25.- Risks and Incentives in Contracting

- This brief chapter focuses on incomplete information games where nature moves at the end.
- These games represent situations where players have uncertainty about their payoffs, where these payoffs are at least partially determined by some state of nature which becomes revealed only after players have made their moves.

- In games that involve actual **monetary payments or transactions**, uncertainty about payoffs brings about the issue of **risk aversion**.
- Risk aversion has to do with the attitudes of individuals towards uncertain monetary outcomes.
- Specifically, risk aversion has to do with the attitudes of individuals towards “monetary lotteries”.

- **Risk aversion and monetary lotteries:** Suppose we present an individual with the following two options:
- **First option:** We give the individual \$1,000 for sure.
- **Second option:** We flip a fair coin. If it falls heads, we give the individual \$2,000. If it falls tails, we give the individual \$0. This is a **lottery** where the individual wins \$2,000 with probability 50% and gets nothing with probability 50%.

- Notice that the expected payment from the lottery is:

$$\frac{1}{2} \cdot \$2{,}000 + \frac{1}{2} \cdot \$0 = \$1{,}000$$

- Thus, the lottery has the same expected payment as the first option (receiving $1,000 with certainty), but there is a 50% chance the individual gets $2,000 (and a 50% chance the individual gets nothing).
- We say that the individual is <u>**risk neutral**</u> **if he is indifferent between both options**.
- We say that the individual is <u>**risk averse**</u> **if he strictly prefers the first option ($1,000 with certainty) to the lottery of the second option (a lottery whose expected value is $1,000)**.

- More general definition of risk aversion:
- Take any monetary amount $\bar{M}$.
- We say that an individual is **risk neutral** if he is indifferent between any lottery whose expected monetary payment is $\bar{M}$ and receiving a monetary payment of $\bar{M}$ with certainty.
- We say that an individual is **risk averse** if he strictly prefers a monetary payment of $\bar{M}$ with certainty to any lottery whose expected monetary payment is $\bar{M}$.

- How do we represent risk aversion mathematically? We need to pay attention to the way money translates into utility (payoff) for an individual.
- Let v represent the function that maps money to utility for an individual.
- That is, let v denote the individual's utility or payoff function for money. For any amount of money x, the value $v(x)$ measures the utility or payoff derived from x.
- Risk aversion and risk neutrality are indicated by the features of the function v.

- Consider lotteries that yield an amount x_0 with probability p and x_1 with probability $1 - p$.

- The expected payoff (expected utility) of this lottery is:

$$p \cdot v(x_0) + (1 - p) \cdot v(x_1)$$

- On the other hand, the expected value of this lottery is:

$$p \cdot x_0 + (1 - p) \cdot x_1$$

- Therefore the utility of receiving the expected value of this lottery with certainty is:

$$v(p \cdot x_0 + (1 - p) \cdot x_1)$$

- An individual whose payoff function is v is **risk neutral** if:

$$p \cdot v(x_0) + (1-p) \cdot v(x_1) = v(p \cdot x_0 + (1-p) \cdot x_1)$$

- As it turns out, an individual is **risk neutral** if we can represent his payoff function simply as the linear function $\boldsymbol{v(x) = x}$
- An individual whose payoff function is v is **risk averse** if:

$$p \cdot v(x_0) + (1-p) \cdot v(x_1) < v(p \cdot x_0 + (1-p) \cdot x_1)$$

- As it turns out, an individual is **risk averse** if we can represent $v(x)$ as a concave function. We can represent a concave function simply as $\boldsymbol{v(x) = x^{\alpha}}$ **for some** $\boldsymbol{0 < \alpha < 1}$**.**

- In summary:
- The payoff function for **risk-neutral** individuals will be represented simply as the linear function:

