4: Beliefs, Mixed Strategies and Expected Payoffs

- In a strategic-interaction setting players must try to **anticipate** the actions that others in the game will choose.
- We model this with the assumption that players construct **beliefs** about the strategies of others.
- A formal mathematical representation of decision making requires a **well-defined notion of beliefs**.
- Beliefs for player i are summarized by a **probability distribution over S_{-i}**.
- That is, when constructing his beliefs, player i assigns a probability to each of the profiles $s_{-i} \in S_{-i}$

- The textbook represents **beliefs for player $\boldsymbol{i}$** using the Greek letter "theta", as $\boldsymbol{\theta}_{-\boldsymbol{i}}$.

- $\boldsymbol{\theta}_{-\boldsymbol{i}}$ is an element of $\Delta \boldsymbol{S}_{-\boldsymbol{i}}$, the space of all probability distributions defined over S_{-i}.

- If S_{-i} is finite (more rigorously, if it is *countable*), then for each profile $s_{-i} \in S_{-i}$, the corresponding belief $\boldsymbol{\theta}_{-\boldsymbol{i}}(\boldsymbol{s}_{-\boldsymbol{i}})$ **represents player $\boldsymbol{i}'\boldsymbol{s}$ assessment of the probability that all other players in the game will select the profile $\boldsymbol{s}_{-\boldsymbol{i}}$.**

- If S_{-i} is finite, then since θ_{-i} is a well-defined probability distribution, it must satisfy:

$$\theta_{-i}(s_{-i}) \geq 0 \text{ for each } s_{-i} \in S_{-i},$$

and

$$\sum_{s_{-i} \in S_{-i}} \theta_{-i}(s_{-i}) = 1$$

- **Mixed strategy:** A mixed strategy for player i is given by a probability distribution over the space of **his own actions**, S_i.
- More formally, a mixed strategy is the act of choosing actions over S_i **according to a probability distribution.**
- The book uses the Greek letter sigma to denote mixed strategies, as $\boldsymbol{\sigma_i}$**.**
- Since mixed strategies are well-defined probability distributions, they belong to $\Delta \boldsymbol{S_i}$, the space of probability distributions over S_i.

- Strategies where a player does not randomize and instead chooses a given action with probability one are called **pure strategies** (they are just a special case of mixed strategies, with a degenerate probability distribution).
- **Expected Payoff:** Suppose player i is uncertain about the actions the other players will choose but has formed beliefs θ_{-i}. The expected payoff to player i of choosing a particular action s_i is computed as **the expected value of u_i if i chooses s_i and the rest of the players played according to the distribution θ_{-i}.**

- This expected payoff is therefore given by:

$$u_i(s_i, \theta_{-i}) = \sum_{s_{-i} \in S_{-i}} \theta_{-i}(s_{-i}) \cdot u_i(s_i, s_{-i})$$

- **Example:** Consider the following matrix-form game:

1 \ 2	L	M	R
U	8, 1	0, 2	4, 0
C	3, 3	1, 2	0, 0
D	5, 0	2, 3	8, 1

- Next consider the following belief for player 1, labeled as θ_2:

$$\theta_2(L) = 1/2, \theta_2(M) = 1/4, \theta_2(R) = 1/4$$

- According to these beliefs, player 1 thinks that player 2 will choose "L" with probability ½, "M" with probability ¼ and "R" with probability ¼.
- Given these beliefs, we can compute the expected payoff for player 1 of choosing each one of his actions: U, C and D.

- We have:

$$u_1(U, \theta_2) = u_1(U, L) \cdot \theta_2(L) + u_1(U, M) \cdot \theta_2(M) + u_1(U, R) \cdot \theta_2(R) = 8 \cdot \frac{1}{2} + 0 \cdot \frac{1}{4} + 4 \cdot \frac{1}{4} = 5$$

- Similarly,

$$u_1(C, \theta_2) = 3 \cdot \frac{1}{2} + 1 \cdot \frac{1}{4} + 0 \cdot \frac{1}{4} = \frac{7}{4} = 1.75$$

$$u_1(D, \theta_2) = 5 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} + 8 \cdot \frac{1}{4} = \frac{20}{4} = 5$$

- According to these beliefs, choosing either “U” or “D” yields a higher expected payoff than choosing “C”. Player 1 would be indifferent between “U” and “D” (in an expected-payoff sense).

