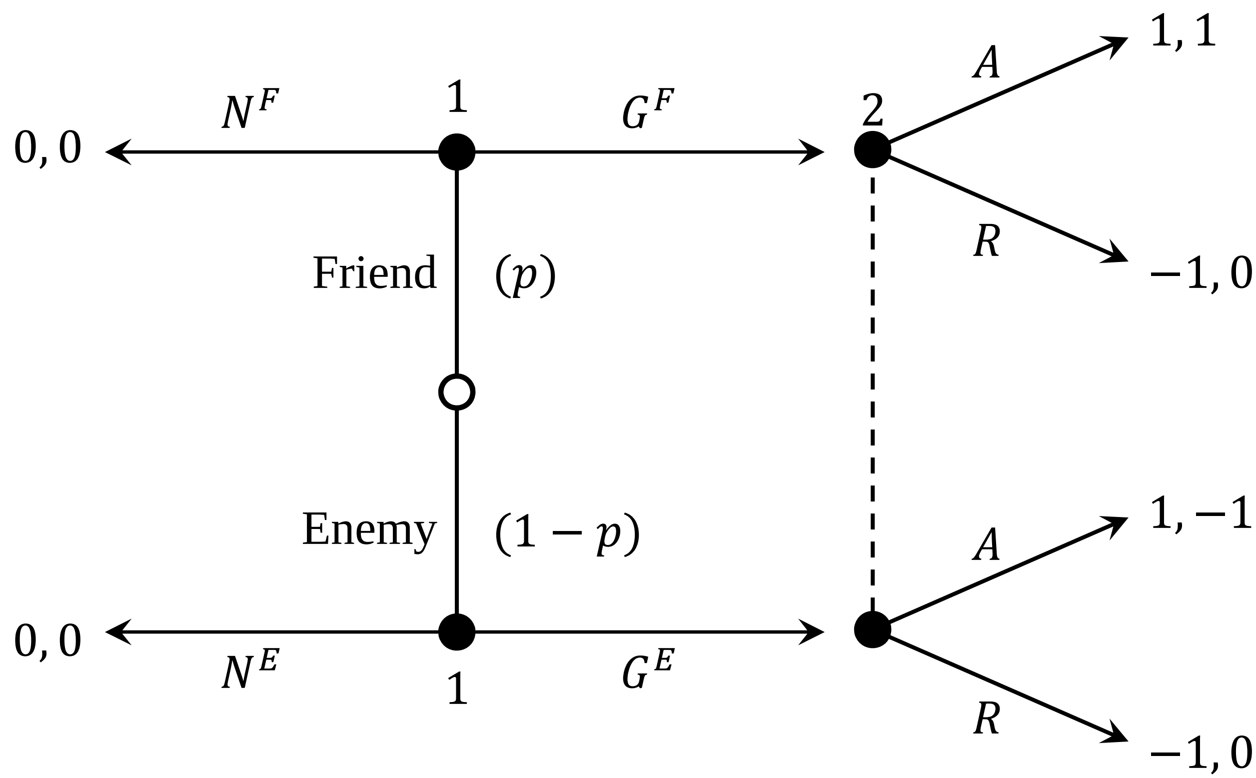
26.- Bayesian Nash Equilibrium and Rationalizability

- In games where nature moves, once we have computed the expected payoffs and constructed the normal form of the game, we can find the set of Nash equilibria (pure and mixed), and the set of rationalizable strategies exactly in the same way we learned before.
- We append the term "Bayesian" to these concepts and thus we obtain **Bayesian Nash equilibria** and **Bayesian rationalizable strategies**.

- ***Bayesian:*** For our purposes, refers to the act of assigning and applying probabilities to events.
- We can revisit the examples from two chapters ago to illustrate Bayesian Nash equilibria and Bayesian rationalizability.
- Once we have figured out the Bayesian normal form, there is nothing new to these notions. They are straightforward applications of the notions of Nash equilibrium and rationalizability in normal form games.