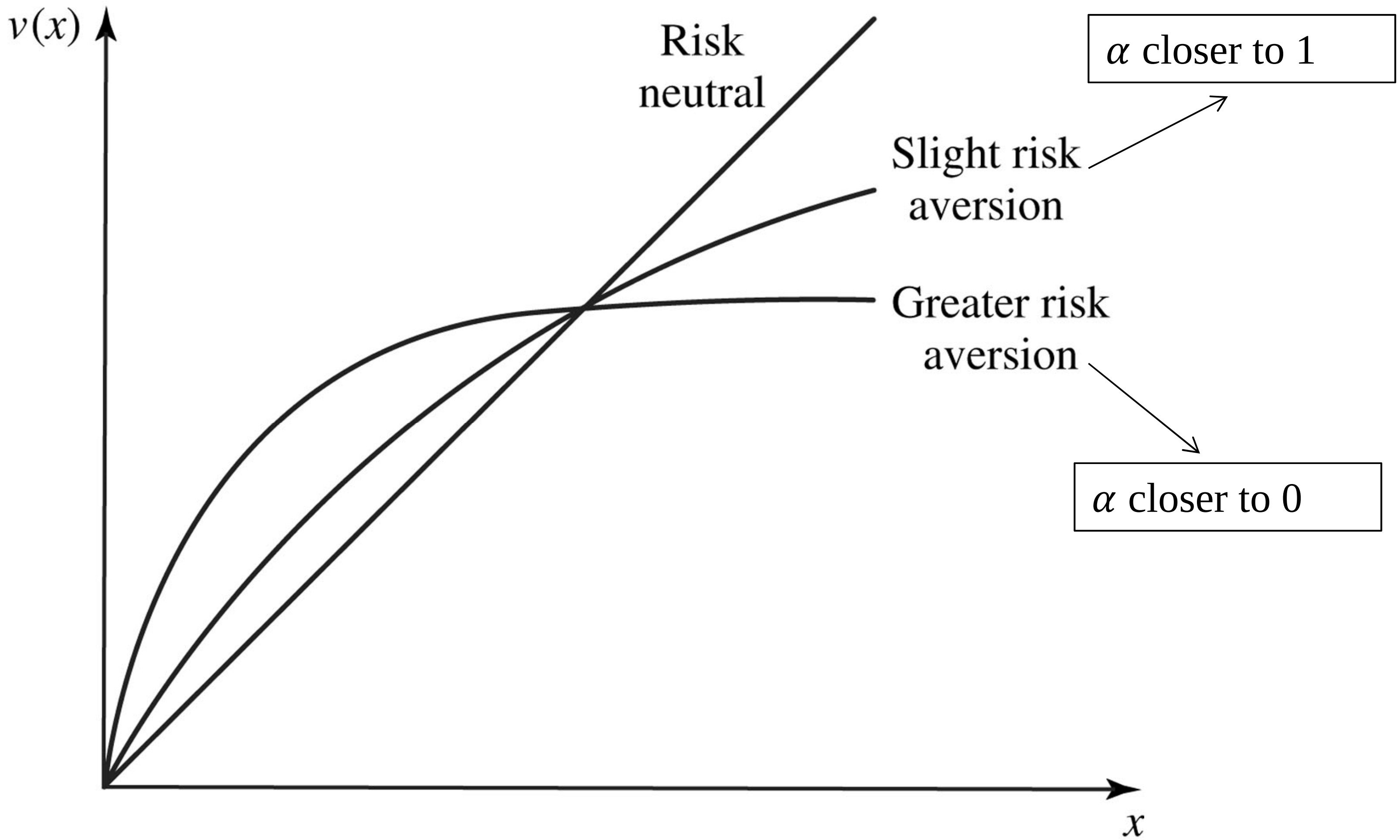
$$\boldsymbol{v(x) = x}$$

- The payoff function for **risk-averse** individuals will be represented as a concave function of the form:

$$v(x) = x^{\alpha} \text{ for some } 0 < \alpha < 1$$

- **The closer $\boldsymbol{\alpha}$ is to zero, the more risk averse the individual is**. If α is close to 1, then the individual is close to being risk neutral.

