24.- Random Events and Incomplete Information

- So far we have seen many examples of games where different players have different information as the game progresses. This produces **asymmetric information among players**.
- However, in all the games we have studied the only source of asymmetric information between players has to do with their **actions**. Namely, with the fact that some actions are not observed by all the players in the game.
- In many instances, asymmetric information may arise because **some events outside of the control of the players may be observed only by a subset of players**.

- For example, certain characteristics of a subset of players may be **private information.** In the real world, **for example, the production costs of a firm may be known only to that firm but not to its competitors**.
- Events outside of the control of the players in the game are viewed as **random events** whose realization may be observed only by a subset of players.
- We model these random events by introducing **"nature"** as an additional player in the game. But we treat nature as a **non-strategic player**.

- Thus, we treat these random events as **moves of nature which are observed only by a subset of players. This introduces a new source of asymmetric information between players in the game.**
- We depict nature moves in an extensive form game through the use of **"chance nodes"** which are **depicted as open circles.**
- The moves made by nature in the chance nodes are observed only by a subset of players.

- **Nature is a nonstrategic player** in the sense that:

1. Nature does not have a payoff function.
2. Nature’s moves are non-strategic. They are viewed simply as random events that follow a certain probability distribution (like the flip of a coin).

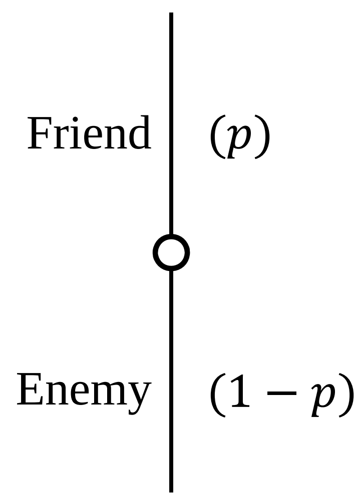
- The **information sets of players should be drawn carefully** in an extensive form game **to describe precisely which players observe the moves of nature and which players don’t**.

- **Example: The gift game.-** Consider a game between two players. In this game, player 1 decides whether to offer a wrapped gift to player 2 or not. If a gift is made, player 2 has to decide whether to accept (A) or reject (R) the gift.
- Player 1 can either be a “Friend” or an “Enemy”. We refer to this as player 1’s “type”. Player 1 is a friend with probability p and he is an enemy with probability $1-p$. **The type of player 1 is known only to player 1**.
- Whether player 1 is an enemy or a friend will be modeled as a **nature move whose outcome is observed only to player 1**.

- If player 1 is a friend, he gives a nice present to player 2. If player 2 accepts the present his payoff is 1.
- If player 1 is an enemy, he gives an undesirable present to player 2. If player 2 accepts the present his payoff is -1.
- If the gift from player 1 is accepted, his payoff is 1, if it is rejected his payoff is -1 regardless of his type.
- If player 2 rejects a gift, his payoff is zero regardless of player 1’s type.
- If player 1 does not offer a gift, then the payoff is zero to both players.

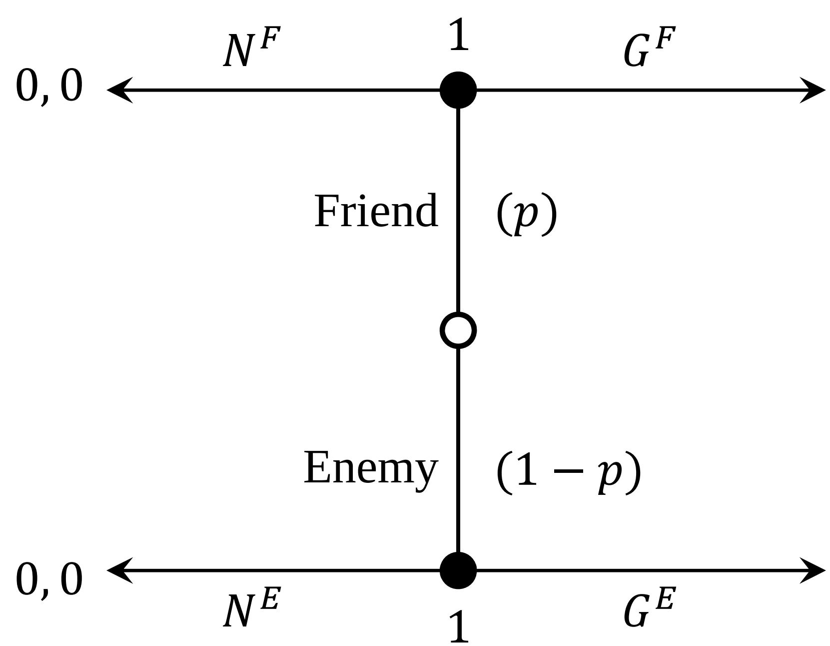
- **Extensive form of this game:** Before drawing the extensive form, let's go over what is observed by players 1 and 2:
  - **Player 1:** Observes his own type (that is, he observes the move made by "nature"). Then he decides whether to make a gift or not to player 2.
  - **Player 2:** Only observes whether player 1 made a gift to him or not, but does not know the true type of player 1. That is, he does not observe the move made by nature.
- Let us construct the extensive form step by step.

- **Step 1.- Drawing nature's move:** We use **open circles** to depict **chance nodes.** In this case, the chance node has two possible branches: "Friend" (with probability p) and "Enemy" (with probability $1 - p$).
- We draw it like this: (you will see why using this type of depiction is graphically convenient)



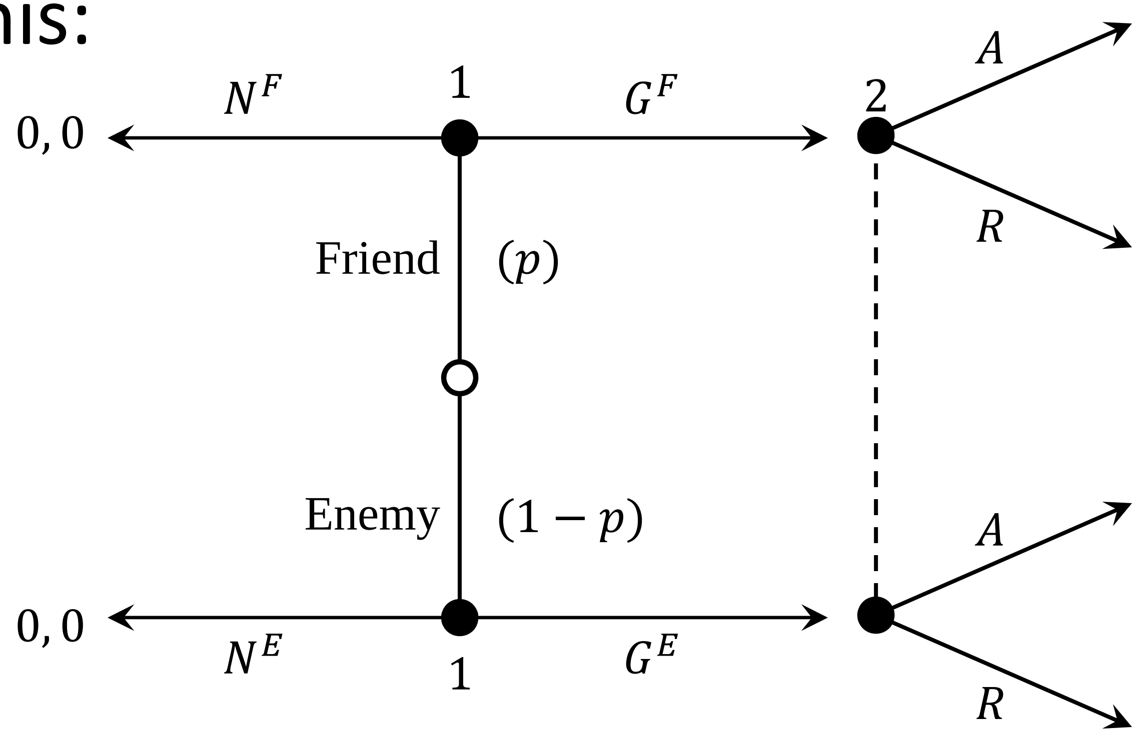
- **Step 2.- Player 1's decision:** Player 1 observes his type; that is, he observes the move of nature and then he decides whether to offer a gift or not. Let us label the possible actions by player 1 as:
  - N^F: Don't offer a gift given that the type is "friend".
  - G^F: Offer a gift given that the type is "friend".
  - N^E: Don't offer a gift given that the type is "enemy".
  - G^E: Offer a gift given that the type is "enemy".

- Graphically, we can represent this as:

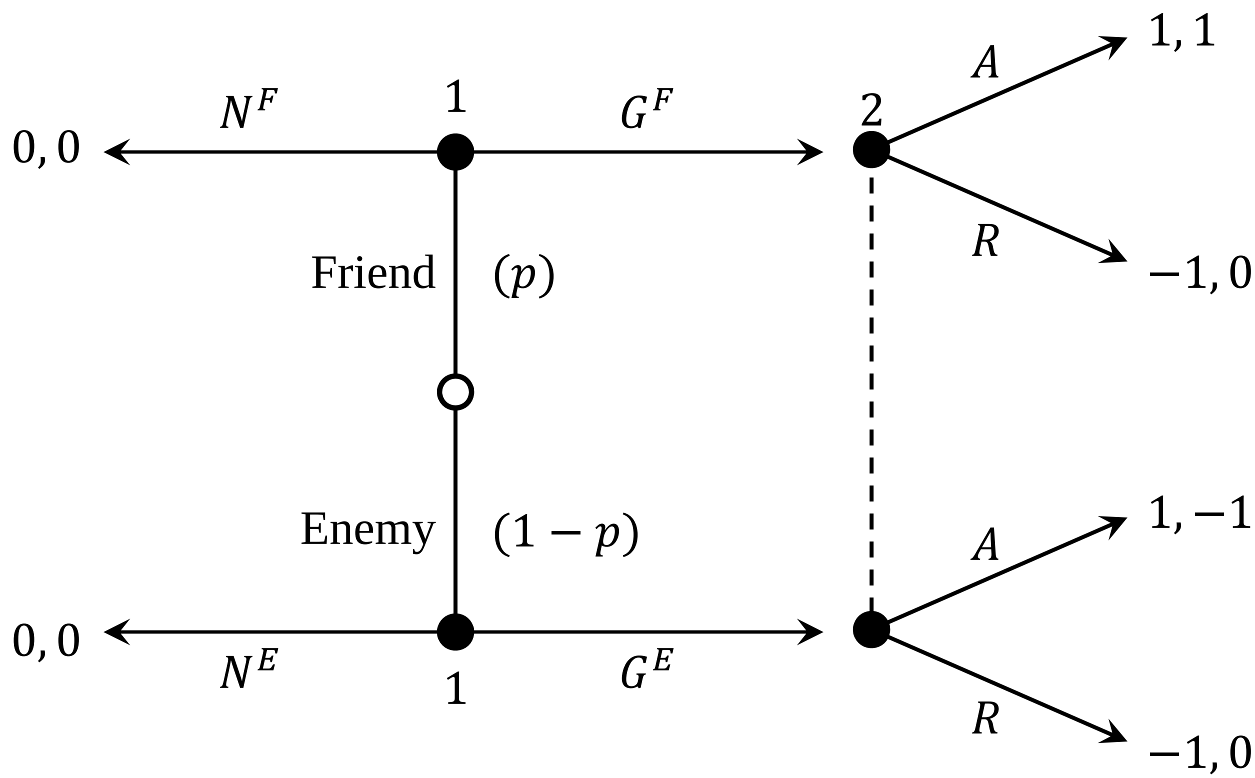


- Above we already described that payoffs are zero to both players if there is no gift offered. **We show the payoffs to player 1 first and player 2 second**. **<u>Note that nature does not have a payoff function!</u>**

- **Step 3.- Player 2's decision:** If a gift was offered, player 2 must decide whether to accept (A) or reject (R) it. Player 2 does not know the true type of player 1. Therefore player 2 does not observe the outcome of nature's move.
- Therefore the information set of player 2 must look like this:



- All that is left is just to indicate the numerical payoffs which were described above:



- This is the extensive form representation of the game.