- **Example:** Let us revisit the gift game which we studied two chapters ago…



- We showed that the Bayesian normal form of this game was given by:

1 \ 2	A	R
$N^E N^F$	$0\,,0$	$0\,,0$
$N^E G^F$	p, p	$-p, 0$
$G^E N^F$	$1-p, p-1$	$p-1, 0$
$G^E G^F$	$1, 2p-1$	$-1, 0$

- Suppose $p = 2/3$ (the probability of being a "friend" is 2/3).

- Find the pure-strategy Bayesian Nash equilibria.

- If $p = 2/3$, the normal form becomes:

1 \ 2	A	R
$N^E N^F$	$0, 0$	$0, 0$
$N^E G^F$	$\frac{2}{3}, \frac{2}{3}$	$-\frac{2}{3}, 0$
$G^E N^F$	$\frac{1}{3}, -\frac{1}{3}$	$-\frac{1}{3}, 0$
$G^E G^F$	$1, \frac{1}{3}$	$-1, 0$

- The pure-strategy Bayesian Nash equilibria are:

$$\left(\boldsymbol{N^E N^F}, \boldsymbol{R}\right) \text{ and } \left(\boldsymbol{G^E G^F}, \boldsymbol{A}\right)$$

- In the first equilibrium, player 1 never gives a gift regardless of his type and player 2 rejects the gift. In the second equilibrium, player 1 always gives a gift and player 2 accepts it.

- Note that in these two equilibria, both types of player 1 do the same. In the first equilibrium, player 1 does not give a gift regardless of his type. In the second equilibrium, player 1 gives a gift regardless of his type.

- Equilibria where players always do the same regardless of their type are called **pooling equilibria**. Equilibria where different types of players do different things are called **separating equilibria**.

- Suppose instead that $p = 0$. The normal form becomes:

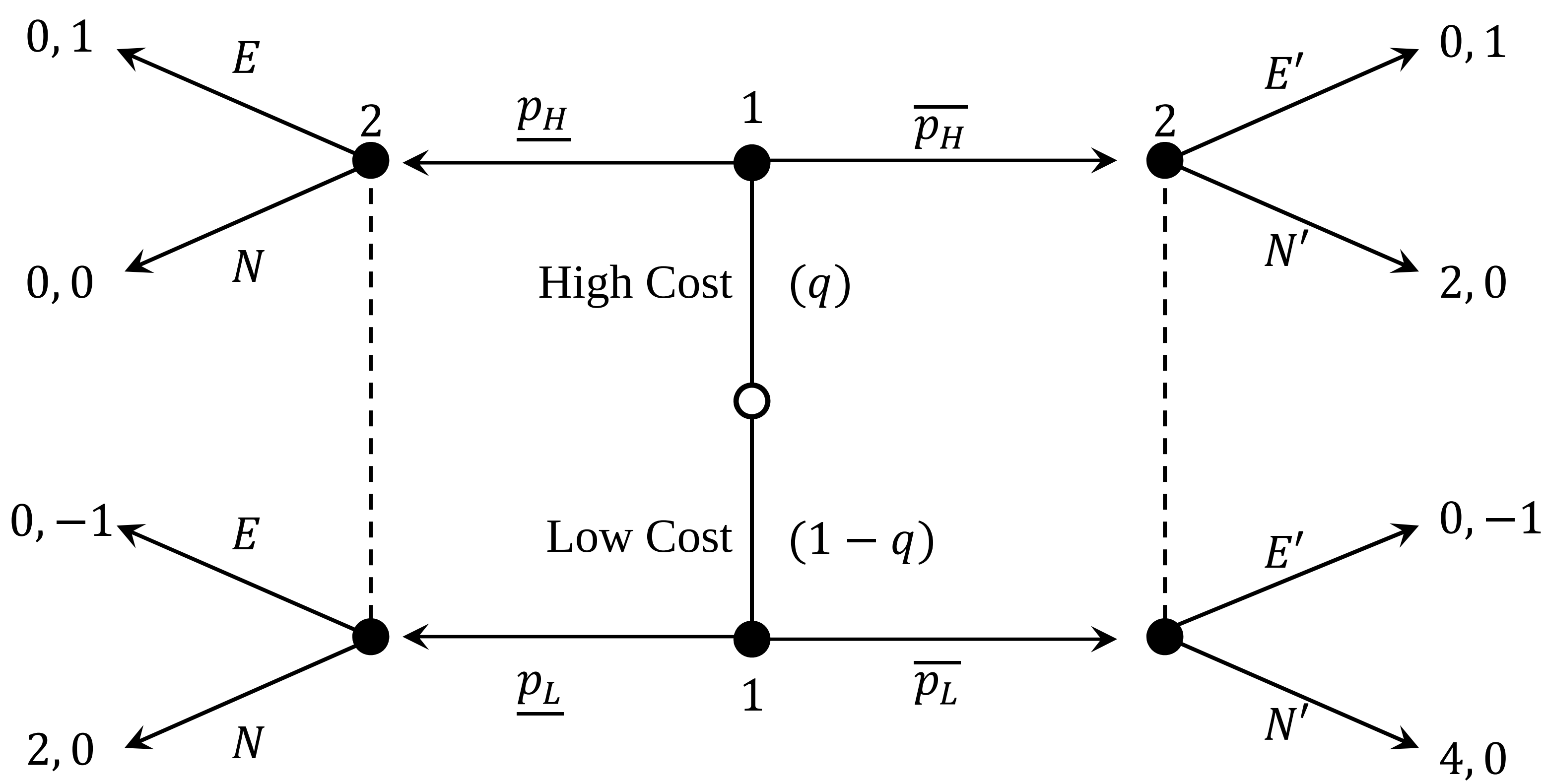
1 \ 2	A	R
$N^E N^F$	$0, 0$	$0, 0$
$N^E G^F$	$0, 0$	$0, 0$
$G^E N^F$	$1, -1$	$-1, 0$
$G^E G^F$	$1, -1$	$-1, 0$

- The game has two pure-strategy Bayesian Nash equilibria:

$$\left(N^E N^F, R\right) \text{ and } \left(N^E G^F, R\right)$$

- The first one is a pooling equilibrium and the second one is a separating equilibrium.

- Example: Let us revisit the example about the incumbent and the potential entrant, where the incumbent can be of one of two types: “High cost” or “Low cost” incumbent.



- We described the normal form of this game as:

1 \ 2	EE'	EN'	NE'	NN'
$\overline{p_H}\,\overline{p_L}$	$0, 2q-1$	$4-2q, 0$	$0, 2q-1$	$4-2q, 0$
$\overline{p_H}\underline{p_L}$	$0, 2q-1$	$2q, q-1$	$2-2q, q$	$2, 0$
$\underline{p_H}\overline{p_L}$	$0, 2q-1$	$4-4q, q$	$0, q-1$	$4-4q, 0$
$\underline{p_H}\underline{p_L}$	$0, 2q-1$	$0, 2q-1$	$2-2q, 0$	$2-2q, 0$

- Suppose $q = 1/4$ (only a 25% chance of being a high cost incumbent). Find the pure-strategy Bayesian Nash equilibria.