- Graphically: The more concave the function is, the more risk averse the individual is. This will happen for values of α that are closer to zero:



- **Example: A principal-agent game.-** One of the best known examples of games with uncertain payoffs is the principal-agent game with moral hazard.
- A principal-agent model with moral hazard arises whenever we have situations where:
- An individual (the principal) wants to hire another individual (the agent) to perform a task.
- The agent’s effort is not verifiable. Only the outcome of the task is verifiable. In addition, the outcome of the project depends not only on the agent’s effort but also on some exogenous random events. This scenario gives rise to what is called “moral hazard”.

- Consider the particular example, where the sequence of moves is the following:
- Stage 1.- The principal sets a wage w to be paid to the agent regardless of the outcome, and a bonus b to be paid to the principal in the event that the project's outcome is successful.
- Stage 2.- Observing w and b, the agent decides whether to accept the job. Then the agent decides how much effort to exert. Assume this can be "High" or "Low" effort.

- Stage 3.- If the effort is Low, then the outcome of the project is certainly unsuccessful. If the effort is High, then some exogenous event (nature) determines whether the outcome is successful or not.
- Specifically, the outcome will be successful with probability 50% and unsuccessful with probability 50%.
- A successful project yields a revenue of $6 to the principal. An unsuccessful project yields a revenue of $2 to the principal.
- High effort has a <u>utility</u> cost of 1 to the agent. Low effort is costless to the agent.

- The degree of risk aversion of the principal and the agent will play a crucial role in this game.
- Assume that the **principal is risk neutral** but that the **agent may be risk averse**. This is the prototypical assumption because a principal typically represents a firm, and firms are typically assumed to be risk neutral. Individuals on the other hand are allowed to be risk averse.

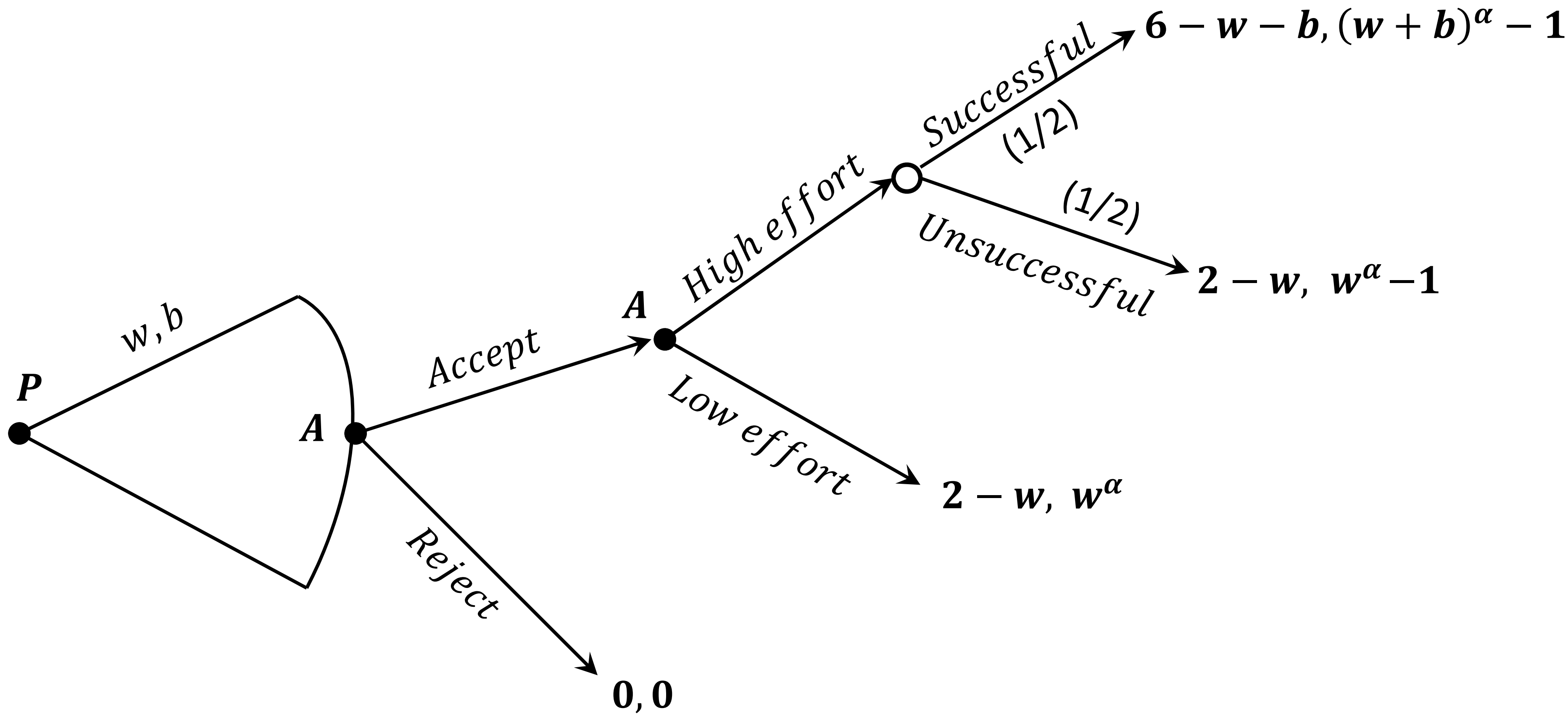
- Therefore, the payoff or utility functions from monetary payments are assumed to be of the form:

$$v_P(x) = x \text{ (for the principal)}$$

$$v_A(x) = x^{\alpha} \text{ with } 0 < \alpha \leq 1 \text{ (for the agent)}$$

- OK, we are ready to represent the extensive form of this game. Notice that this is an example where nature moves at the end, so there is uncertainty about payoffs.

- Extensive form:



- Find the subgame perfect equilibrium (SPE).

- If the agent exerts high effort the expected continuation payoffs are:
- **For the principal:**

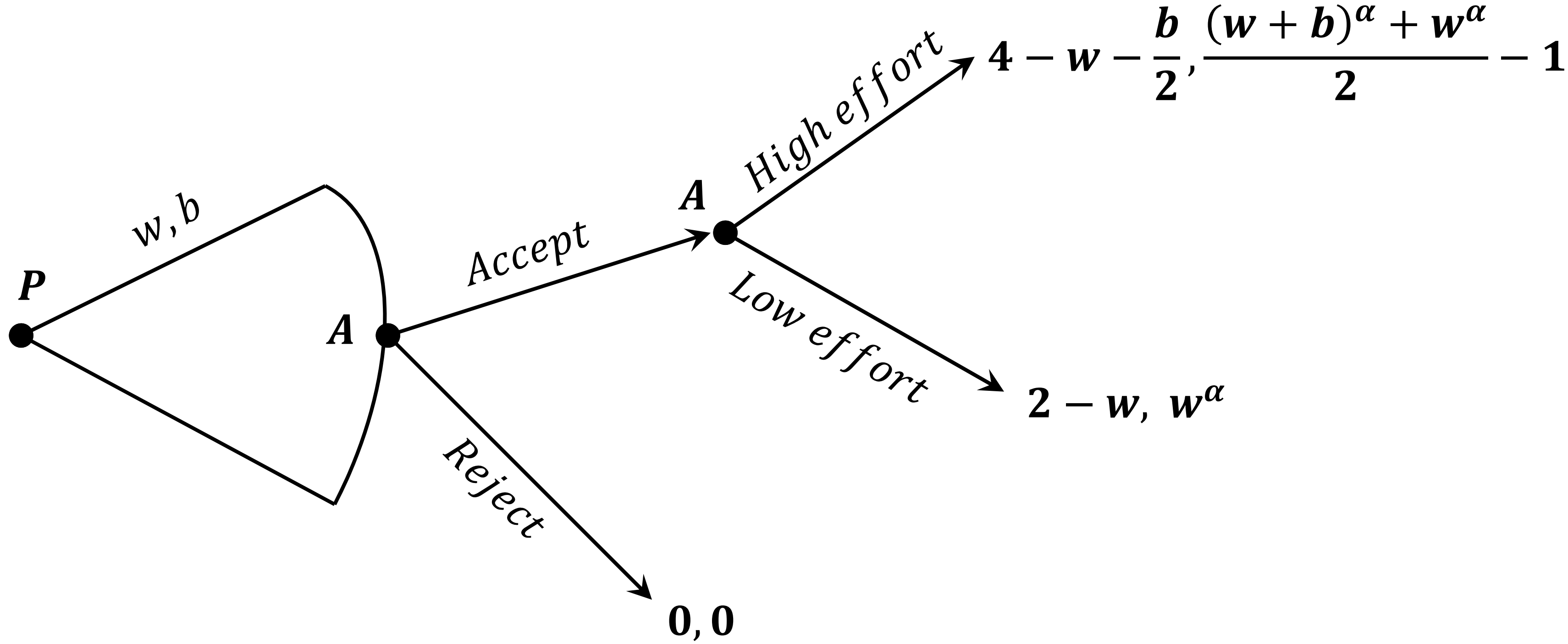
$$\frac{1}{2} \cdot (6 - w - b) + \frac{1}{2} \cdot (2 - w) = 4 - w - \frac{b}{2}$$