- What about the **normal form** representation?
- First, let us enumerate the strategies available to each player:
- Player 2 simply has TWO strategies:

$$(A, R)$$

- Player 1 in contrast has FOUR strategies:

$$(N^E N^F, N^E G^F, G^E N^F, G^E G^F)$$

- $N^E N^F$: “don’t give a gift if “enemy”, don’t give a gift if “friend”.
- $N^E G^F$: “don’t give a gift if “enemy”, give a gift if “friend”.
- $G^E N^F$: “give a gift if “enemy”, don’t give a gift if “friend”.
- $G^E G^F$: “give a gift if “friend”, give a gift if “enemy”.

- **Recall:** Strategies must be complete contingent plans. Therefore they must contain instructions for player 1 of what to do in each possible move of nature.
- OK, so the normal form must be a matrix that looks like this:

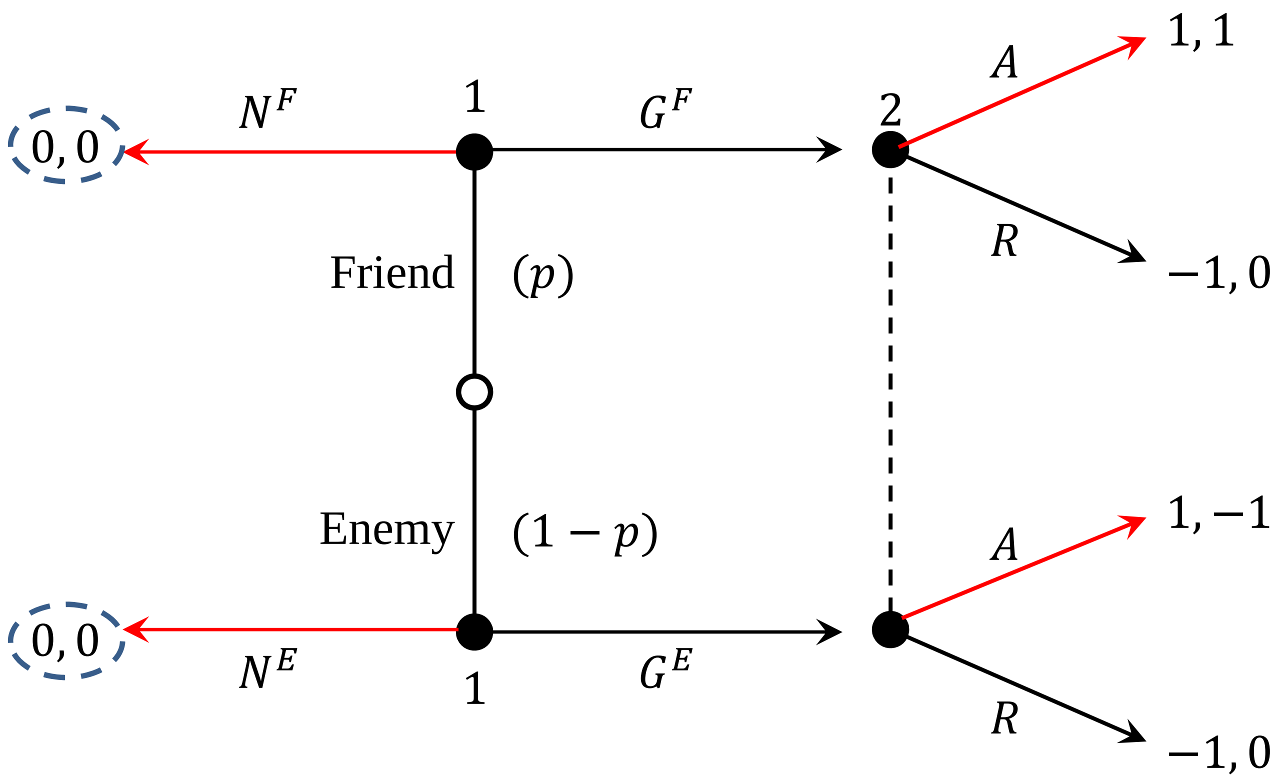
1 \ 2	A	R
$N^E N^F$		
$N^E G^F$		
$G^E N^F$		
$G^E G^F$		

- The entries in the normal form game must correspond to the **expected payoffs that result from the probability distribution of nature's moves**.

- So, for each of the eight entries in the matrix we need to compute the corresponding expected payoffs for players 1 and 2.

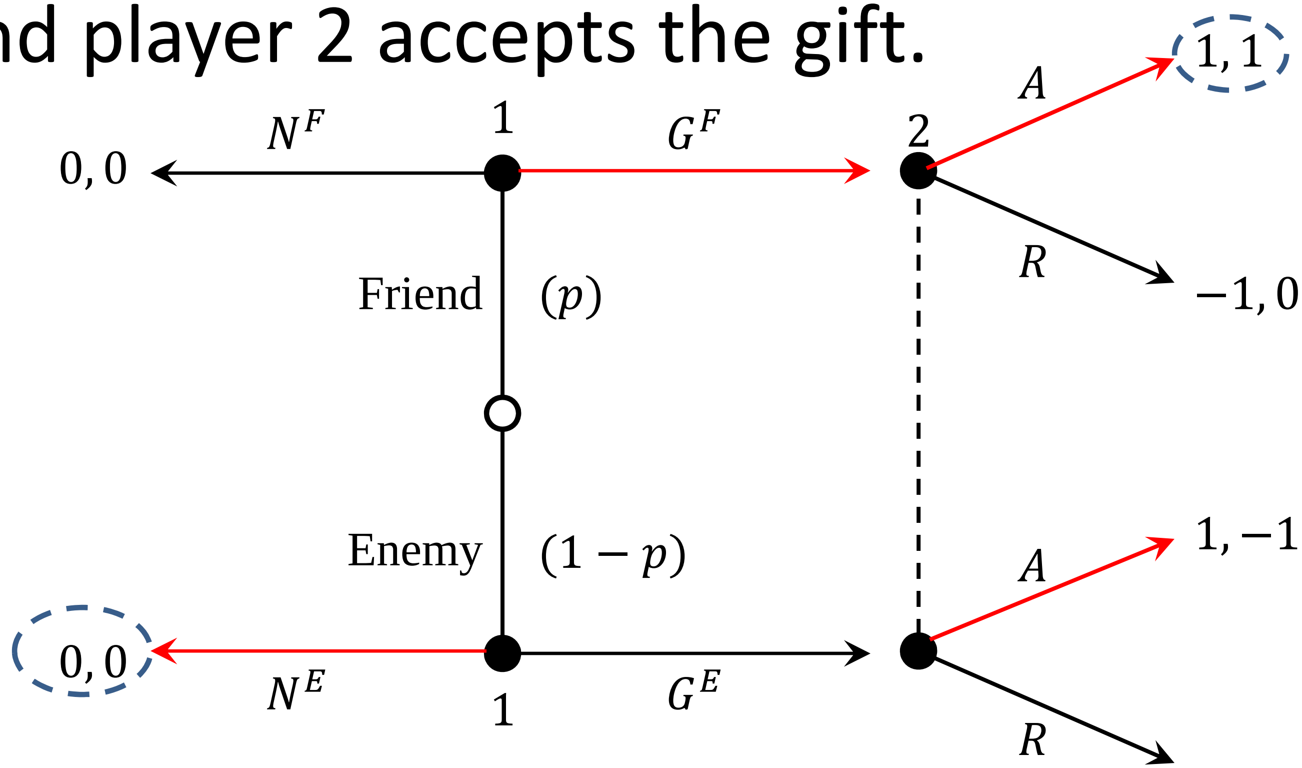
- These expected payoffs depend on the probabilities p (the probability of being "friend") and $1-p$ (the probability of being :enemy").

- We will compute each entry at a time:
- **Profile $\left(\boldsymbol{N^E N^F}, \boldsymbol{A}\right)$:** Here, player 1 does not offer a gift in either case and player 2 accepts.



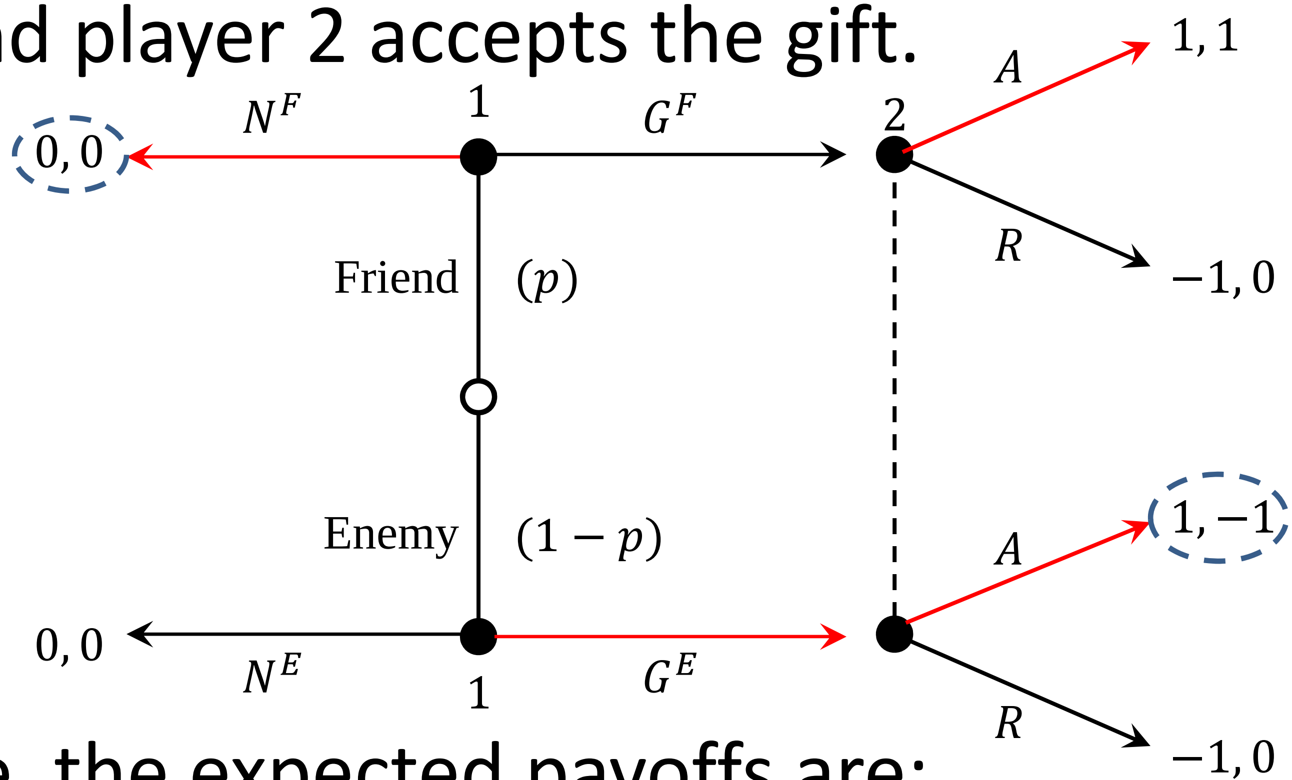
- Trivially both players earn a payoff of <u>zero.</u>