- If $q = 1/4$ the normal form becomes:

1 \ 2	EE'	EN'	NE'	NN'
$\overline{p_H}\,\overline{p_L}$	$0, -\frac{1}{2}$	$\frac{7}{2}, 0$	$0, -\frac{1}{2}$	$\frac{7}{2}, 0$
$\overline{p_H}\underline{p_L}$	$0, -\frac{1}{2}$	$\frac{1}{2}, -\frac{3}{4}$	$\frac{3}{2}, \frac{1}{4}$	$2, 0$
$\underline{p_H}\overline{p_L}$	$0, -\frac{1}{2}$	$3, \frac{1}{4}$	$0, -\frac{3}{4}$	$3, 0$
$\underline{p_H p_L}$	$0, -\frac{1}{2}$	$0, -\frac{1}{2}$	$\frac{3}{2}, 0$	$\frac{3}{2}, 0$

- This game has four pure-strategy Bayesian Nash equilibria. Three of them are pooling equilibria:

$$(\overline{p_H}\,\overline{p_L}, EN'), (\overline{p_H}\,\overline{p_L}, NN'), \left(\underline{p_H p_L}, NE'\right)$$

- And one of them is a separating equilibrium:

$$\left(\overline{p_H}\underline{p_L}, NE'\right)$$

- Find the set of Bayesian rationalizable strategies in this game.- We know how to do this, first we identify the best responses:

1 \ 2	EE'	EN'	NE'	NN'
$\overline{p_H}\,\overline{p_L}$	$0, -\frac{1}{2}$	$\frac{7}{2}, 0$	$0, -\frac{1}{2}$	$\frac{7}{2}, 0$
$\overline{p_H}\,\underline{p_L}$	$0, -\frac{1}{2}$	$\frac{1}{2}, -\frac{3}{4}$	$\frac{3}{2}, \frac{1}{4}$	$2, 0$
$\underline{p_H}\,\overline{p_L}$	$0, -\frac{1}{2}$	$3, \frac{1}{4}$	$0, -\frac{3}{4}$	$3, 0$
$\underline{p_H}\,\underline{p_L}$	$0, -\frac{1}{2}$	$0, -\frac{1}{2}$	$\frac{3}{2}, 0$	$\frac{3}{2}, 0$

- There is only one dominated strategy: $\boldsymbol{EE'}$ **is dominated by** $\boldsymbol{NN'}$ for player 2

- The **reduced game** $\boldsymbol{R_1}$ looks as follows (with best-responses indicated):

1 \ 2	EN'	NE'	NN'
$\overline{p_H}\,\overline{p_L}$	$\frac{7}{2}, 0$	$0, -\frac{1}{2}$	$\frac{7}{2}, 0$
$\overline{p_H}\underline{p_L}$	$\frac{1}{2}, -\frac{3}{4}$	$\frac{3}{2}, \frac{1}{4}$	$2, 0$
$\underline{p_H}\overline{p_L}$	$3, \frac{1}{4}$	$0, -\frac{3}{4}$	$3, 0$
$\underline{p_H}\underline{p_L}$	$0, -\frac{1}{2}$	$\frac{3}{2}, 0$	$\frac{3}{2}, 0$

- There is only one strategy that is not a best response: $\underline{p_H}\overline{p_L}$ for player 1. We need to check if it is dominated.

- By inspection we see that $\underline{\boldsymbol{p}_H}\overline{\boldsymbol{p}_L}$ is not dominated by any pure strategy. Therefore we need to check if a mixed-strategy dominates it…
- Consider a mixed-strategy where player 1 mixes between $\overline{\boldsymbol{p}_H}\,\overline{\boldsymbol{p}_L}$ and $\overline{\boldsymbol{p}_H}\underline{\boldsymbol{p}_L}$. It is easy to verify that any such mixed strategy will dominate $\underline{\boldsymbol{p}_H}\overline{\boldsymbol{p}_L}$ as long as:

$$\frac{5}{6} < \Pr(\overline{\boldsymbol{p}_H}\,\overline{\boldsymbol{p}_L}) < 1$$

- Therefore $\underline{\boldsymbol{p}_H}\overline{\boldsymbol{p}_L}$ IS a dominated strategy and we can eliminate it.

