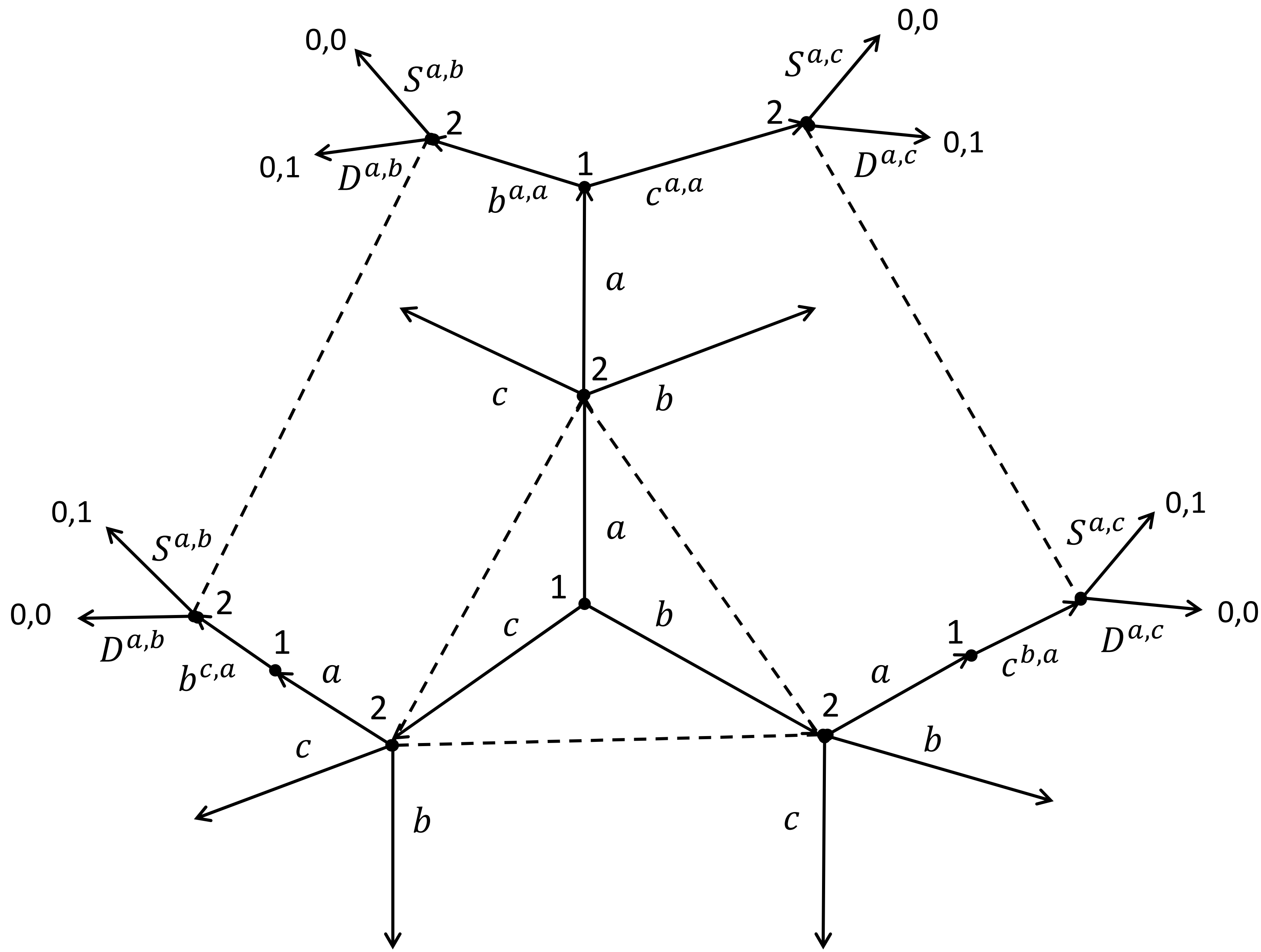
- This can be illustrated as:



- Suppose player 1 is indifferent between all possible outcomes, and normalize his payoff to zero.

- Suppose player 2 prefers to win the prize than not. Suppose his payoff is 1 if he wins the prize, and zero otherwise.