- **Profile $\left(N^E G^F, A\right)$:** Here, player 1 does not offer a gift if he is an enemy but offers a gift if he is a friend, and player 2 accepts the gift.



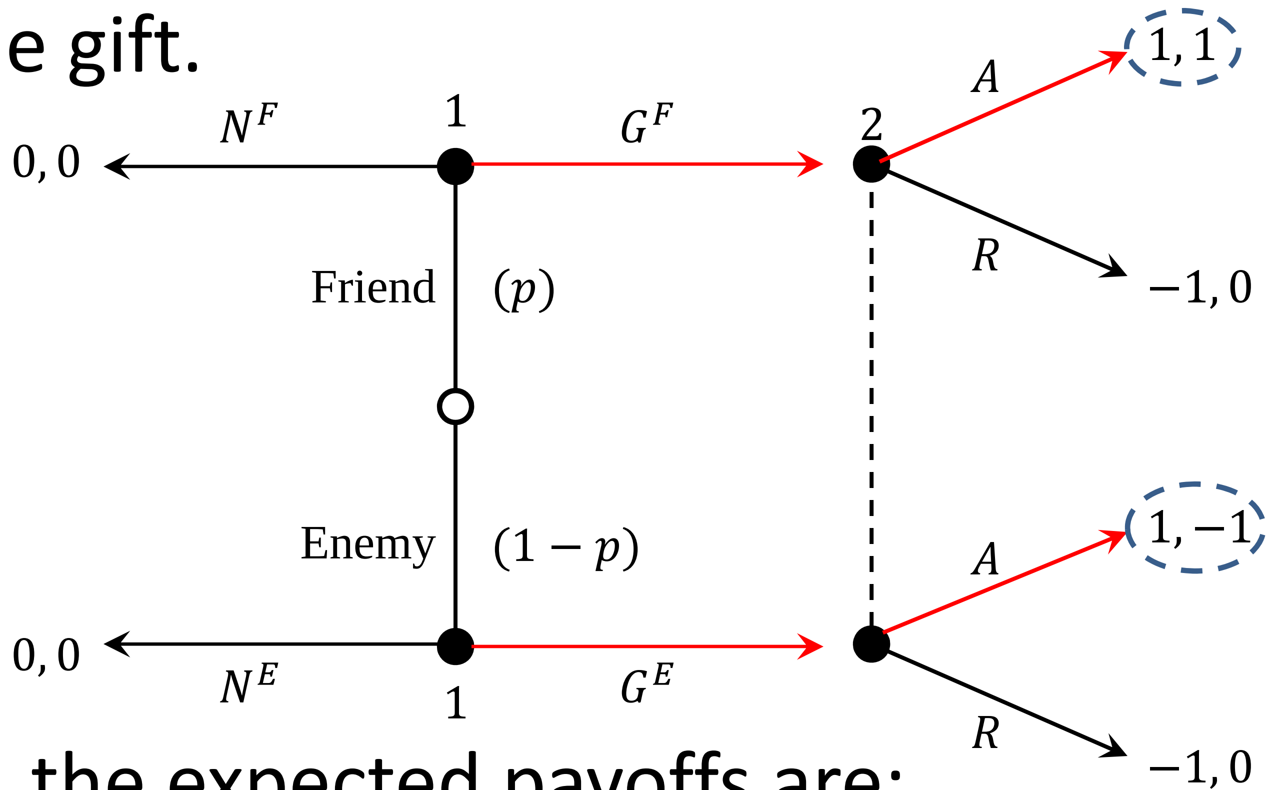
- Therefore, the expected payoffs are:
- For player 1: $0 \cdot (1-p) + 1 \cdot p = p$
- For player 2: $0 \cdot (1-p) + 1 \cdot p = p$

- **Profile $(G^E N^F, A)$:** Here, player 1 offers a gift if he is an enemy and does not offer a gift if he is a friend, and player 2 accepts the gift.



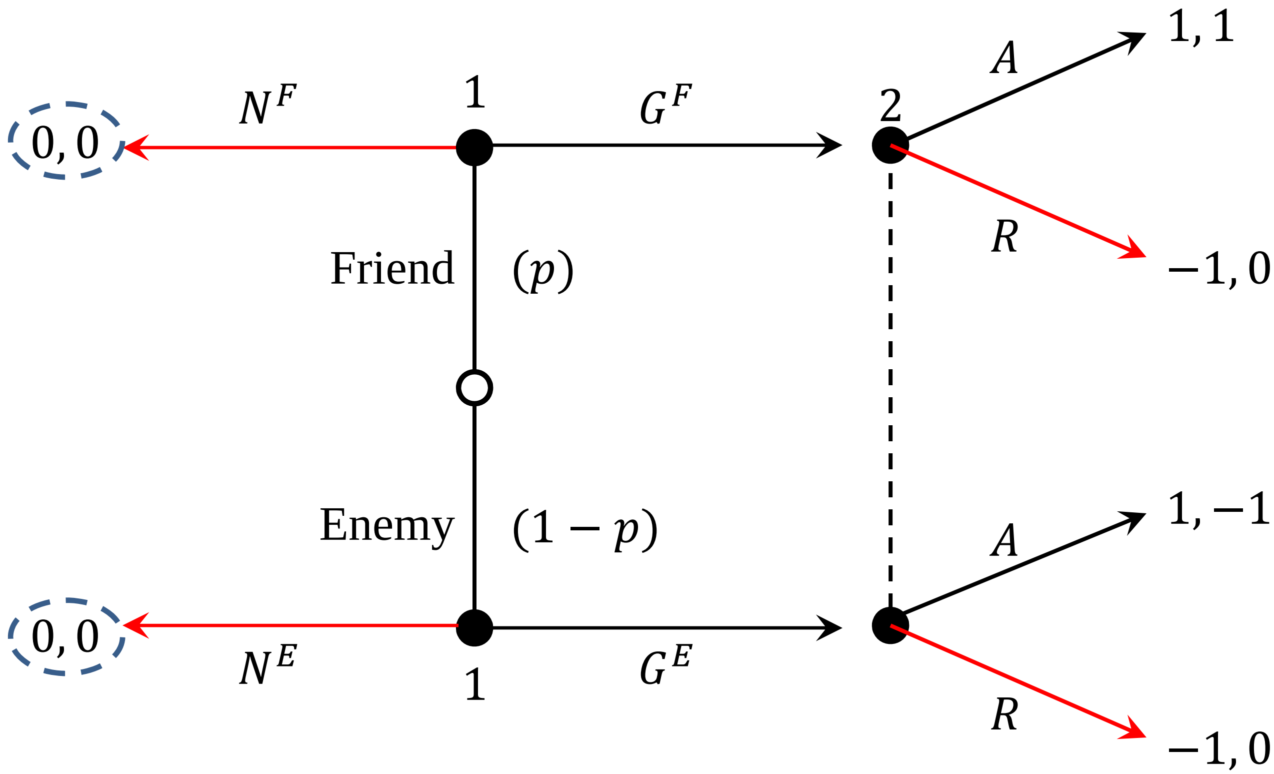
- Therefore, the expected payoffs are:
- <u>For player 1:</u> $1 \cdot (1 - p) + 0 \cdot p = 1 - p$
- <u>For player 2:</u> $-1 \cdot (1 - p) + 0 \cdot p = p - 1$

- **Profile $(G^E G^F, A)$:** Here, player 1 offers a gift if he is an enemy and also if he is a friend, and player 2 accepts the gift.



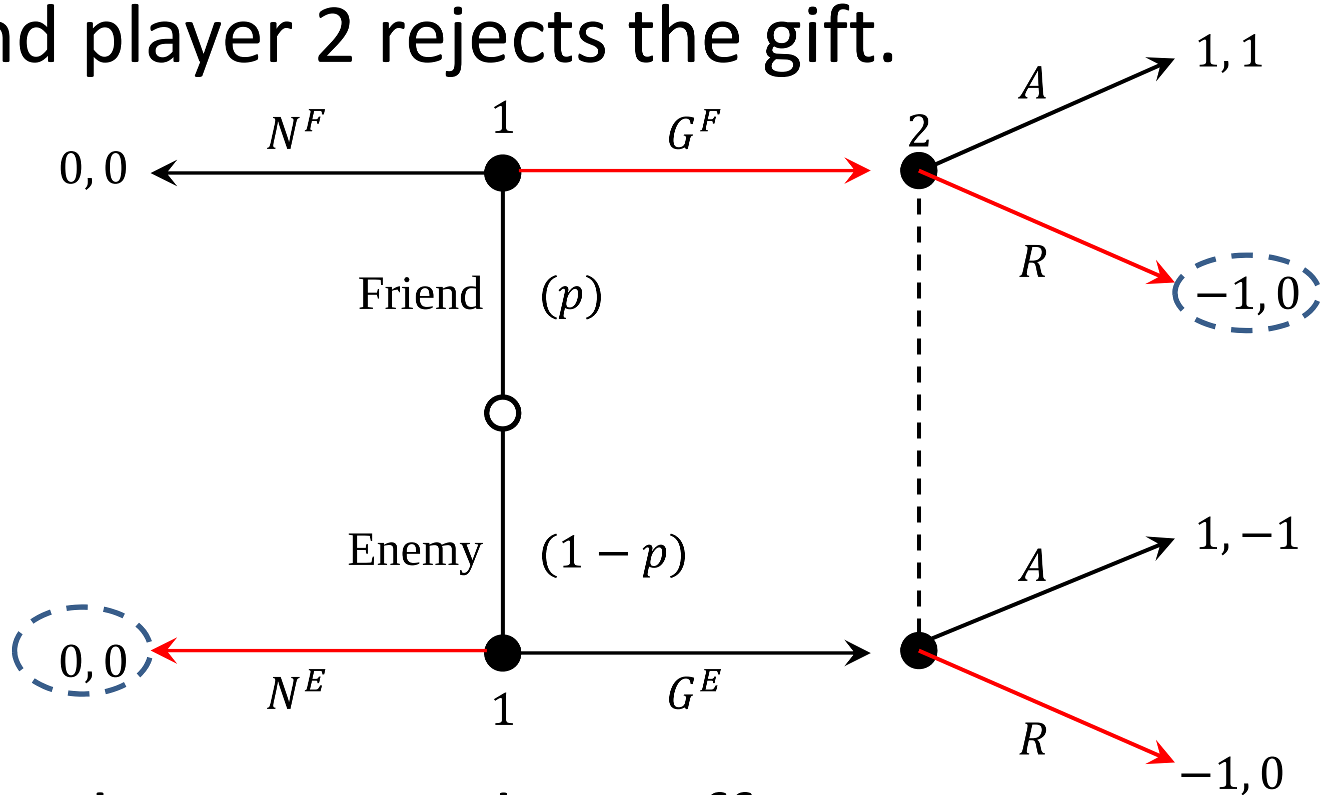
- Therefore, the expected payoffs are:
- <u>For player 1:</u> $1 \cdot (1-p) + 1 \cdot p = 1$
- <u>For player 2:</u> $-1 \cdot (1-p) + 1 \cdot p = 2p - 1$