- **For the agent:**

$$\frac{1}{2} \cdot \left((w + b)^{\alpha} - 1\right) + \frac{1}{2} \cdot (w^{\alpha} - 1)$$

$$= \frac{(w + b)^{\alpha} + w^{\alpha}}{2} - 1$$

- We can input these continuation payoffs into the extensive form, which yields:



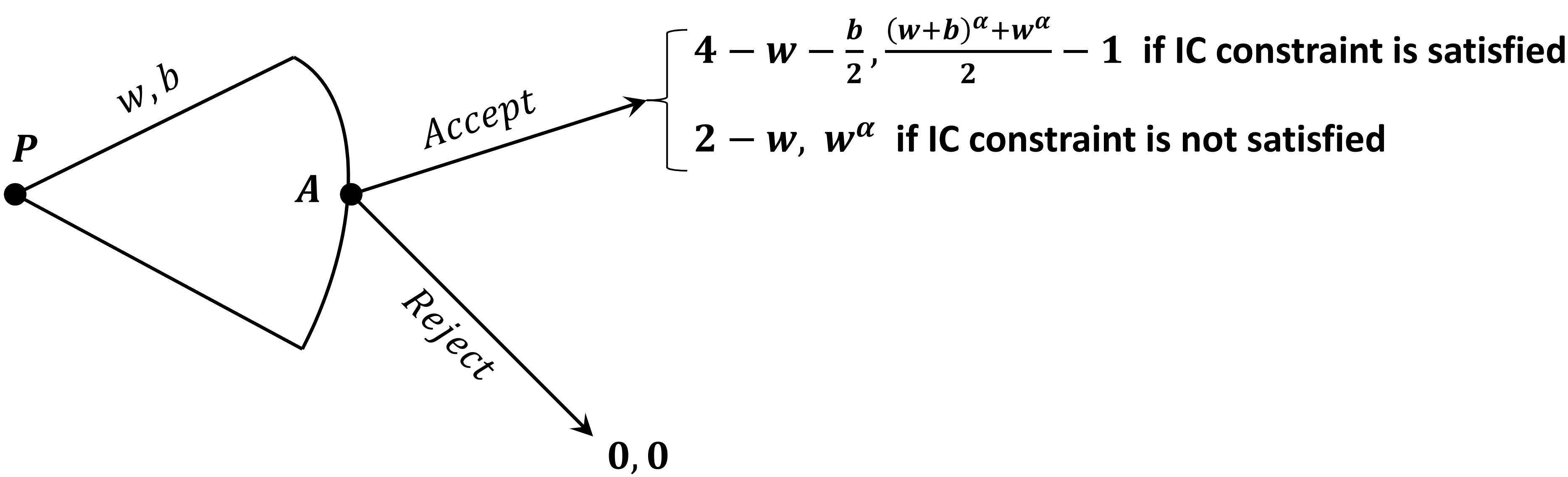
- From here we continue doing backward induction…

