

3.- Strategies and the Normal Form

- We begin by defining the concept of ***strategy***, which will also explain the name of our course.
- A **strategy** is a *complete contingent plan for a player in the game*.
- In words, a strategy describes the actions the player is supposed to take in **each possible decision point of the game**.
- That is, the actions the player is supposed to take in each one of his information sets.

- We refer to the collection of all strategies available to a player as his **strategy space**.
- Typically we denote the strategy space of player “i” as S_i (upper case “S”).
- In the previous example, we have:

$$S_1 = \{H, L\} \text{ and } S_2 = \{HH', HL', LH', LL'\}$$
- Typically, we denote elements of a strategy space (i.e, **strategies**) with **lower case** letters. Thus, $s_i \in S_i$ denotes strategy s_i , which belongs in the strategy space S_i of player “i”.

- A **strategy profile** is a vector (i.e, collection) of strategies by a subset of players in the game. Each element in the strategy profile denotes a particular strategy for each player.
- In a game with “n” players, a strategy profile for all players in the game is typically denoted as:

$$s = (s_1, s_2, \dots, s_n)$$

- Where (as before) s_i denotes the strategy of player i in the strategy profile s .
- The set of all strategy profiles in the game is typically denoted by S , and is given by the **Cartesian product** of the individual strategy spaces S_1, S_2, \dots, S_n . That is,

$$S = S_1 \times S_2 \times \dots \times S_n$$

- The Cartesian product $S_1 \times S_2 \times \cdots \times S_n$ is defined as:
 $\{(s_1, \dots, s_n) \text{ such that } s_1 \in S_1, s_2 \in S_2, \dots, s_n \in S_n\}$
- That is, the collection of all n-tuples (s_1, \dots, s_n) such that each element of the n-tuple belongs to each of the sets S_1, S_2, \dots, S_n .
- For example, suppose $n=2$, $S_1 = \{a, b\}$, $S_2 = \{u, d\}$.
Then, $S_1 \times S_2 = \{(a, u), (a, d), (b, u), (b, d)\}$.

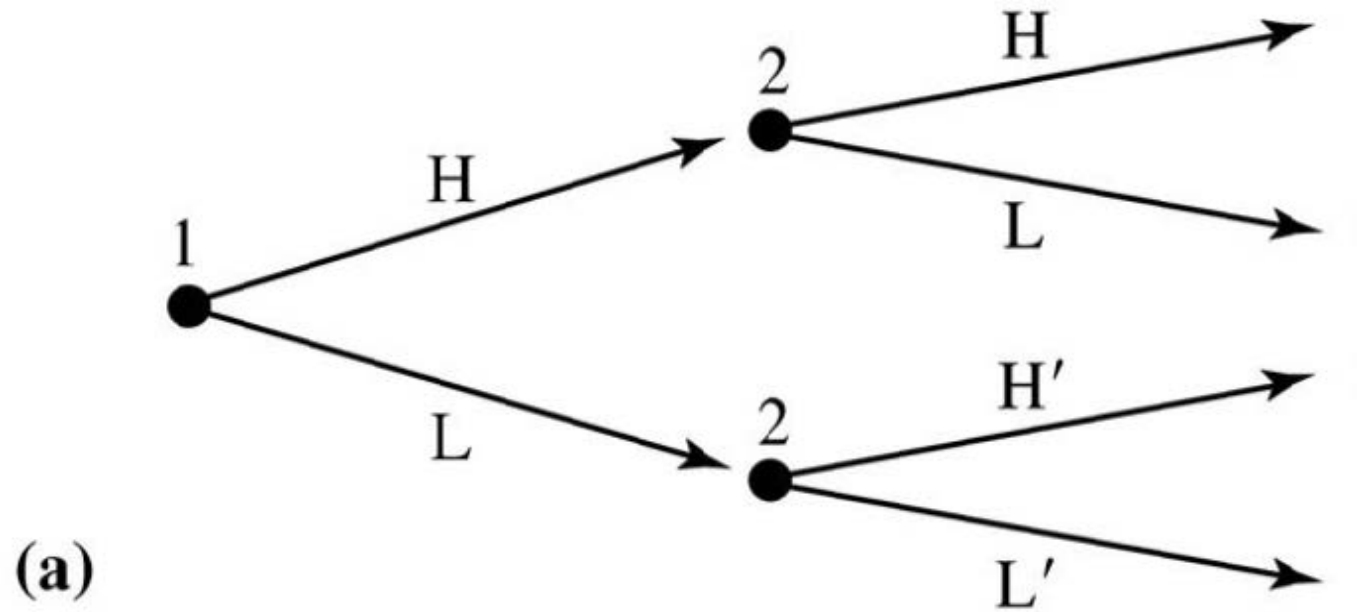
- We typically use the subscript “ $-i$ ” to refer to “every player except ‘ i ’”.
- In particular we will use s_{-i} to denote a strategy profile of **everyone except player i** . With:

$$s_{-i} = (s_1, s_2, \dots, s_{i-1}, s_{i+1}, \dots, s_n)$$

- For instance, consider the strategy profile $s = (H, L, P)$. Then,

$$s_{-1} = (L, P), \quad s_{-2} = (H, P), \quad s_{-3} = (H, L)$$

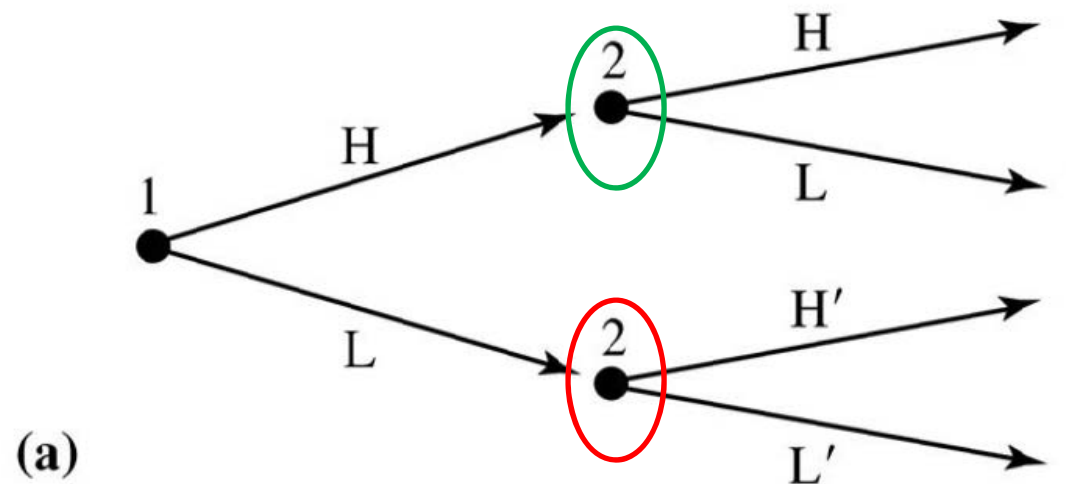
- For example, let us revisit the extensive form game:



- Let us examine each player:
- **Player 1**: Has only **one** decision point in the game (the initial node). Therefore, his strategy space is simply:

$$S_1 = \{H, L\}$$

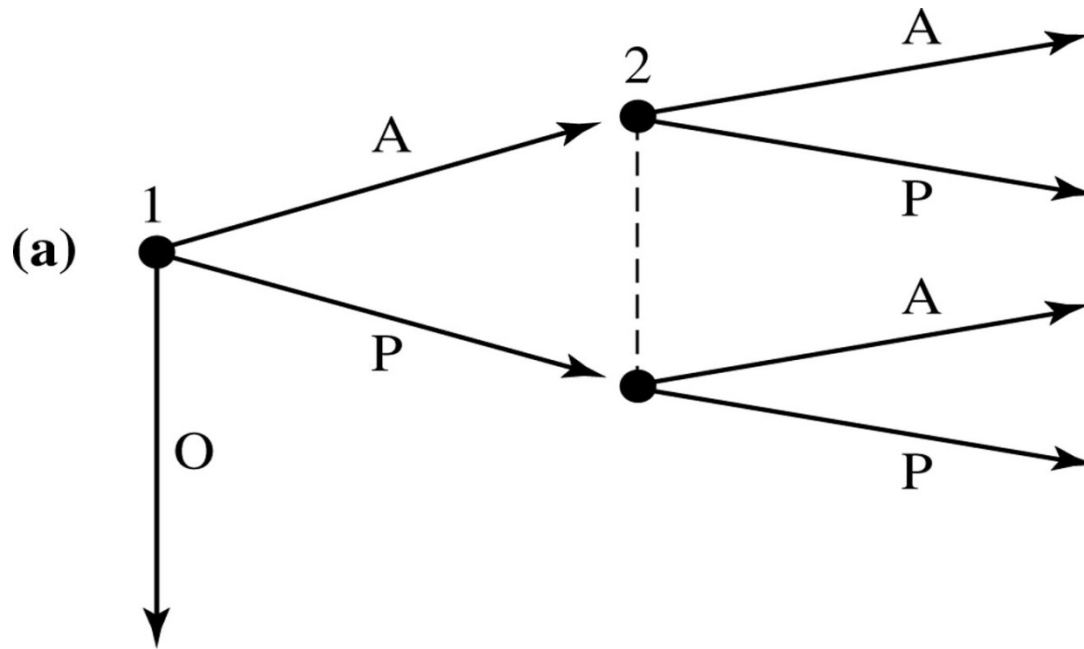
- **Player 2**: Has **two** decision points in the game, indicated below in green and red:



- A strategy for player 2 must tell him what to do in the green and in the red decision points.
- The actions available in the green node are
H and L
- The actions available in the red node are
H' and L'
- Therefore the strategy space for player 2 is:
$$S_2 = \{HH', HL', LH', LL'\}$$