- **Profile $(\boldsymbol{N^E N^F, R})$:** Here, player 1 does not offer a gift in either case and player 2 rejects.



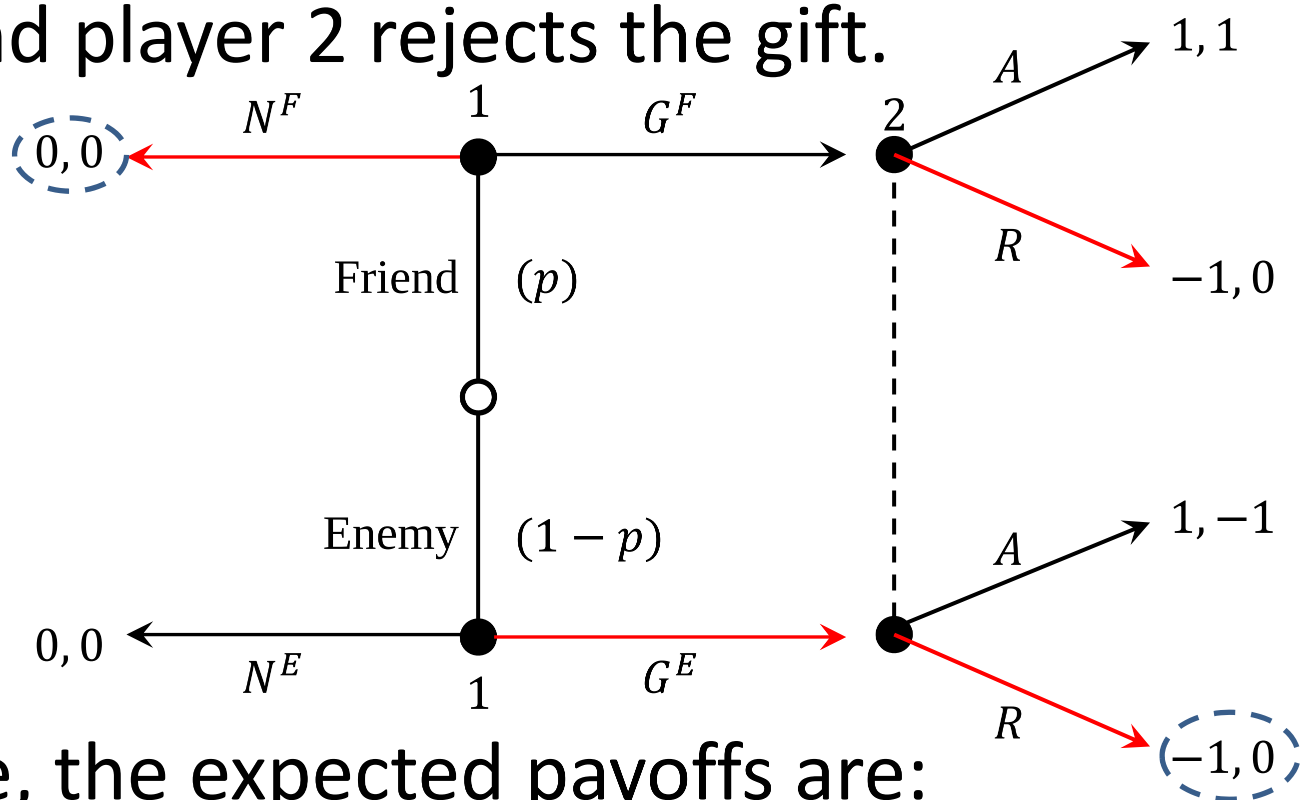
- Trivially both players earn a payoff of <u>zero.</u>

- **Profile $\left(N^E G^F, R\right)$:** Here, player 1 does not offer a gift if he is an enemy but offers a gift if he is a friend, and player 2 rejects the gift.



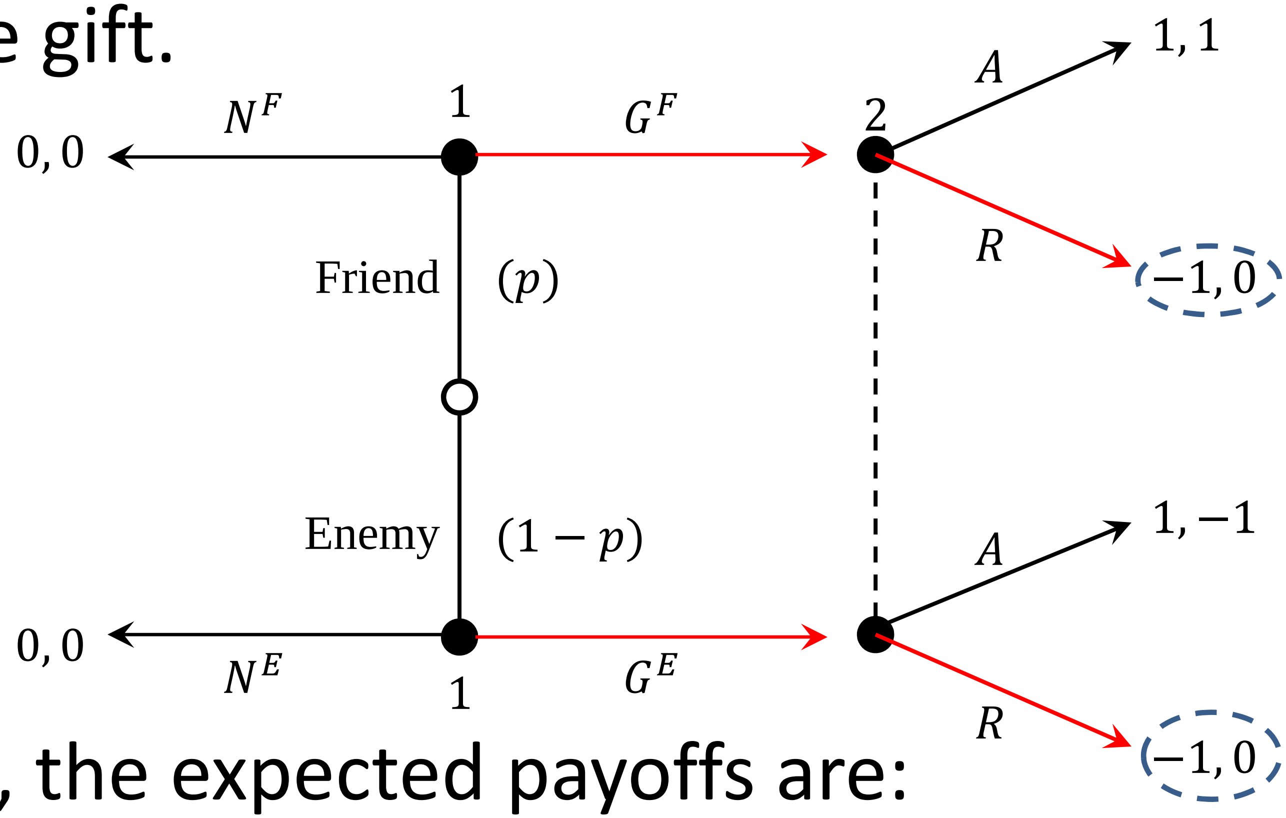
- Therefore, the expected payoffs are:
- <u>For player 1:</u> $0 \cdot (1-p) - 1 \cdot p = -p$
- <u>For player 2:</u> $0 \cdot (1-p) + 0 \cdot p = 0$

- **Profile $(G^E N^F, R)$:** Here, player 1 offers a gift if he is an enemy and does not offer a gift if he is a friend, and player 2 rejects the gift.



- Therefore, the expected payoffs are:
- <u>For player 1:</u> $-1 \cdot (1 - p) + 0 \cdot p = p - 1$
- <u>For player 2:</u> $0 \cdot (1 - p) + 0 \cdot p = 0$

- **Profile $(\boldsymbol{G^E G^F, R})$:** Here, player 1 offers a gift if he is an enemy and also if he is a friend, and player 2 rejects the gift.



- Therefore, the expected payoffs are:
- <u>For player 1:</u> $-1 \cdot (1-p) - 1 \cdot p = -1$
- <u>For player 2:</u> $0 \cdot (1-p) + 0 \cdot p = 0$

- OK now we are ready to enter the expected payoffs into the normal form matrix:

1 \ 2	A	R
$N^E N^F$	$0, 0$	$0, 0$
$N^E G^F$	p, p	$-p, 0$
$G^E N^F$	$1-p, p-1$	$p-1, 0$
$G^E G^F$	$1, 2p-1$	$-1, 0$

- Normal form representations of games with nature moves are called **Bayesian normal forms**.