- Then the previous figure becomes…



0,0
$S^{a,b}$
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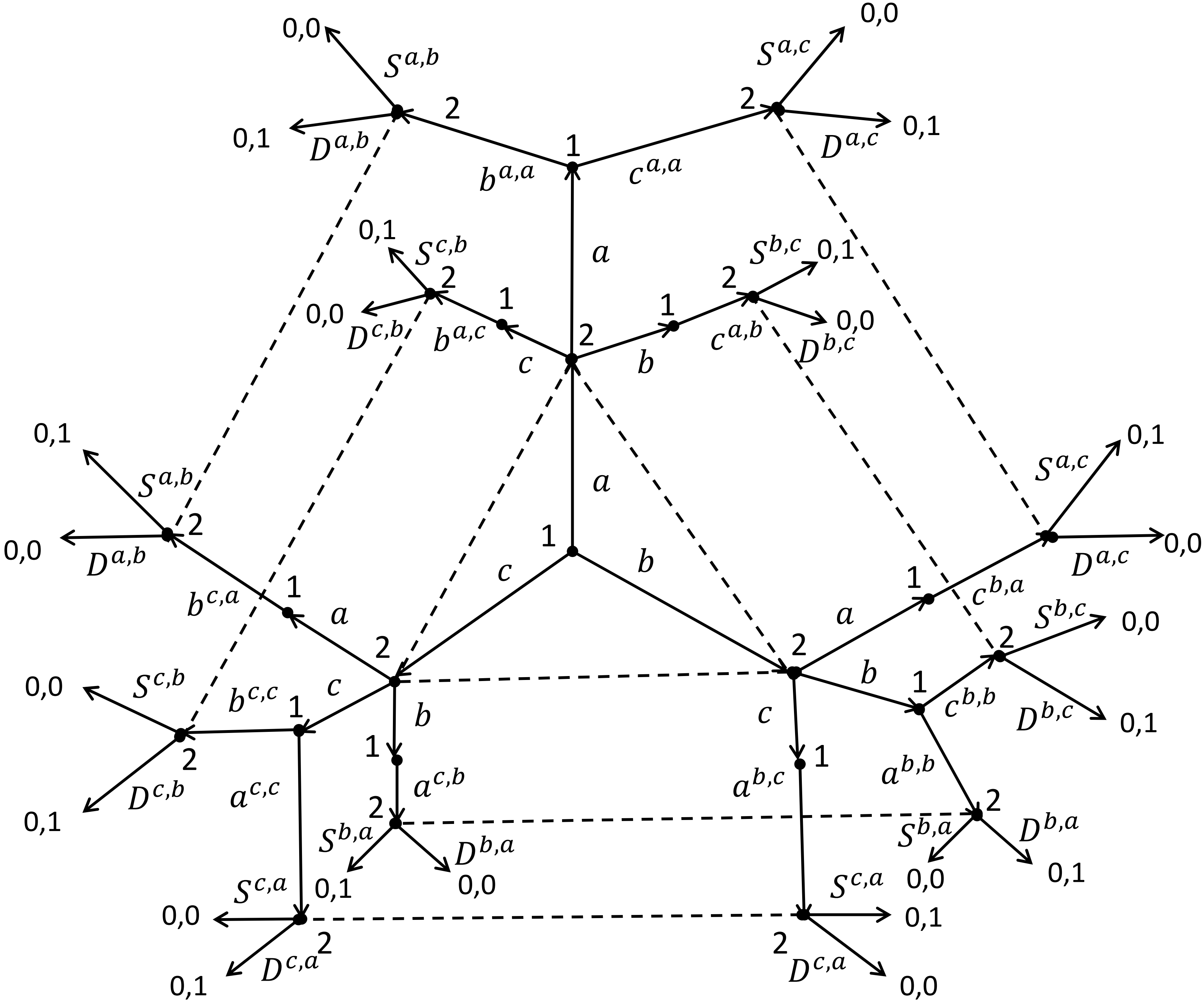
0,1
$D^{a,b}$
1
$b^{a,a}$
$c^{a,a}$
2
$S^{a,c}$
0,0
0,1
$D^{a,c}$
a
2
c
b
0,1
$S^{a,b}$
2
0,0
$D^{a,b}$
$b^{c,a}$
1
a
c
1
a
b
2
c
b
$S^{a,c}$
0,1
0,0
$D^{a,c}$
1
$c^{b,a}$
a
2
b
c

- To complete the decision tree and the extensive form representation, we apply the same logic to the remaining two scenarios:

  i. Player 2 chooses “b” in the first stage.
  ii. Player 2 chooses “c” in the first stage.

- The full extensive form representation of the game looks as follows…



0,0
$S^{a,b}$
2
0,1
$D^{a,b}$
1
$b^{a,a}$
$c^{a,a}$
$S^{a,c}$
0,0
2
0,1
$D^{a,c}$
a
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$S^{c,b}$
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$S^{c,a}$
0,1
2
$D^{c,a}$
0,0

- The previous example illustrates that extensive form games do not necessarily have to have the shape of a “tree” in a conventional sense.

- Extensive form games can be drawn in alternative ways. However, any representation must convey the same features.

- “Homework”: Try drawing at home an alternative extensive form representation of this game.