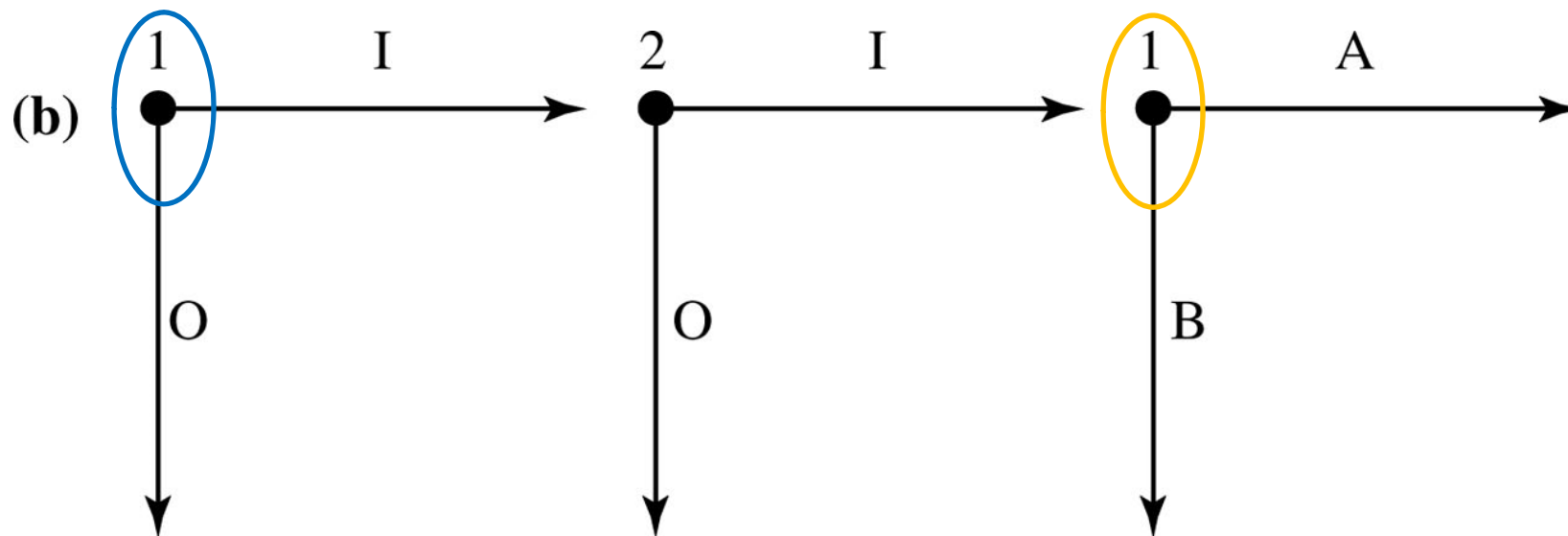
- HH' represents “Play H if player 1 chooses H, and play L' if player 1 chooses L”.
- HL' represents “Play H if player 1 chooses H, and play L' if player 1 chooses L”.
- LH' represents “Play L if player 1 chooses H, and play H' if player 1 chooses L”.
- LL' represents “Play L if player 1 chooses H, and play L' if player 1 chooses L”.

- In the following example, each player has only one decision point in the game:



$$S_1 = \{A, P, O\}, \quad S_2 = \{A, P\}$$

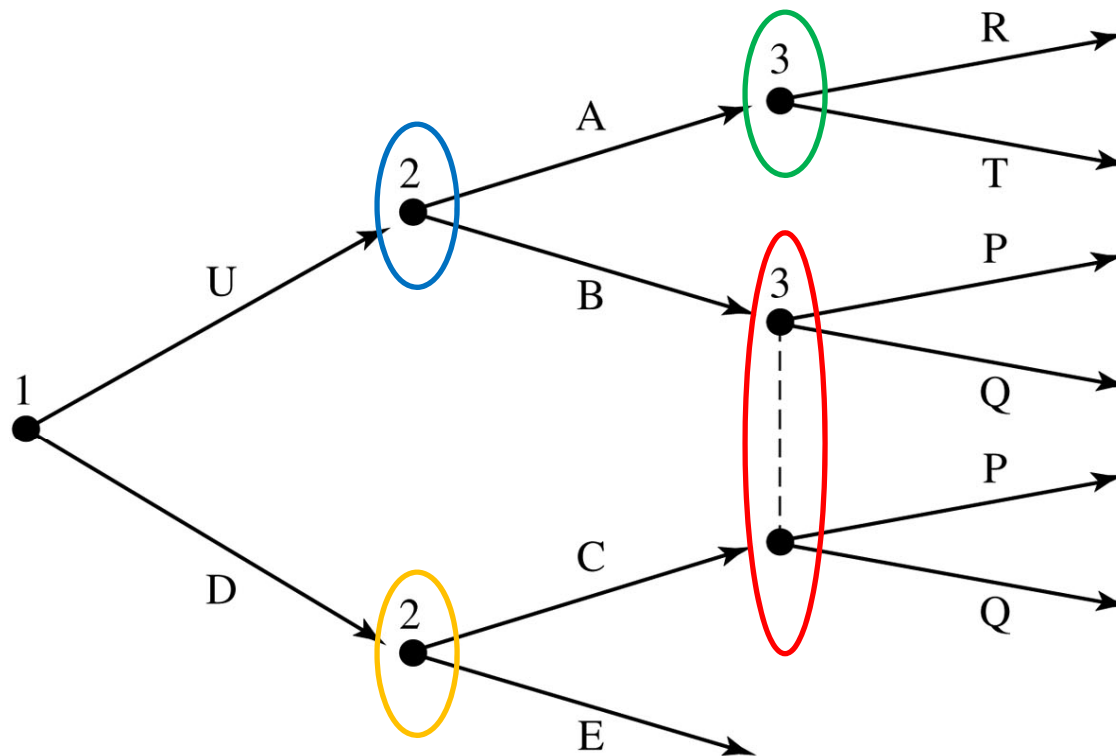
In this example, Player 1 has two decision points (indicated in blue and orange) and player 2 has only one.