- **Computing the expected payoff for a mixed strategy**: Let us generalize the construction of expected payoffs. Suppose player i has **beliefs** $\boldsymbol{\theta_{-i}}$ and wants to compute the expected payoff of a **mixed strategy** $\boldsymbol{\sigma_i}$**.** This is given by:

$$\boldsymbol{u_i(\sigma_i, \theta_{-i}) = \sum_{s_i \epsilon S_i} u_i(s_i, \theta_{-i}) \cdot \sigma_i(s_i)}$$

- **Example (cont):** Consider the same set of beliefs θ_2 as in the previous example. We had figured out that:

$$u_1(U,\theta_2)=5,\ u_1(C,\theta_2) = 1.75,\ u_1(D,\theta_2) = 5$$

- Consider now a **mixed strategy $\boldsymbol{\sigma_1}$** for player 1 given by:

$$\sigma_1(U) = \frac{1}{2}, \sigma_1(C) = \frac{1}{8}, \sigma_1(D) = \frac{3}{8}$$

- The expected payoff for this mixed strategy is with these beliefs is:

$$u_1(\sigma_1, \theta_2) = u_1(U,\theta_2) \cdot \sigma_1(U) + u_1(C,\theta_2) \cdot \sigma_1(C)$$
$$+u_1(D,\theta_2) \cdot \sigma_1(D) = 5 \cdot \frac{1}{2} + 1.75 \cdot \frac{1}{8} + 5 \cdot \frac{3}{8} = 4.59$$

- Similarly, we will let

$$\boldsymbol{u_i(\sigma_i, \sigma_{-i})}$$

Denote the expected payoff of player i of choosing the mixed strategy σ_i if all other players are using the mixed strategy σ_{-i}.

- We compute $\boldsymbol{u_i(\sigma_i, \sigma_{-i})}$ analogously to $\boldsymbol{u_i(\sigma_i, \theta_{-i})}$, simply replacing $\boldsymbol{\theta_{-i}}$ with $\boldsymbol{\sigma_{-i}}$.

- Also, we will abbreviate mixed strategies simply by listing the numerical probabilities used in each one of the strategies.

- For example, consider the following game:

1 \ 2	I	O
OA	2, 2	2, 2
OB	2, 2	2, 2
IA	4, 2	1, 3
IB	3, 4	1, 3

- Then the mixed strategy $\sigma_1 = (\frac{1}{8}, \frac{1}{4}, \frac{1}{4}, \frac{3}{8})$ refers to the mixing distribution:
$\sigma_1(OA) = \frac{1}{8}, \sigma_1(OB) = \frac{1}{4}, \sigma_1(IA) = \frac{1}{4}, \sigma_1(IB) = \frac{3}{8}$

- Compute the following expected payoffs for the previous game:

a) $u_1(\sigma_1, I)$ for $\sigma_1 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{10}, \frac{7}{30})$.

b) $u_2(\sigma_1, O)$ for $\sigma_1 = (\frac{1}{8}, \frac{1}{4}, \frac{1}{4}, \frac{3}{8})$.

c) $u_1(\sigma_1, \sigma_2)$ for $\sigma_1 = (\frac{1}{16}, \frac{1}{4}, \frac{1}{2}, \frac{3}{16})$ and $\sigma_2 = (\frac{1}{3}, \frac{2}{3})$

d) $u_2(\sigma_1, \sigma_2)$ for $\sigma_1 = (\frac{1}{16}, \frac{1}{4}, \frac{1}{2}, \frac{3}{16})$ and $\sigma_2 = (\frac{1}{3}, \frac{2}{3})$

a) $\boldsymbol{u_1(\sigma_1, I)}$ **for** $\boldsymbol{\sigma_1 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{10}, \frac{7}{30})}$**.-** This is player 1's expected payoff if player 2 chooses strategy "I" with probability one, and player 1 uses a mixed strategy where:

$$\sigma_1(OA) = \frac{1}{3}, \sigma_1(OB) = \frac{1}{3}, \sigma_1(IA) = \frac{1}{10},$$

$$\sigma_1(IB) = \frac{7}{30}$$

- Using the definition of expected payoff, we have:

$$u_1(\sigma_1, I)$$
$$= u_1(OA, I) \cdot \sigma_1(OA) + u_1(OB, I) \cdot \sigma_1(OB)$$
$$+ u_1(IA, I) \cdot \sigma_1(IA) + u_1(IB, I) \cdot \sigma_1(IB)$$
$$= 2 \cdot \frac{1}{3} + 2 \cdot \frac{1}{3} + 4 \cdot \frac{1}{10} + 3 \cdot \frac{7}{30} = \frac{73}{30} = 2.43$$

b) $\boldsymbol{u_2(\sigma_1, O)}$ **for** $\boldsymbol{\sigma_1 = (\frac{1}{8}, \frac{1}{4}, \frac{1}{4}, \frac{3}{8})}$**.-** This is player 2's expected payoff if player 2 chooses strategy "O" with probability one, and player 1 uses a mixed strategy where:

$$\sigma_1(OA) = \frac{1}{8}, \sigma_1(OB) = \frac{1}{4}, \sigma_1(IA) = \frac{1}{4},$$
$$\sigma_1(IB) = \frac{3}{8}$$

- Using the definition of expected payoff, we have:

$$u_2(\sigma_1, O)$$
$$= u_2(OA, O) \cdot \sigma_1(OA) + u_2(OB, O) \cdot \sigma_1(OB)$$
$$+ u_2(IA, O) \cdot \sigma_1(IA) + u_2(IB, O) \cdot \sigma_1(IB)$$
$$= 2 \cdot \frac{1}{8} + 2 \cdot \frac{1}{4} + 3 \cdot \frac{1}{4} + 3 \cdot \frac{3}{8} = \frac{21}{8} = 2.62$$

c) $\boldsymbol{u_1(\sigma_1, \sigma_2)}$ **for** $\boldsymbol{\sigma_1 = (\frac{1}{16}, \frac{1}{4}, \frac{1}{2}, \frac{3}{16})}$ **and** $\boldsymbol{\sigma_2 = (\frac{1}{3}, \frac{2}{3})}$**.-**

This is player 1's expected payoff if player 2 chooses the mixed strategy σ_2, and player 1 uses the mixed strategy σ_1

$$\sigma_1(OA) = \frac{1}{16}, \sigma_1(OB) = \frac{1}{4}, \sigma_1(IA) = \frac{1}{2},$$

$$\sigma_1(IB) = \frac{3}{16}$$

and

$$\sigma_2(I) = \frac{1}{3}, \qquad \sigma_2(O) = \frac{2}{3}$$

- We know how to compute this expected payoff. It is given by:

$$u_1(\sigma_1, \sigma_2)$$
$$= u_1(OA, \sigma_2) \cdot \sigma_1(OA) + u_1(OB, \sigma_2) \cdot \sigma_1(OB)$$
$$+ u_1(IA, \sigma_2) \cdot \sigma_1(IA) + u_1(IB, \sigma_2) \cdot \sigma_1(IB)$$
$$= u_1(OA, \sigma_2) \cdot \frac{1}{16} + u_1(OB, \sigma_2) \cdot \frac{1}{4} + u_1(IA, \sigma_2) \cdot \frac{1}{2}$$
$$+ u_1(IB, \sigma_2) \cdot \frac{3}{16}$$

- Therefore, we first need to compute:
  - $u_1(OA, \sigma_2)$
  - $u_1(OB, \sigma_2)$
  - $u_1(IA, \sigma_2)$
  - $u_1(IB, \sigma_2)$