- When we have games with moves by nature, all the concepts we have studied before are modified to include the word "Bayesian" at the beginning. For example:
  - Bayesian normal form.
  - Bayesian Nash equilibrium.
  - Bayesian rationalizability.
  - Bayesian perfect equilibrium

- For instance, once we have derived the Bayesian normal form of the previous example we can find the Bayesian Nash equilibria of the game in the usual way.
- The specific equilibria would depend on the specific value of the probability p that governs nature's move:

1 \ 2	A	R
$N^E N^F$	$0 , 0$	$0 , 0$
$N^E G^F$	p, p	$-p, 0$
$G^E N^F$	$1 - p, p - 1$	$p - 1, 0$
$G^E G^F$	$1, 2p - 1$	$-1, 0$

- **Example:** Consider an industry where there is an incumbent firm (player 1) and a potential entrant (player 2). The incumbent can be of two types depending on his manufacturing costs: **low cost incumbent** or **high cost incumbent**.
- The true type of the incumbent is known only to the incumbent itself, but not to the entrant. Suppose the **incumbent is a high cost type with probability q and low cost type with probability $1 - q$.**

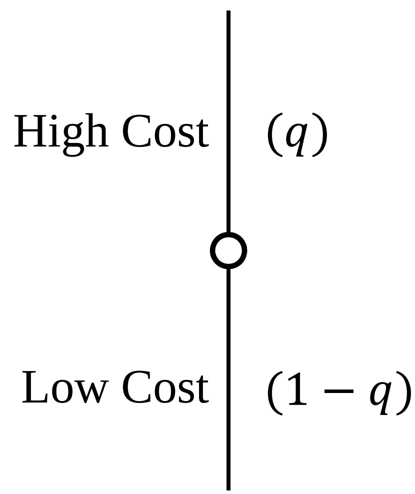
- The game proceeds as follows:
- Once the incumbent learns his true type, he must decide whether to **set a low price** ($\underline{p}$) or **set a high price** $(\overline{p})$.
- The potential entrant observes the price decision by the incumbent (but not the incumbent's type) and then the entrant decides whether to enter the market (E) or not enter (N).

- Suppose payoffs are given as follows:
- If the entrant decides not to enter, his payoff is always zero. In this case, the payoff to the incumbent is equal to:
  - 2 if the incumbent is a high cost type and chooses a high price.
  - 0 if the incumbent is a high cost type and chooses a low price.
  - 4 if the incumbent is low cost type and chooses a high price.
  - 2 if the incumbent is a low cost type and chooses a low price.

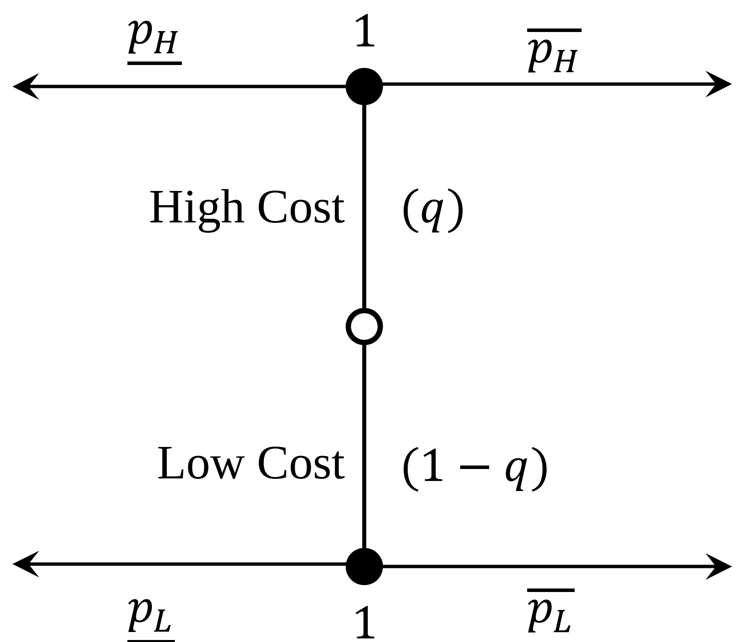
- If the entrant decides to enter, then the payoff to the incumbent is always zero regardless of his type. The payoff to the entrant is equal to:
  - 0 if the incumbent is high cost, regardless of the price set by the incumbent.
  - -1 if the incumbent is low cost, regardless of the price set by the incumbent.

- **Draw the extensive form of this game.-** Again, this can be represented as a game where nature moves first, determining the type of the incumbent.

- **Step 1:** Draw nature's initial move.- Again, it will be graphically convenient to represent it like this:

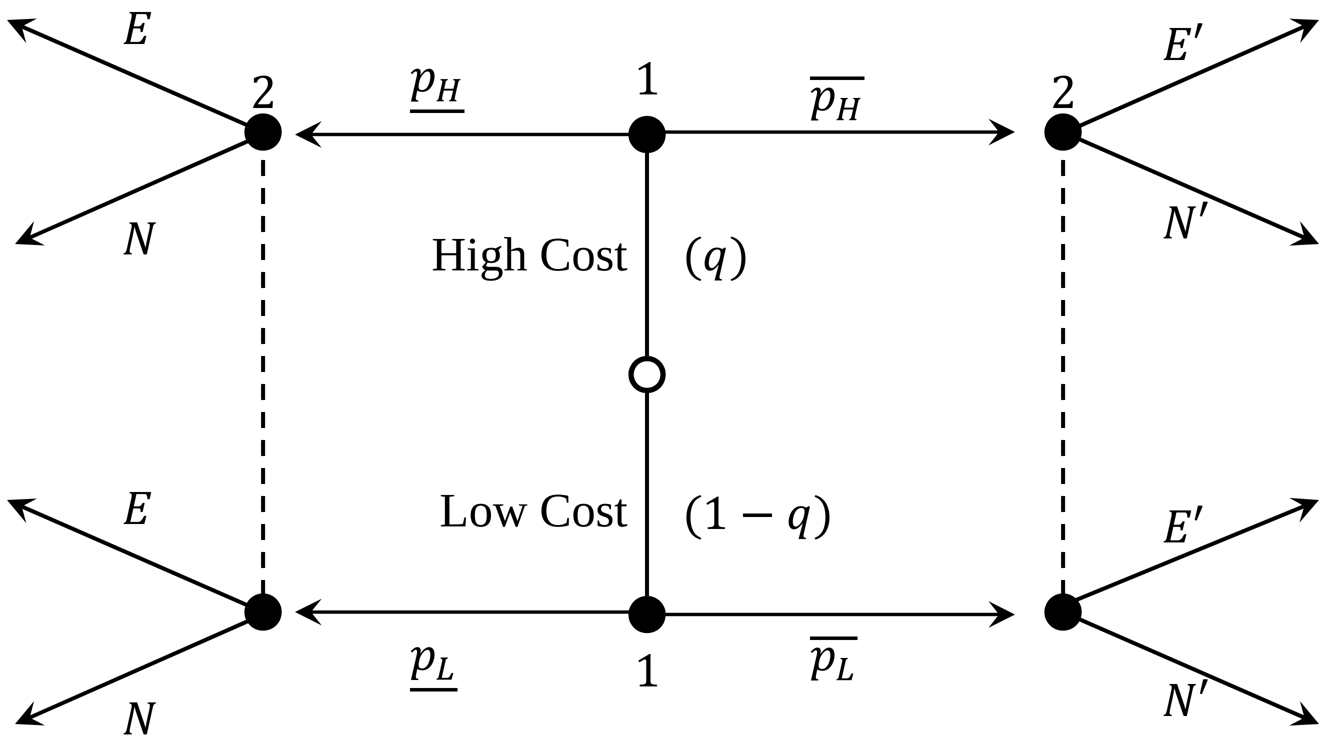


- **Step 2:** Represent the incumbent's actions.- The incumbent learns his true type and then decides whether to set a high price or a low price:



- $\overline{p_H}$ represents "set a high price if high cost type, $\overline{p_L}$ is "set a high price if low cost type", etc.,

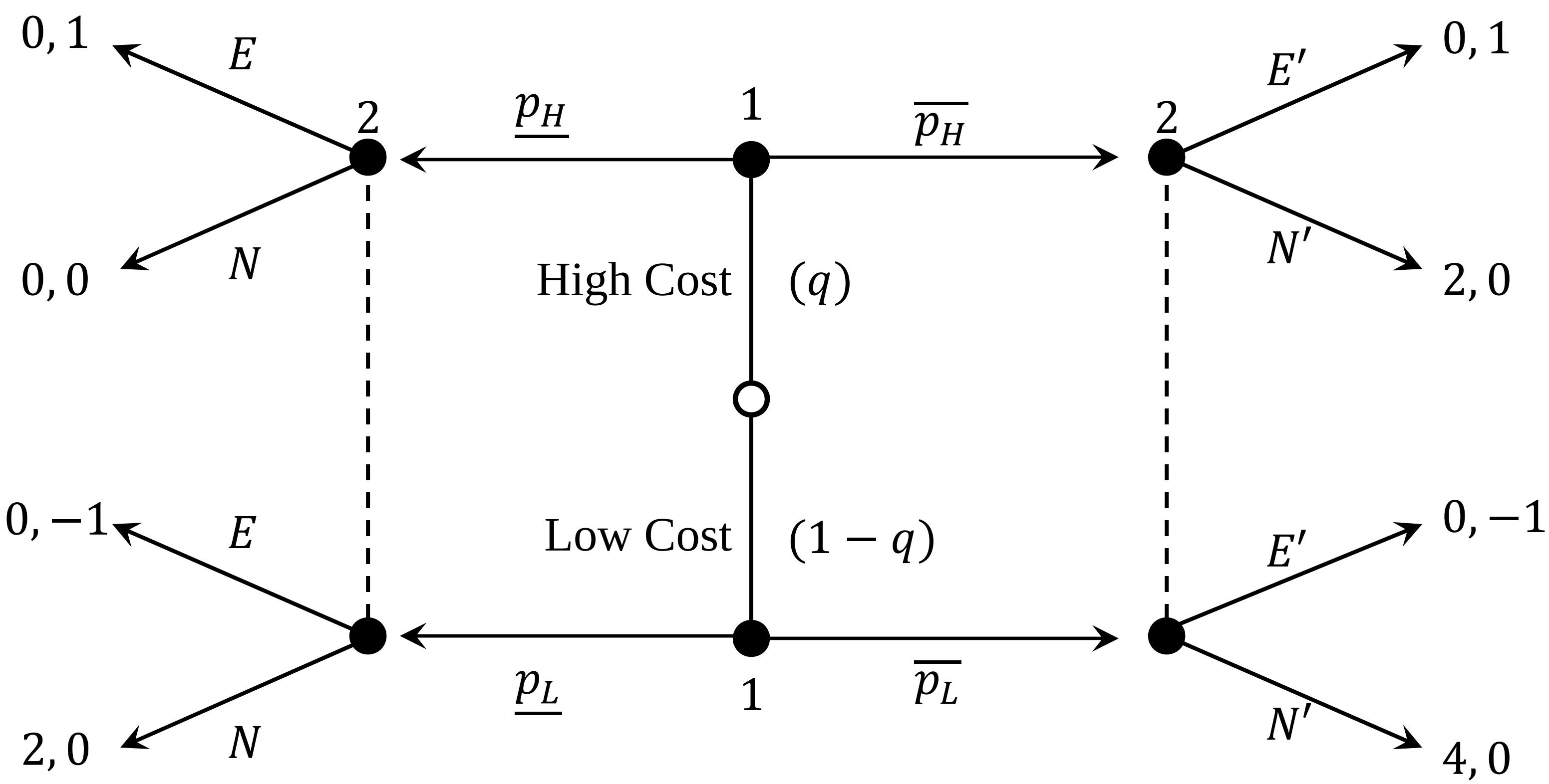
- **Step 3:** Represent the entrant's actions.- The entrant observes if the incumbent set a low or a high price but does not know the true type of the incumbent. The entrant then decides whether to enter or not. Graphically:



- Where:
- E represents the strategy "enter given that the incumbent set a low price".
- N represents the strategy "do not enter given that the incumbent set a low price".