We have: $S_1 = \{\text{OA}, \text{OB}, \text{IA}, \text{IB}\}$ and $S_2 = \{O, I\}$

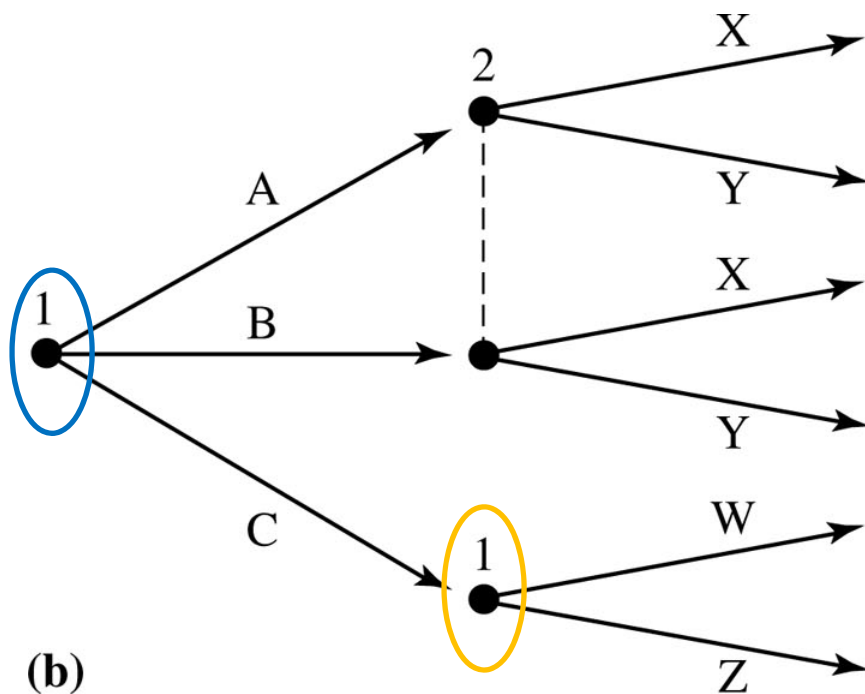
Notice that the definition of a strategy requires a description of what player 1 would do in his second decision node even in the event that he chooses “O” in the first stage, which would ensure that he never reaches the second node.

In this example, Player 1 has one decision point, Player 2 has **two** decision points (indicated in blue and orange) and player 3 has **two** (indicated in green and red).



(a)

We have: $S_1 = \{U, D\}$, $S_2 = \{AC, AE, BC, BE\}$ and
 $S_3 = \{RP, RQ, TP, TQ\}$



$$S_1 = \{AW, BW, CW, AZ, BZ, CZ\}$$

$$S_2 = \{X, Y\}$$

Again, note that the definition of strategy requires us to specify the action that each player would take at each and every information set in the game, including those that cannot be reached. For example, if Player 1 chooses “A” or “B”, he will never reach his second information set. However, every strategy (even those where he plays “A” or “B” in the first node) must describe what he would do in the second information set.

- **Payoff function:** An essential component of a normal form representation of a game, the **payoff function assigns a numerical value to each possible strategy profile (outcome) of the game.**
- Each player has a payoff function. We let $u_i: S \rightarrow \mathbf{R}$ denote player i 's payoff function.
- For each strategy profile $s \in S$, player i 's payoff function $u_i(s)$ indicates the numerical payoff for player i .
- **Recall** that a strategy profile is a vector of the type

$$s = (s_1, s_2, \dots, s_n)$$

- **Normal (or strategic) form representation:** A game in normal form consists of the following:
 1. A set of players, $\{1, 2, \dots, n\}$.
 2. A description of strategy spaces for all players, S_1, S_2, \dots, S_n
 3. A specification of payoff functions for all players, u_1, u_2, \dots, u_n
- Note that the strategies available to each player already summarize the information available to them throughout the game since they are a description of actions to take in each of their information sets.