- We have:

$$u_1(OA,\sigma_2)$$
$$= u_1(OA,I\ )\cdot\sigma_2(I)+u_1(OA,O\ )\cdot\sigma_2(O)$$
$$= 2\cdot\frac{1}{3}+2\cdot\frac{2}{3}=2$$

$$u_1(OB,\sigma_2)$$
$$= u_1(OB,I\ )\cdot\sigma_2(I)+u_1(OB,O\ )\cdot\sigma_2(O)$$
$$= 2\cdot\frac{1}{3}+2\cdot\frac{2}{3}=2$$

$$u_1(IA,\sigma_2)=u_1(IA,I\ )\cdot\sigma_2(I)+u_1(IA,O\ )\cdot\sigma_2(O)$$
$$= 4\cdot\frac{1}{3}+1\cdot\frac{2}{3}=2$$

$$u_1(IB,\sigma_2)=u_1(IB,I\ )\cdot\sigma_2(I)+u_1(IB,O\ )\cdot\sigma_2(O)$$
$$= 3\cdot\frac{1}{3}+1\cdot\frac{2}{3}=\frac{5}{3}$$

- Therefore,

$$u_1(\sigma_1, \sigma_2)$$
$$= u_1(OA, \sigma_2) \cdot \frac{1}{16} + u_1(OB, \sigma_2) \cdot \frac{1}{4}$$
$$+ u_1(IA, \sigma_2) \cdot \frac{1}{2} + u_1(IB, \sigma_2) \cdot \frac{3}{16}$$
$$= 2 \cdot \frac{1}{16} + 2 \cdot \frac{1}{4} + 2 \cdot \frac{1}{2} + \frac{5}{3} \cdot \frac{3}{16} = \frac{93}{48}$$
$$= 1.93$$

d) $\boldsymbol{u_2(\sigma_1, \sigma_2)}$ **for** $\boldsymbol{\sigma_1 = (\frac{1}{16}, \frac{1}{4}, \frac{1}{2}, \frac{3}{16})}$ **and** $\boldsymbol{\sigma_2 = (\frac{1}{3}, \frac{2}{3})}$**.-**

This is player 2's expected payoff for the same mixed strategies as in the previous part. It is given by:

$$u_2(\sigma_1, \sigma_2) = u_2(\sigma_1, I) \cdot \sigma_2(I) + u_2(\sigma_1, O) \cdot \sigma_2(O)$$
$$= u_2(\sigma_1, I) \cdot \frac{1}{3} + u_2(\sigma_1, O) \cdot \frac{2}{3}$$

- So now we have to compute:
  - $u_2(\sigma_1, I)$
  - $u_2(\sigma_1, O)$

- We have:

$$u_2(\sigma_1, I)$$
$$= u_2(OA, I) \cdot \sigma_1(\text{OA}) + u_2(OB, I) \cdot \sigma_1(\text{OB})$$
$$+ u_2(IA, I) \cdot \sigma_1(\text{IA}) + u_2(IB, I) \cdot \sigma_1(\text{IB})$$
$$= 2 \cdot \frac{1}{16} + 2 \cdot \frac{1}{4} + 2 \cdot \frac{1}{2} + 4 \cdot \frac{3}{16} = \frac{38}{16} = 2.37$$

$$u_2(\sigma_1, O)$$
$$= u_2(OA, O) \cdot \sigma_1(\text{OA}) + u_2(OB, O) \cdot \sigma_1(\text{OB})$$
$$+ u_2(IA, O) \cdot \sigma_1(\text{IA}) + u_2(IB, O) \cdot \sigma_1(\text{IB})$$
$$= 2 \cdot \frac{1}{16} + 2 \cdot \frac{1}{4} + 3 \cdot \frac{1}{2} + 3 \cdot \frac{3}{16} = \frac{43}{16} = 2.68$$

- Finally, from here we obtain:

$$u_2(\sigma_1, \sigma_2) = u_2(\sigma_1, I) \cdot \frac{1}{3} + u_2(\sigma_1, O) \cdot \frac{2}{3}$$
$$= \frac{38}{16} \cdot \frac{1}{3} + \frac{43}{16} \cdot \frac{2}{3} = \frac{124}{48} = 2.58$$