- E' represents the strategy "enter given that the incumbent set a high price".
- N' represents the strategy "do not enter given that the incumbent set a high price".

- The last step is to include the payoffs which were described before. We have:



- What about the **Bayesian normal form?**
- **As always we first need to enumerate the strategies available to each player:**
- **Player 1:** Has four possible strategies.-

$$\left(\overline{p_H}\ \overline{p_L}, \overline{p_H}\underline{p_L}, \underline{p_H}\overline{p_L}, \underline{p_H p_L}\right)$$

- **Player 2:** Has four possible strategies.-

$$(EE', EN', NE', NN')$$

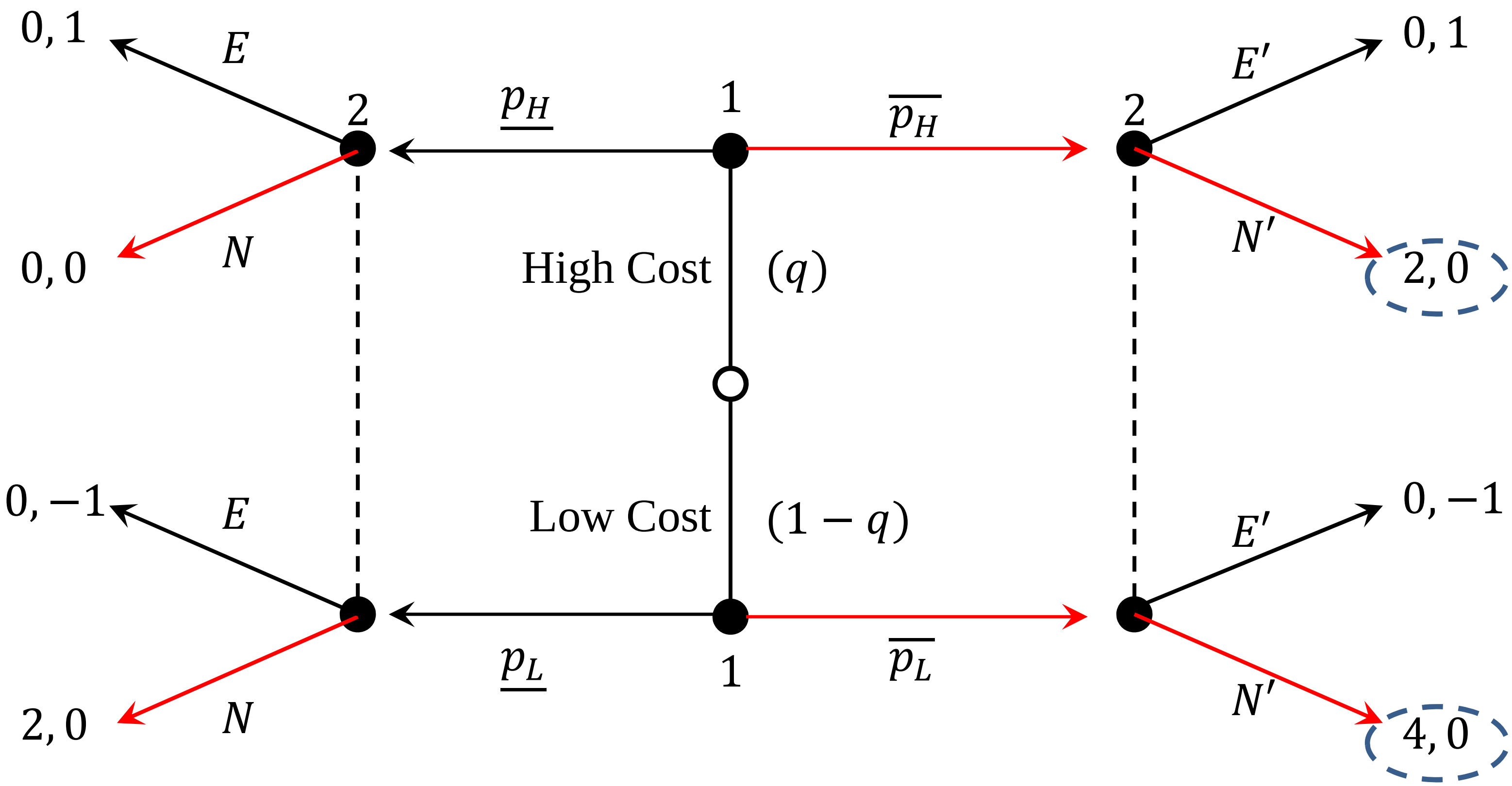
- The Bayesian normal form therefore has 16 total cells.

- Bayesian Normal form structure:

1 \ 2	EE'	EN'	NE'	NN'
$\overline{p_H}\,\overline{p_L}$				
$\overline{p_H}\,\underline{p_L}$				
$\underline{p_H}\,\overline{p_L}$				
$\underline{p_H}\,\underline{p_L}$				

- Once again, the entries in this matrix correspond to the expected payoffs for each profile. We will derive the expected payoffs for four profiles and leave the derivation of the rest as an exercise.

- **Profile $(\overline{p_H}\,\overline{p_L}, NN')$:**



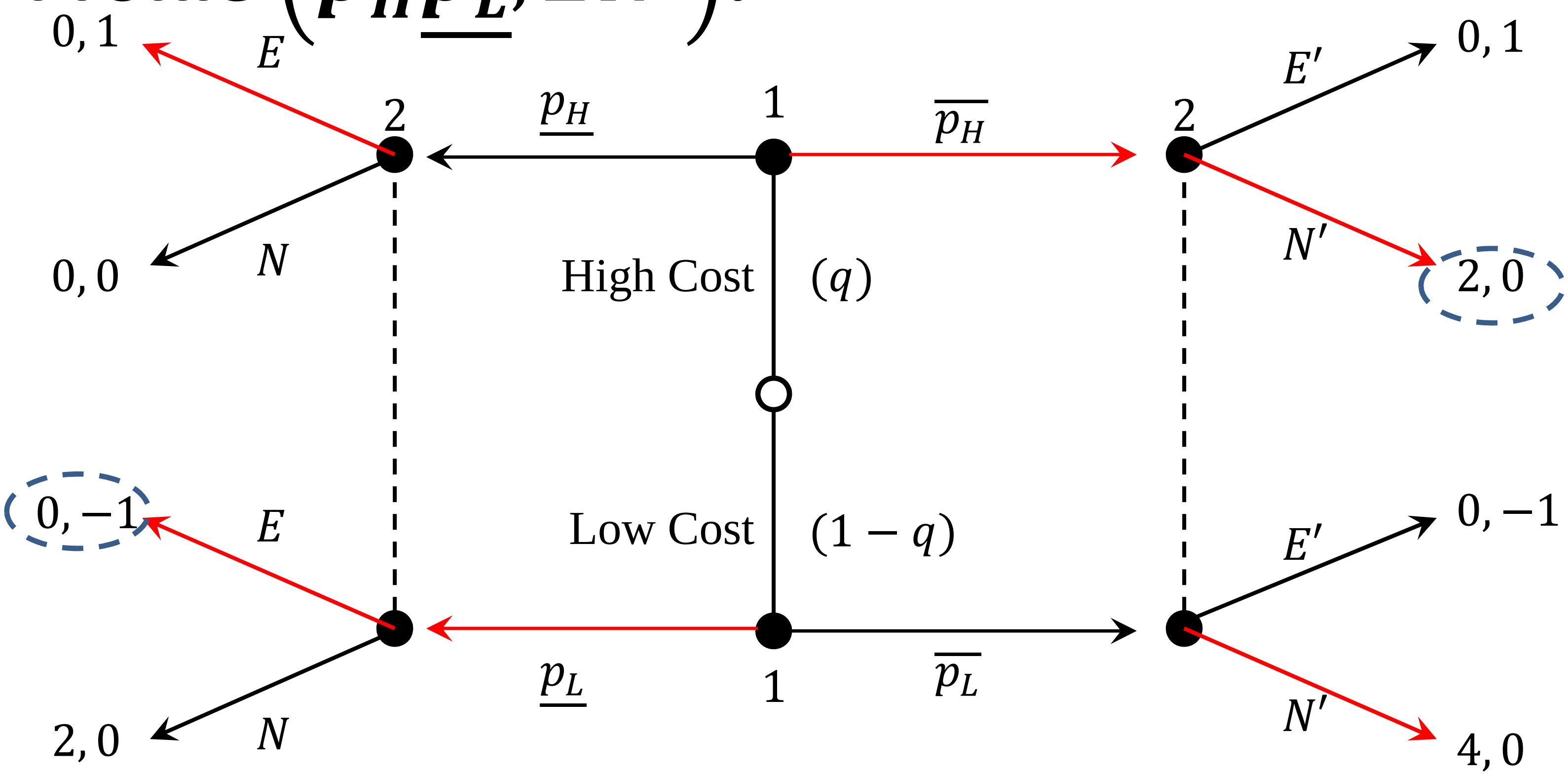
- Expected payoff for player 1:

$$2 \cdot q + 4 \cdot (1 - q) = 4 - 2 \cdot q$$

- Expected payoff for player 2:

$$0 \cdot q + 0 \cdot (1 - q) = 0$$

- **Profile $\left(\overline{p_H}\underline{p_L}, EN'\right)$:**



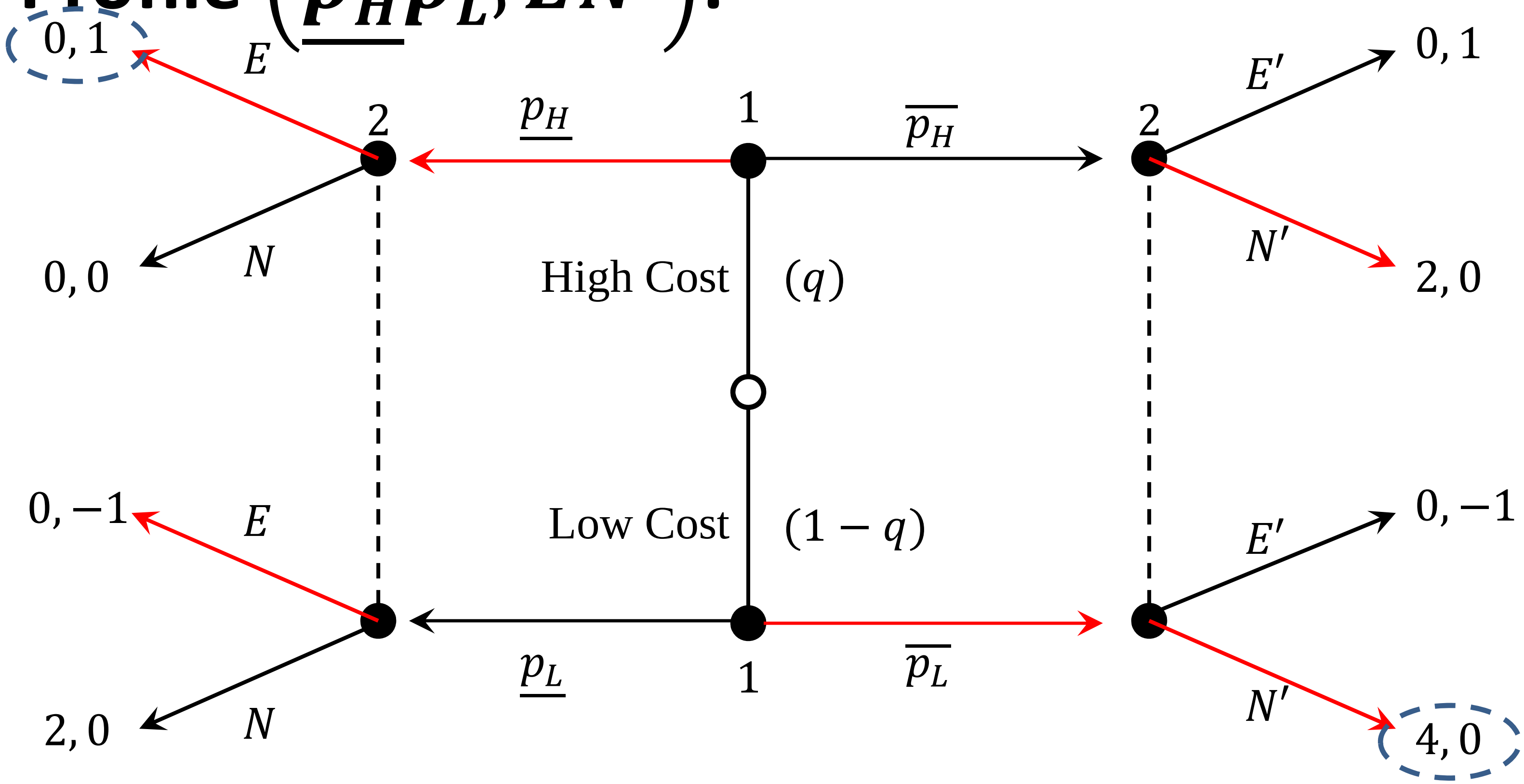
- Expected payoff for player 1:

$$2 \cdot q + 0 \cdot (1 - q) = 2 \cdot q$$

- Expected payoff for player 2:

$$0 \cdot q - 1 \cdot (1 - q) = q - 1$$

- **Profile $\left(\underline{p_H}\overline{p_L}, EN'\right)$:**



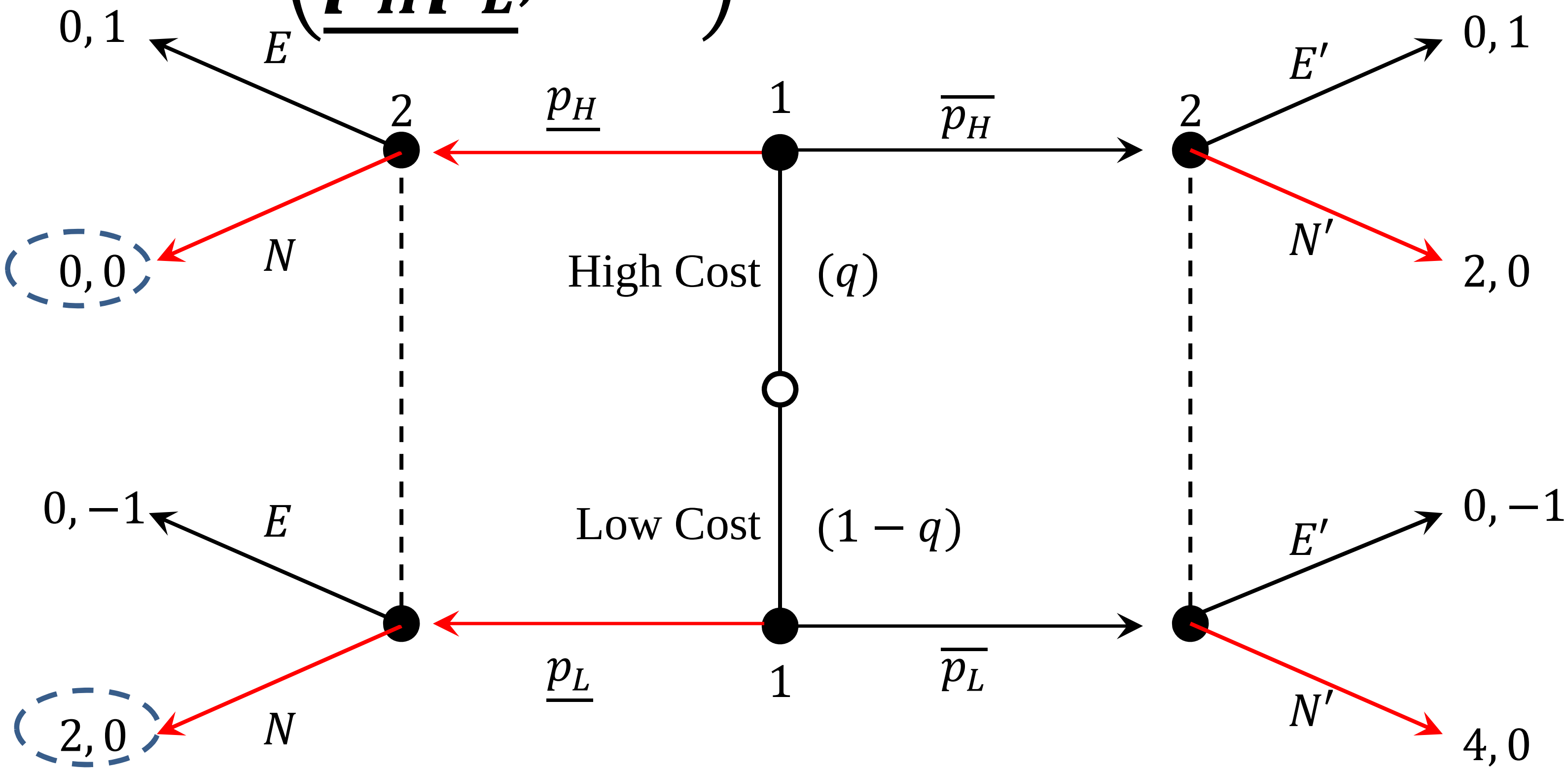
- Expected payoff for player 1:

$$0 \cdot q + 4 \cdot (1 - q) = 4 - 4 \cdot q$$

- Expected payoff for player 2:

$$1 \cdot q + 0 \cdot (1 - q) = q$$

- **Profile $(\underline{p_H p_L}, NN')$:**



- Expected payoff for player 1:

$$0 \cdot q + 2 \cdot (1 - q) = 2 - 2 \cdot q$$

- Expected payoff for player 2:

$$0 \cdot q + 0 \cdot (1 - q) = 0$$

- Entering these payoffs into the normal form:

1 \ 2	$\boldsymbol{EE'}$	$\boldsymbol{EN'}$	$\boldsymbol{NE'}$	$\boldsymbol{NN'}$
$\overline{\boldsymbol{p_H}}\,\overline{\boldsymbol{p_L}}$				$4-2q, 0$
$\overline{\boldsymbol{p_H}}\underline{\boldsymbol{p_L}}$		$2q, q-1$		
$\underline{\boldsymbol{p_H}}\overline{\boldsymbol{p_L}}$		$4-4q, q$		
$\underline{\boldsymbol{p_H}}\underline{\boldsymbol{p_L}}$				$2-2q, 0$

- In the same fashion we can compute every payoff (verify this as an exercise):

1 \ 2	$\boldsymbol{EE'}$	$\boldsymbol{EN'}$	$\boldsymbol{NE'}$	$\boldsymbol{NN'}$
$\overline{\boldsymbol{p_H}}\,\overline{\boldsymbol{p_L}}$	$0, 2q-1$	$4-2q, 0$	$0, 2q-1$	$4-2q, 0$
$\overline{\boldsymbol{p_H}}\underline{\boldsymbol{p_L}}$	$0, 2q-1$	$2q, q-1$	$2-2q, q$	$2, 0$
$\underline{\boldsymbol{p_H}}\overline{\boldsymbol{p_L}}$	$0, 2q-1$	$4-4q, q$	$0, q-1$	$4-4q, 0$
$\underline{\boldsymbol{p_H}}\underline{\boldsymbol{p_L}}$	$0, 2q-1$	$0, 2q-1$	$2-2q, 0$	$2-2q, 0$

- The Bayesian Nash equilibria in this game can be found from its Bayesian normal form representation. These equilibria will depend on the value of the probability q (the probability that the incumbent is of the "high cost" type).