- In games involving **two players**, and **finite strategy spaces**, normal form representations can be summarized through **matrices**. We call these **matrix-form games**.
- **Classic examples of two-player, two-action matrix-form games** include:
 - Matching Pennies.
 - Prisoners' Dilemma.
 - Battle of the Sexes.
 - Hawk-Dove/Chicken.
 - Coordination.
 - Pareto Coordination.
 - Pigs.
- Each of these classic examples is used to highlight particular properties of games.

- **Matching Pennies:** Two players, each with a penny. They independently choose a side of the coin and then simultaneously show the other the chosen side. If the sides match, then Player 1 gains Player 2's penny. If the sides differ, Player 2 wins Player 1's penny.

		2	
		H	T
1	H	1, -1	-1, 1
	T	-1, 1	1, -1

Matching Pennies

- This is an example of a **zero-sum game** (where the **payoffs of all players offset each other** in each possible outcome of the game). It is also a classic example of a game where equilibrium exists only in *mixed strategies* (we will study this later in the course).

- **Prisoners' Dilemma:** Two criminals have been caught. DA has evidence to convict each of a minor crime, but knows that they committed a major crime together but cannot prove it. DA offers each a deal:
 - If both confess to the major crime, they get a reduced sentence.
 - If only one confesses, that person gets an even bigger reduction in his sentence.
- Thus, each player has two strategies:
 - To squeal on the other guy or “defect” (D) from his fellow criminal.
 - Not to squeal on the other guy or “cooperate” (C) with his fellow criminal.

- Matrix game representation for prisoners' dilemma:

		2	
		C	D
1	C	2, 2	0, 3
	D	3, 0	1, 1

Prisoners' Dilemma

- The outcome “Neither defects” yields a higher payoff (2,2) to each player than the outcome “Both defect” (1,1). However, it turns out that “Both defect” is the only equilibrium of this game.
- Prisoners' dilemma illustrates situations with a **fundamental tension between conflict and cooperation**. That is, the **tension between individual and group interests**.

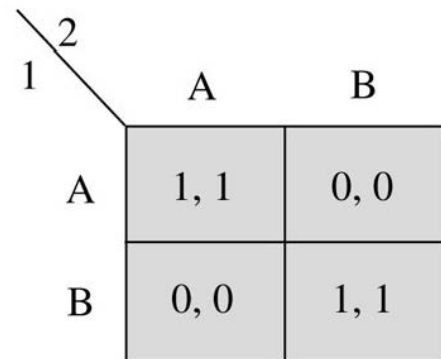
- **Battle of the Sexes:** A couple must decide to go to the Opera or to the Movies. They both want to be together, but P1 prefers the opera and P2 prefers the movie.

		2	
		Opera	Movie
1	Opera	2, 1	0, 0
	Movie	0, 0	1, 2

Battle of the Sexes

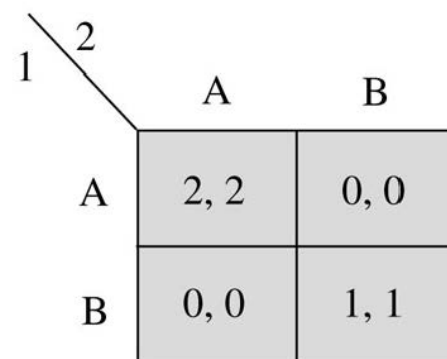
- This game has two equilibria: Both go to the Opera or both go to the Movies. It is a classical example of a **coordination game**.
- **Coordination games** are characterized by having **multiple equilibria** in which players choose (or coordinate) to the **same corresponding strategies**.

- More examples of **coordination games**:



		2	
		A	B
1	A	1, 1	0, 0
	B	0, 0	1, 1

Coordination



		2	
		A	B
1	A	2, 2	0, 0
	B	0, 0	1, 1

Pareto Coordination

- The second panel shows that some equilibria can be more “efficient” than others. Outcome (A,A) is more efficient than (B,B) because it leads to a higher aggregate payoff (4 vs. 2). We also say that (A,A) “Pareto-dominates” (B,B).

- **Hawk-Dove/Chicken Game:** This game represents generically a situation of “conflict” between two players. It has been used to model, e.g, situations of geopolitical conflict.
- Each player has two actions: yield to the other and be a “dove” (D), or not yield to the other and be a “hawk” (H). The key to the game is that:
 1. Neither player wants to be the only one to yield.
 2. However, the worst possible outcome is one where neither player yields (leading to a situation of “mutual destruction”).