- …we proceed to analyze the decision of the agent of whether to exert high or low effort. Given the continuation payoffs, the **agent will exert high effort if**:

$$\frac{(w+b)^{\alpha}+w^{\alpha}}{2}-1 \geq w^{\alpha}$$

- This condition is called the **effort constraint** or **incentive compatibility constraint.** Is the condition that w and b must satisfy in order to induce high effort from the agent.
- From here we continue doing backward induction…

- Plugging in the continuation payoffs implied by the incentive compatibility constraint (**IC constraint**) we have:



- Next step in backward induction is to analyze the agent’s decision of whether to accept or not the job offer.

- The agent will accept the job offer as long as his expected payoff is at least equal to zero, which is what he would obtain if he rejects the offer.
- Therefore, if the IC constraint is satisfied, the agent will accept the job offer if:

$$\frac{(w+b)^{\alpha}+w^{\alpha}}{2}-1 \geq 0$$

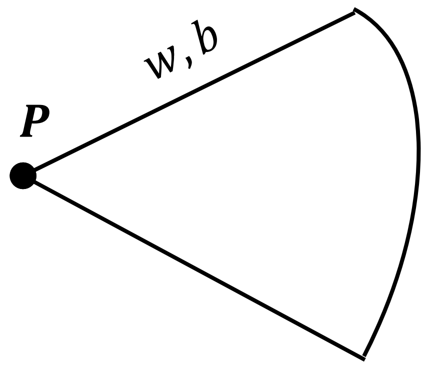
- This is called the **participation constraint.** It describes the condition under which an agent will accept a job offer that ultimately leads him to exert high effort.

- If the IC is violated, then the agent will accept the job offer if:

$$w^{\alpha} \geq 0$$

- Note that this condition is simply $w \geq 0$.
- Thus, if the IC is violated, the agent will still take the job as long as the salary is nonnegative, but the agent will exert low effort.
- We can take these continuation payoffs and go back to the initial node where the principal sets w and b…

- Continuation expected payoffs for the Principal at the initial node:



$$\begin{cases} \mathbf{4 - w - \frac{b}{2}} & \textbf{if } \boldsymbol{w, b} \textbf{ satisfy the IC and participation constraints} \\ \mathbf{2 - w} & \textbf{if } \boldsymbol{w, b} \textbf{ violate IC constraint but } \boldsymbol{w \geq 0} \\ \mathbf{0} & \textbf{if } \boldsymbol{w, b} \textbf{ violate IC constraint and } \boldsymbol{w < 0} \end{cases}$$

- What is the optimal strategy for the principal? That is, what are the optimal values of w and b for the principal given these expected continuation payoffs? There are only **two relevant cases**.