- The **reduced game $\boldsymbol{R_2}$** looks as follows (with best-responses indicated):

1 \ 2	$\boldsymbol{EN'}$	$\boldsymbol{NE'}$	$\boldsymbol{NN'}$
$\overline{\boldsymbol{p_H}}\,\overline{\boldsymbol{p_L}}$	$\frac{7}{2}, 0$	$0, -\frac{1}{2}$	$\frac{7}{2}, 0$
$\overline{\boldsymbol{p_H}}\underline{\boldsymbol{p_L}}$	$\frac{1}{2}, -\frac{3}{4}$	$\frac{3}{2}, \frac{1}{4}$	$2, 0$
$\underline{\boldsymbol{p_H}}\underline{\boldsymbol{p_L}}$	$0, -\frac{1}{2}$	$\frac{3}{2}, 0$	$\frac{3}{2}, 0$

- As we can see, every strategy in R_2 is a best response. Therefore we cannot eliminate any more strategies. **The set of (Bayesian) rationalizable strategies $\boldsymbol{R}$ is equal to $\boldsymbol{R_2}$.**

- Is there a mixed-strategy Bayesian-NE where player 1 randomizes between $\overline{\boldsymbol{p_H}}\,\overline{\boldsymbol{p_L}}$ and $\overline{\boldsymbol{p_H}}\underline{\boldsymbol{p_L}}$ and player 2 randomizes between $\boldsymbol{NE'}$ and $\boldsymbol{NN'}$?

- Let us denote:

$$\pi_1 = Prob(\overline{\boldsymbol{p_H}}\,\overline{\boldsymbol{p_L}})\,,\, 1-\pi_1 = Prob(\overline{\boldsymbol{p_H}}\underline{\boldsymbol{p_L}})$$

and

$$\pi_2 = Prob(\boldsymbol{NE'})\,,\, 1-\pi_2 = Prob(\boldsymbol{NN'})$$

- By the conditions for a mixed strategy NE, these probabilities must satisfy:

$$\underbrace{0 \cdot \pi_2 + \frac{7}{2} \cdot (1 - \pi_2)}_{\text{Expected payoff for 1 of } \overline{\boldsymbol{p}_H}\,\overline{\boldsymbol{p}_L}} = \underbrace{\frac{3}{2} \cdot \pi_2 + 2 \cdot (1 - \pi_2)}_{\text{Expected payoff for 1 of } \overline{\boldsymbol{p}_H}\underline{\boldsymbol{p}_L}}$$

$$\underbrace{-\frac{1}{2} \cdot \pi_1 + \frac{1}{4} \cdot (1 - \pi_1)}_{\text{Expected payoff for 2 of } \boldsymbol{NE'}} = \underbrace{0 \cdot \pi_1 + 0 \cdot (1 - \pi_1)}_{\text{Expected payoff for 2 of } \boldsymbol{NN'}}$$

- Solving these equations we obtain:

$$\pi_1 = \frac{1}{3} \quad \text{and} \quad \pi_2 = \frac{1}{2}$$

- Therefore there exists a mixed-strategy Bayesian-NE where player 1 randomizes between $\overline{\boldsymbol{p_H}}\,\overline{\boldsymbol{p_L}}$ and $\overline{\boldsymbol{p_H}}\underline{\boldsymbol{p_L}}$ and player 2 randomizes between $\boldsymbol{N\overline{E}'}$ and $\boldsymbol{NN'}$. In this equilibrium we must have:

$$Prob(\overline{p_H}\,\overline{p_L}) = \frac{1}{3}, Prob\left(\overline{p_H}\underline{p_L}\right) = \frac{2}{3}$$

and

$$Prob(NE') = \frac{1}{2}, Prob(NN') = \frac{1}{2}$$

- Note that this Bayesian-NE can be described in words in the following way:
- **Incumbent firm (player 1):**
  - Always sets a high price if his own type is “High Cost”.
  - If his type is “Low Cost”, set a high price with probability 1/3 and a low price with probability 2/3.
- **Entrant firm (player 2):**
  - Never enter if he observes the incumbent set a low price.
  - If he observes the incumbent set a high price, enter with probability ½ and stay out with probability ½.