- Matrix-form for Hawk-Dove/Chicken Game:

<div style="display: flex; align-items: center; justify-content: center;"> <div style="text-align: right; margin-right: 10px;">1</div> <div style="text-align: left; margin-left: 10px;">2</div> </div>		H	D
		H	D
H		0, 0	3, 1
D		1, 3	2, 2

Hawk-Dove/Chicken

- This game has two equilibria: (H,D) and (D,H). In each one of these equilibria, players must choose opposite strategies. In this sense it is the “opposite” of a coordination game.

- **Rational Pigs Game:** A dominant and a subordinate pig share a pen. Each has to decide whether to push the lever for food.
 - If neither presses the lever, neither of them gets any food.
 - If only the dominant pig presses the lever, the subordinate one eats most of the food before the dominant pushes it away.
 - If only the subordinate pig presses the lever, the dominant pig eats all of the food.
 - If both press the lever, the subordinate pig can eat a small portion of the food before the dominant pushes it away.

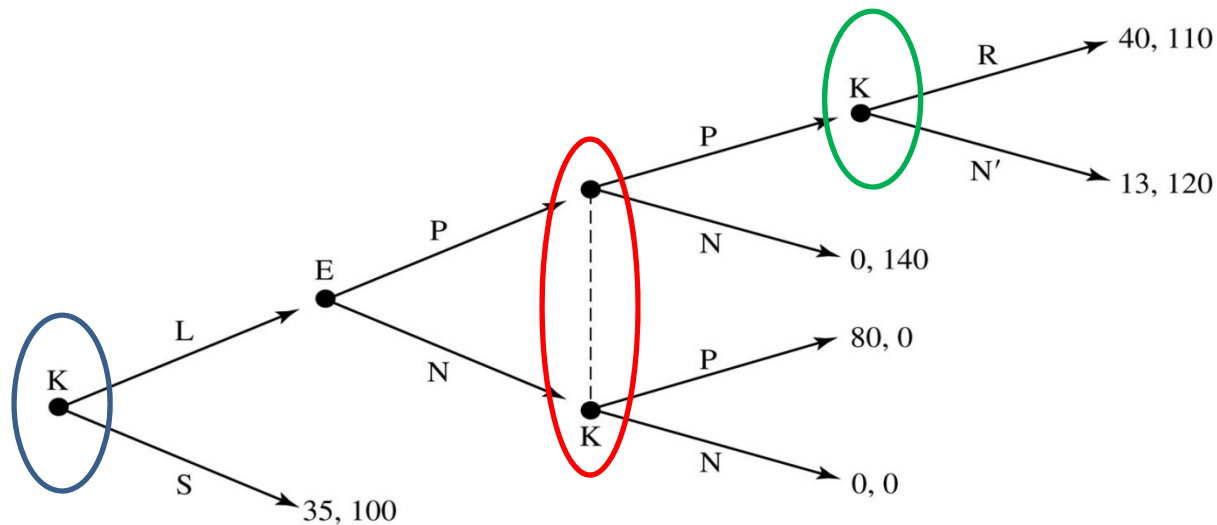
- Matrix form of Pigs game:

		S	
		P	D
D	P	4, 2	2, 3
	D	6, -1	0, 0

Pigs

- The subordinate pig has a **dominant strategy** in this game (i.e, a strategy that dominates all others regardless of what the opponent does). This is “Don’t push the lever”. As a result, the only equilibrium of this game is: “Dominant pig is the only one who pushes the lever”.
- The equilibrium outcome in this game yields the second-lowest possible payoff (2) to the dominant pig. This classic example is meant to illustrate that **sometimes “strength” can be a strategic weakness.**

- **Example (cont): Katzenberg-Eisner game.-** Let us revisit the extensive-form of this game,



- Going to the normal-form representation requires us to carefully characterize the strategy spaces for both players.

- The strategy spaces for both players are:

$$S_E = \{P, N\}$$

$$S_K = \{LPR, LPN', LNR, LNN', SPR, SPN', SNR, SNN'\}$$

- The next step is to write down the payoffs correctly for each strategy profile s in the strategy space

$$S = S_E \times S_K$$

- For instance, note that

$$u_K(LPR, P) = 40, u_E(LPR, P) = 110$$

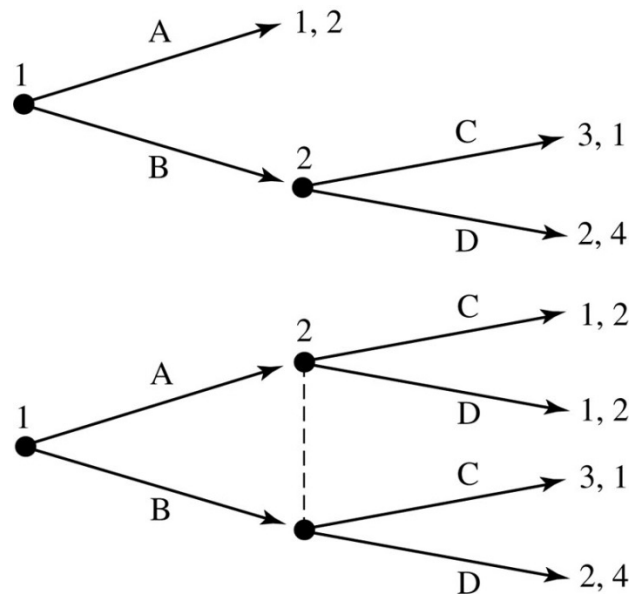
$$u_K(LPR, N) = 80, u_E(LPR, N) = 0$$

$$u_K(LNR, P) = 0, u_E(LNR, N) = 140$$

- The full normal-form matrix is:

E K \		P	N
LPR		40, 110	80, 0
LPN'		13, 120	80, 0
LNR		0, 140	0, 0
LNN'		0, 140	0, 0
SPR		35, 100	35, 100
SPN'		35, 100	35, 100
SNR		35, 100	35, 100
SNN'		35, 100	35, 100

- **Going back and forth from the extensive to the normal forms:** Once a game is written in extensive form, writing down the corresponding normal-form representation is straightforward.
- However, a normal-form game can have multiple extensive-form representations:



		2	
1		C	D
A	B	1, 2	1, 2
		3, 1	2, 4

Remark: Infinite Strategy Spaces and the Normal Form

- Many examples of economic interest involve **continuous decision variables** (e.g, price, quantity produced, etc.).
- In this case the strategy space would be infinite, making an **extensive-form** graphical representation **unfeasible**.
- However, **normal-form** representations are perfectly **suitable** for infinite strategy spaces.

- **Example: A Cournot duopoly model.-**

Consider an industry with a duopoly, where firms 1 and 2 simultaneously and independently select quantities to produce in a market.

- The quantity selected by firm $i = 1, 2$ is denoted by q_i . The **market price** (inverse demand) is given by:

$$p = 100 - 2 \cdot q_1 - 2 \cdot q_2$$

- Suppose that each firm produces at a **cost** of 20 per unit.
- The **payoffs** for firm $i = 1, 2$ are given by its **profits**.

- Profits are given by:

$$\begin{aligned} \text{Profits} &= \text{Total Revenue} - \text{Total Costs} \\ &= \text{Price} \times \text{Quantity Produced} - \text{Total Costs} \end{aligned}$$

- The market price depends on the quantity produced by both firms. For a given profile of production (q_1, q_2) , the market price is

$$p = 100 - 2 \cdot q_1 - 2 \cdot q_2$$

- Therefore, the total revenue for firm $i = 1, 2$ is given by:

$$\underbrace{(100 - 2 \cdot q_1 - 2 \cdot q_2)}_{\text{Price}} \times \underbrace{q_i}_{\text{Quantity produced by firm } i}$$

- Total costs for firm $i = 1, 2$ are given by:

$$20 \cdot q_i$$

- Take a given profile $q = (q_1, q_2)$. The profits for players 1 and 2 are then given by:

$$\begin{aligned} u_1(q_1, q_2) &= (100 - 2 \cdot q_1 - 2 \cdot q_2) \cdot q_1 - 20 \cdot q_1 \\ &= (80 - 2 \cdot q_1 - 2 \cdot q_2) \cdot q_1 \end{aligned}$$

- And

$$\begin{aligned} u_2(q_1, q_2) &= (100 - 2 \cdot q_1 - 2 \cdot q_2) \cdot q_2 - 20 \cdot q_2 \\ &= (80 - 2 \cdot q_1 - 2 \cdot q_2) \cdot q_2 \end{aligned}$$

- Therefore, the Normal-form representation of this game (whose decision variable is continuous) is given by:

- Players: $i = 1, 2$.

- Strategy spaces:

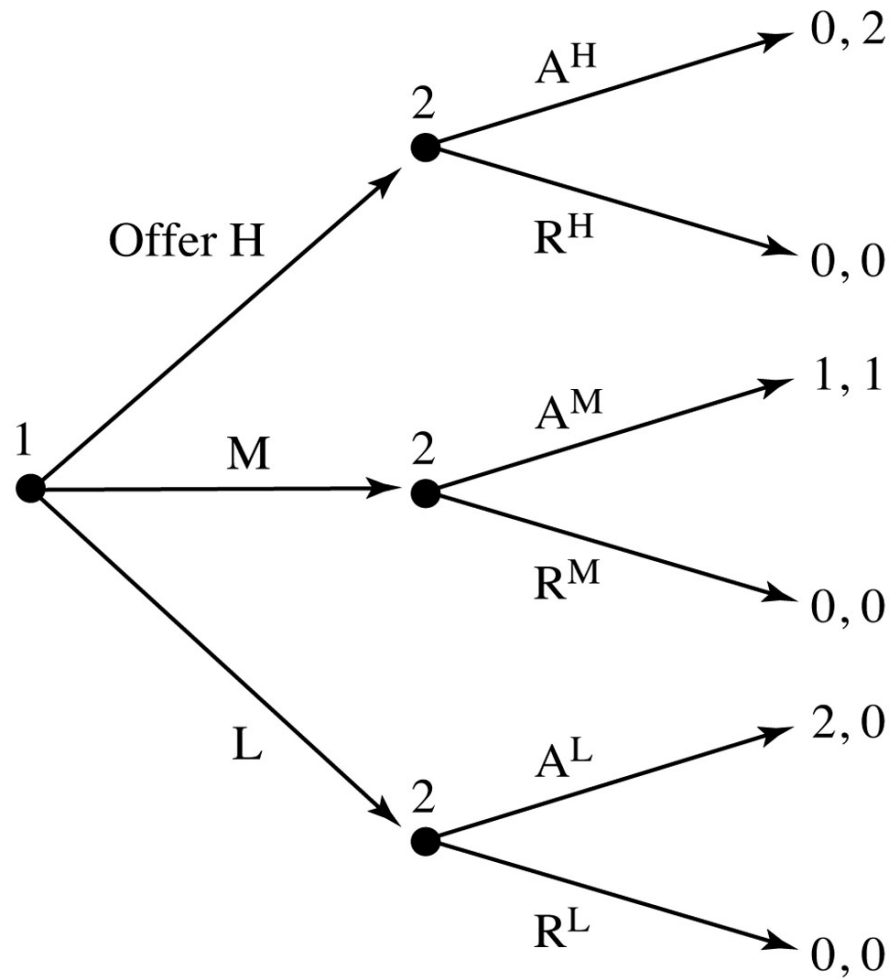
$$S_i = [0, \infty), \text{ for } i = 1, 2$$

- Payoff functions:

$$u_i(q_i, q_{-i}) = (80 - 2 \cdot q_i - 2 \cdot q_{-i}) \cdot q_i$$

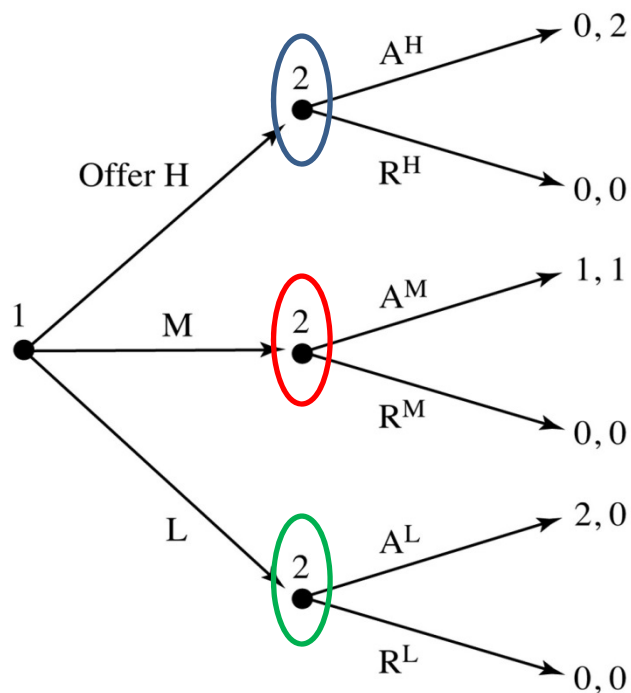
- Example: “Ultimatum-offer” bargaining game.-
Two players negotiate how to divide two dollars:
 - Player 1 chooses whether to offer all of the money (action H), half of the money (action M) or none of the money (action L) to player 2.
 - Player 2 observes the offer made by Player 1 and decides whether to accept or reject the offer.
 - If the offer is rejected, both players get a payoff of zero.

- Extensive form:



- Write this game in Normal form.

- The first step is to characterize the strategy spaces of each player. Player 1 moves only at one point in the game, Player 2 moves at three different points in the game:



$$S_1 = \{H, M, L\}$$

and

$$S_2 = \{A^H A^M A^L, A^H A^M R^L, R^H A^M A^L, R^H A^M R^L, A^H R^M A^L, A^H R^M R^L, R^H R^M A^L, R^H R^M R^L\}$$

- The normal form of this game can be expressed as:

Player 1 Player 2		H	M	L
$A^H A^M A^L$		2, 0	1, 1	0, 2
$A^H A^M R^L$		2, 0	1, 1	0, 0
$R^H A^M A^L$		0, 0	1, 1	0, 2
$R^H A^M R^L$		0, 0	1, 1	0, 0
$A^H R^M A^L$		2, 0	0, 0	0, 2
$A^H R^M R^L$		2, 0	0, 0	0, 0
$R^H R^M A^L$		0, 0	0, 0	0, 2
$R^H R^M R^L$		0, 0	0, 0	0, 0

- Note:** that **Player 1** is represented on the **columns** of the matrix and **Player 2** is represented on the **rows**.