- **Case 1.- Optimal choice of w and b if both the IC and the participation constraints are satisfied:** Here the principal's problem is to **maximize** $\boldsymbol{4 - w - \frac{b}{2}}$ subject to the IC constraint and the participation constraint. That is, **subject to**:

$$\frac{(w+b)^{\alpha}+w^{\alpha}}{2} - 1 \geq w^{\alpha}$$

and

$$\frac{(w+b)^{\alpha}+w^{\alpha}}{2} - 1 \geq 0$$

- Note that the principal wants therefore to **choose the smallest values of w and b that satisfy these two inequalities**.
- It is not hard to see that the smallest values of w and b that satisfy these conditions are the ones that **satisfy these inequalities as equalities**. That is, the values of w and b that satisfy:

$$\frac{(w+b)^{\alpha}+w^{\alpha}}{2}-1=w^{\alpha}$$

and

$$\frac{(w+b)^{\alpha}+w^{\alpha}}{2}-1=0$$

- The left hand sides are the same on both equations. But the right hand side is w^{α} in the first one and 0 in the second one. Therefore we must have $w^{\alpha} = 0$. That is, $w = 0$. In this case, b must satisfy:

$$\frac{(0+b)^{\alpha} + 0^{\alpha}}{2} - 1 = 0$$

- This yields $b = 2^{1/\alpha}$.
- Therefore, the optimal choices for w and b if both IC and participation constraints are satisfied are:

$$w = 0 \quad \text{and} \quad b = 2^{1/\alpha}.$$

- This yields an expected payoff of zero to the agent and an expected payoff to the principal of:

$$\boldsymbol{4 - 0 - \frac{2^{\frac{1}{\alpha}}}{2} = 4 - 2^{\frac{1-\alpha}{\alpha}}}$$

- **Case 2.- Optimal choice of b and w when $\boldsymbol{w}, \boldsymbol{b}$ violate IC constraint but $\boldsymbol{w \geq 0}$**.- Here the principal's problem is to **maximize $\boldsymbol{2 - w}$** subject to the constraint that $\boldsymbol{w \geq 0}$.
- In this case we can simply set b=0 (or to any value of b that violates the IC constraint) and the optimal choice for w is simply $\boldsymbol{w = 0}$.
- Expected payoffs would be: zero to the agent, and $\boldsymbol{2 - 0 = 2}$ to the principal.
- Note that the principal will never want to choose $w < 0$ because that would guarantee him a payoff of zero.

- OK, so what is the optimal decision for the principal?
- In all cases, it is **optimal to choose $\boldsymbol{w = 0}$**.
- The only question is whether the principal wants to set the bonus b to $b = 2^{1/\alpha}$ and satisfy the IC constraint, or set $b = 0$ and violate it.
- The **optimal decision to the principal will depend on the value of α, which measures the degree of risk aversion of the agent.**
- If $b = 2^{1/\alpha}$ the expected payoff to the principal is

$$\boldsymbol{4 - 2^{\frac{1-\alpha}{\alpha}}}$$

- If b=0 the expected payoff to the principal is **2.**

- Therefore the principal will choose $b = 2^{1/\alpha}$ as long as:

$$\mathbf{4 - 2^{\frac{1-\alpha}{\alpha}} \geq 2}$$

- This can be shown to simplify to the condition:

$$\boldsymbol{\alpha \geq \frac{1}{2}}$$

- **<u>(See the next slide for the mathematical details)</u>**
- **Recall that values of α closer to zero denote more risk aversion. Therefore, if the agent is too risk averse, the principal will not find it profitable to design a contract that induces high effort.**
- **If the agent is too risk averse (if $\alpha < \frac{1}{2}$ in this specific example) , uncertainty about payoffs following high effort requires a bonus "b" that is too big for the principal to pay.**

- Mathematical details of why

$$4 - 2^{\frac{1-\alpha}{\alpha}} \geq 2$$

simplifies to the condition:

$$\alpha \geq \frac{1}{2}$$

- If we let $\boldsymbol{ln}(\boldsymbol{X})$ denote the **natural logarithm** of X (where $X > 0$), then the properties of this function imply that: $\ln(X^c) = c \cdot \ln(\mathrm{X})$.
- Rearranging the inequality $4 - 2^{\frac{1-\alpha}{\alpha}} \geq 2$ yields:

$$1 \geq 2^{\frac{1-2\alpha}{\alpha}}$$