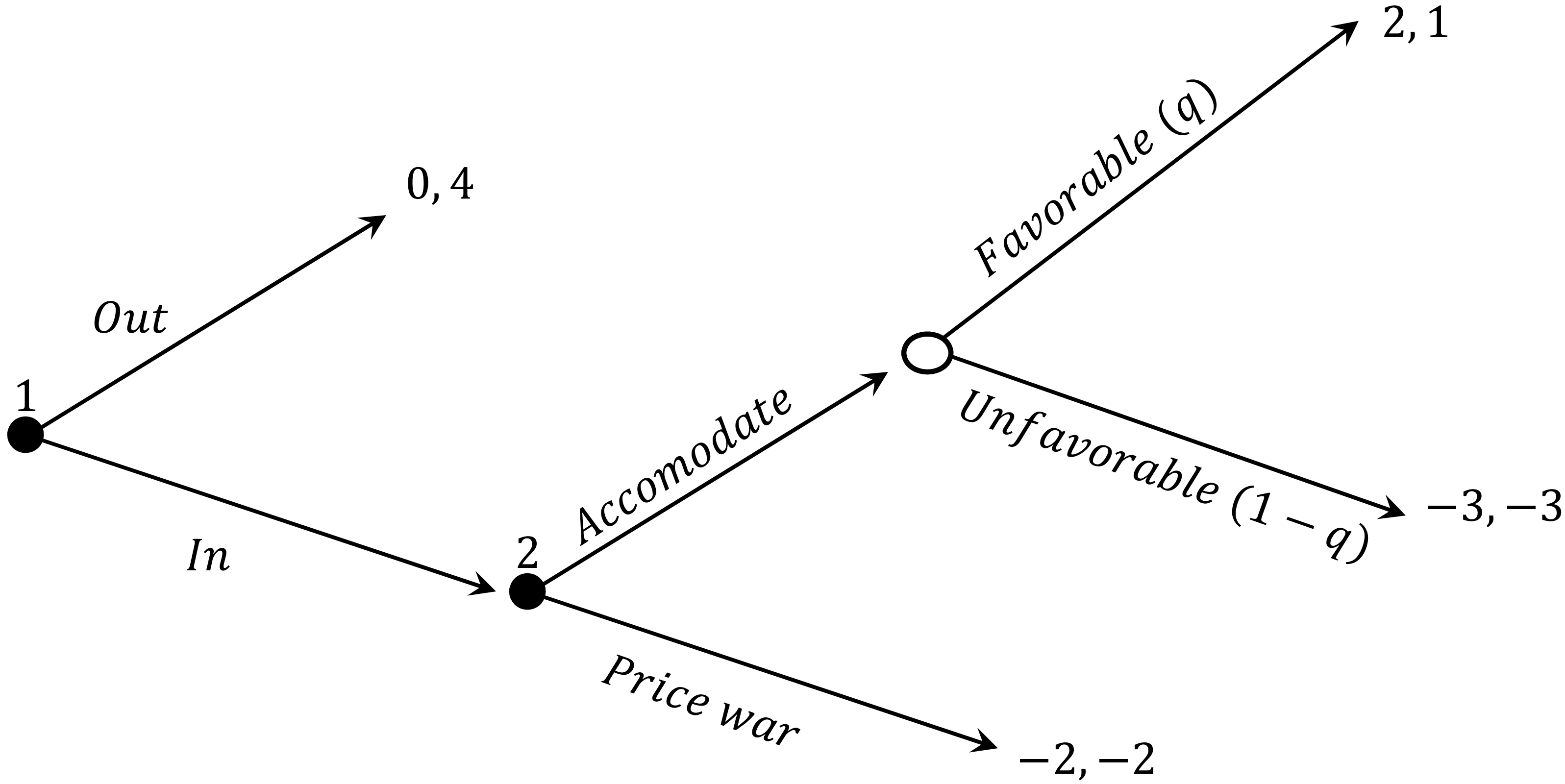
- **Incomplete information games where nature moves at the end of the game.-** We can have games where nature moves only after all players have made their decisions.
- These represent cases where players' payoffs depend on their own actions but also on some state of nature that is not revealed until after all players have made their choices.
- These are games where **players have uncertainty about their own payoffs**.
- Games where nature moves last are the **simplest incomplete information games to analyze**.

- **Example:** Consider the following game, describing an industry with a single incumbent firm (a monopolist).
- A potential entrant (Player 1) decides whether to enter or not into an industry. Suppose the incumbent firm (Player 2) observes the decision of Player 1 and then, if the latter decides to enter, then the incumbent decides whether to initiate a price war or accommodate the new entrant.
- If the entrant chooses not to enter, the payoff to the incumbent is 4 and the payoff to the entrant is zero.

- If the entrant decides to enter and the incumbent decides to start a price war, their payoffs are -2 each.
- If the entrant decides to enter and the incumbent decides to accommodate the entry, the payoffs depend on whether the market is **favorable** or **unfavorable**.
- If the market is favorable, they payoff to the entrant would be 2 and the payoff to the incumbent would be 1. If it is unfavorable, their payoffs would be -3 each.
- Players must make their choices before finding out if the market is favorable or unfavorable.

- This is an example of a game where nature moves at the very end.
- Suppose that the probability of being favorable is q and the probability of being unfavorable is $1 - q$.
- Suppose that these probabilities are known to both players.
- The extensive form of this game is easy to represent..

- Extensive form:



- Games where nature moves last can be solved straightforwardly: Compute the expected payoffs implied by nature’s moves and proceed with backward induction.

- What are the expected payoffs for each player following nature’s move?
- **Player 1:**

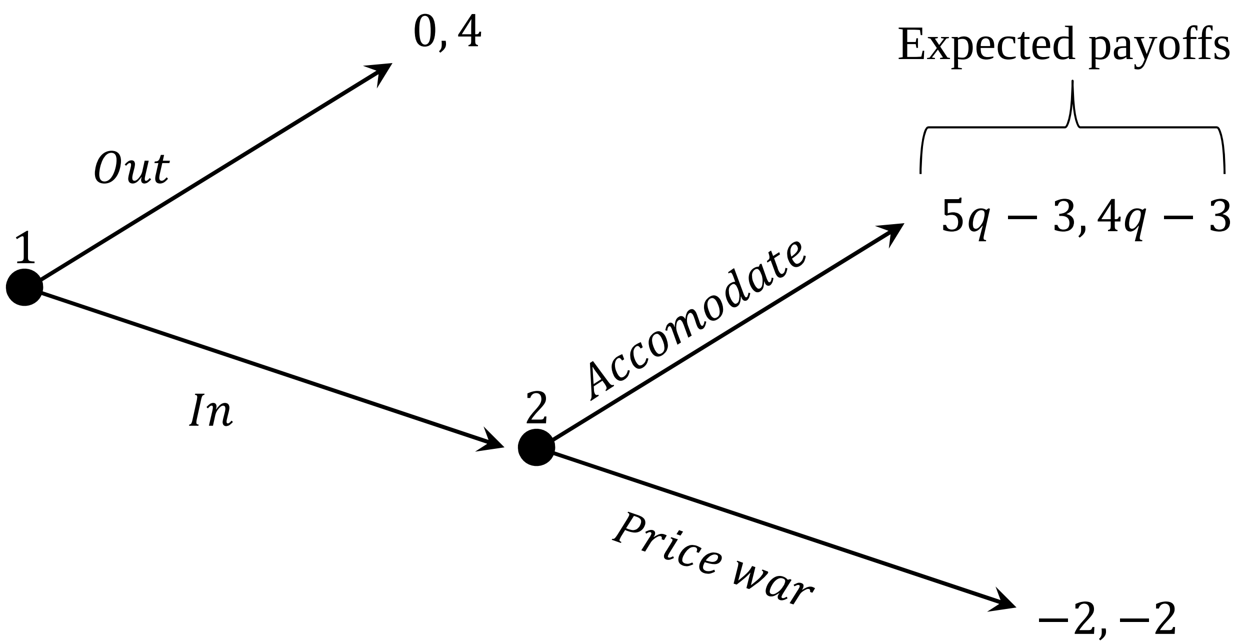
$$2 \cdot q - 3 \cdot (1 - q) = 5 \cdot q - 3$$

- **Player 2:**

$$1 \cdot q - 3 \cdot (1 - q) = 4 \cdot q - 3$$

- These are the expected continuation payoffs for both players implied by nature’s move at the end of the game. To proceed solving the game, we plug in these expected continuation payoffs into the extensive form and continue with backward induction.

- Using the expected continuation payoffs, the game looks like this:



- From here we proceed doing backward induction as usual.

- Backward induction: We start with player 2's final decision node. Let's assume that if he is indifferent between accommodating and going into a price war, player 2 prefers to accommodate.
- Player 2 will accommodate if:

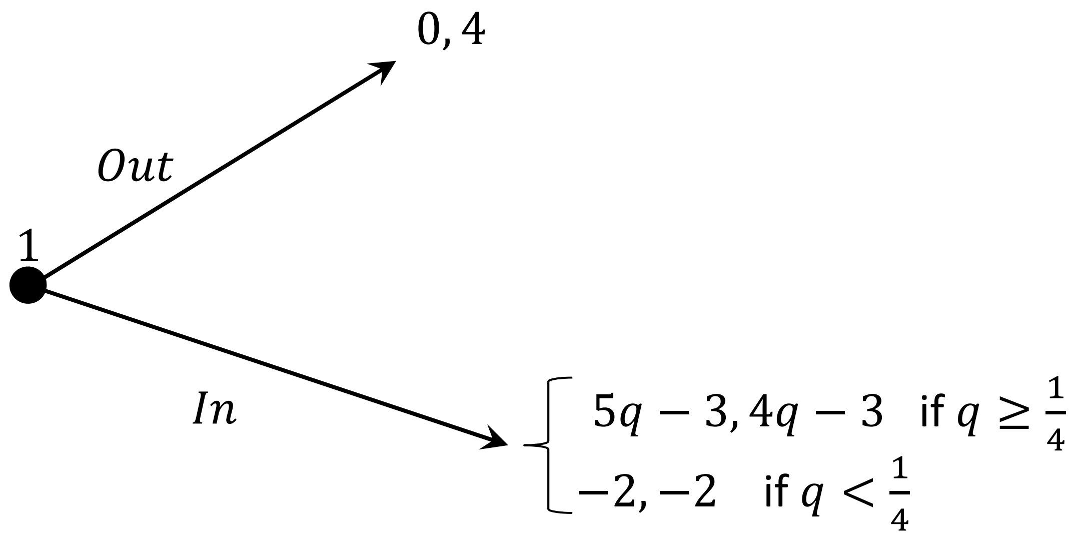
$$4q - 3 \geq -2$$

- Player 2 will choose a price war if:

$$4q - 3 < -2$$

- Expressing these conditions in terms of "q":
- Player 2 will accommodate if $q \geq \frac{1}{4}$
- Player 2 will go to a price war if $q < \frac{1}{4}$

- The continuation payoffs therefore look like this:



- What is the optimal choice for player 1?
- Clearly, if $q < \frac{1}{4}$ player 1 will choose "out" since it leads to a payoff of zero which is larger than the payoff of -2 he would obtain if he enters.

- What about if $q \geq \frac{1}{4}$?
- In this case the continuation payoff for player 1 if he decides to enter would be $5q - 3$.
- Therefore, in this case player 1 will enter if:

$$5q - 3 \geq 0$$

- Otherwise player 1 will choose out if:

$$5q - 3 < 0$$

- That is, player 1 will enter if $q \geq \frac{3}{5}$ and will choose out if $q < \frac{3}{5}$.

- This concludes the backward induction solution. In summary we have:
- Player 1 will decide to enter if $q \geq \frac{3}{5}$.
- Player 1 will decide to stay out if $q < \frac{3}{5}$.
- Player 2 will accommodate if $q \geq \frac{1}{4}$
- Player 2 will choose a price war if $q < \frac{1}{4}$

- Therefore:
- If $q < \frac{1}{4}$: The subgame perfect equilibrium (SPE) of this game is $(Out, Price\ War)$
- If $\frac{1}{4} \leq q < \frac{3}{5}$: The SPE is $(Out, Accommodate)$
- If $q \geq \frac{3}{5}$: The SPE is $(In, Accommodate)$