- **Example: Cournot-duopoly model.-** Let us go back to the Cournot duopoly example described in the previous chapter, where market price is given by:

$$p = 100 - 2 \cdot q_1 - 2 \cdot q_2$$

And total costs for each firm are:

$$20 \cdot q_i \text{ for } i = 1{,}2.$$

- When we introduced this example, we showed that payoff (profit) functions are given by:

$$u_1(q_1, q_2) = (80 - 2 \cdot q_1 - 2 \cdot q_2) \cdot q_1$$
$$u_2(q_1, q_2) = (80 - 2 \cdot q_1 - 2 \cdot q_2) \cdot q_2$$

- Now consider the following **beliefs for player 1,** where he conjectures that: Player 2 will produce $q_2 = 10$ with probability 1/4, $q_2 = 12$ with probability 1/2, $q_2 = 15$ with probability 1/8, and $q_2 = 20$ with probability 1/8.

- Given these beliefs, **compute player 1's expected payoff of producing q_1 units**.

- Note that player 1's beliefs about player 2 are given by the probability distribution:

$$\theta_2(10) = \frac{1}{4}, \theta_2(12) = \frac{1}{2}, \theta_2(15) = \frac{1}{8}, \theta_2(20) = \frac{1}{8}$$

and

$$\theta_2(q_2) = 0 \text{ for all } q_2 \neq 10{,}12{,}15{,}20$$

- Given these beliefs, player 1's expected payoff of producing q_1 units is given by:

$$u_1(q_1, \theta_2) = u_1(q_1, 10) \cdot \theta_2(10) + u_1(q_1, 12) \cdot \theta_2(12)$$
$$+u_1(q_1, 15) \cdot \theta_2(15) + u_1(q_1, 20) \cdot \theta_2(20)$$
$$= (80 - 2 \cdot q_1 - 2 \cdot 10) \cdot q_1 \times \theta_2(10)$$
$$+(80 - 2 \cdot q_1 - 2 \cdot 12) \cdot q_1 \times \theta_2(12)$$
$$+(80 - 2 \cdot q_1 - 2 \cdot 15) \cdot q_1 \times \theta_2(15)$$
$$+(80 - 2 \cdot q_1 - 2 \cdot 20) \cdot q_1 \times \theta_2(20)$$
$$= (80 - 2 \cdot q_1 - 2 \cdot 10) \cdot q_1 \times \frac{1}{4}$$
$$+(80 - 2 \cdot q_1 - 2 \cdot 12) \cdot q_1 \times \frac{1}{2}$$
$$+(80 - 2 \cdot q_1 - 2 \cdot 15) \cdot q_1 \times \frac{1}{8}$$
$$+(80 - 2 \cdot q_1 - 2 \cdot 20) \cdot q_1 \times \frac{1}{8}$$

- Grouping terms we have:

$$u_1(q_1, \theta_2) = (80 - 2 \cdot q_1) \times \left(\frac{1}{4} + \frac{1}{2} + \frac{1}{8} + \frac{1}{8}\right)$$

$$-2 \cdot q_1 \times \left(10 \cdot \frac{1}{4} + 12 \cdot \frac{1}{2} + 15 \cdot \frac{1}{8} + 20 \cdot \frac{1}{8}\right)$$

$$= (80 - 2 \cdot q_1) - \left(\frac{206}{8}\right) \cdot q_1 = \left(80 - \frac{206}{8} - 2 \cdot q_1\right) \cdot q_1$$

$$= (54.25 - 2 \cdot q_1) \cdot q_1$$

- That is, the expected payoff function for player 1 of producing q_1 units given the beliefs described above is:

$$\boldsymbol{u_1(q_1, \theta_2) = (54.25 - 2 \cdot q_1) \cdot q_1}$$

- **Uncertainty and the ordinal nature of payoffs:** When we focused on pure strategies only (previous two chapters), we argued that payoffs only needed to reflect the **ordinal preferences of players over <u>outcomes</u>**.

- In the general case where there can be mixed strategies and uncertainty about others' choices, payoffs should reflect the **ordinal preferences of players over <u>distributions of outcomes</u>**.

- Assigning numerical payoffs is straightforward in games where these payoffs have a monetary interpretation. Otherwise we should always keep in mind that the numerical payoffs in a game represent more than just ordinal preferences over outcomes.