6.- Dominance and Best Response

- The main concepts in this section are:
  - Dominated strategy.
  - Dominance.
  - Weak Dominance.
  - Best Response.

- **Dominated Strategy:** A (pure) strategy s_i is ***dominated*** if there exists some strategy (pure or mixed) σ_i such that:

$$\boldsymbol{u_i(\sigma_i, s_{-i}) > u_i(s_i, s_{-i})} \text{ for every } \boldsymbol{s_{-i} \in S_{-i}}$$

- The strict inequality is key in the definition.
- **Example:**

1 \ 2	L	M	R
U	8, 1	0, 2	4, 0
C	3, 3	1, 2	0, 0
D	5, 0	2, 3	8, 1

- Strategy "D" dominates "C" for player 1.
- Strategy "M" dominates "R" for player 2.

- More generally, let us focus on mixed strategies for player 1 where he **randomizes only between "U" and "D"**.
- Which of those strategies dominate the pure strategy of choosing "C"?
- Let σ_1 denote any such mixed strategy. Let $\boldsymbol{\sigma_1(U) = Pr(U)}$ and $\boldsymbol{1 - \sigma_1(U) = Pr(D)}$
- By definition of dominance, this requires that $\boldsymbol{\sigma_1}$ be such that:

$$u_1(\sigma_1, L) > u_1(C, L)$$
$$u_1(\sigma_1, M) > u_1(C, M)$$
$$u_1(\sigma_1, R) > u_1(C, R)$$

- We have:

$$u_1(\sigma_1, L) = 8 \cdot \sigma_1(U) + 5 \cdot (1 - \sigma_1(U))$$
$$u_1(\sigma_1, M) = 0 \cdot \sigma_1(U) + 2 \cdot (1 - \sigma_1(U))$$
$$u_1(\sigma_1, R) = 4 \cdot \sigma_1(U) + 8 \cdot (1 - \sigma_1(U))$$

- Simple algebra leads to:

$$u_1(\sigma_1, L) = 5 + 3 \cdot \sigma_1(U)$$
$$u_1(\sigma_1, M) = 2 - 2 \cdot \sigma_1(U)$$
$$u_1(\sigma_1, R) = 8 - 4 \cdot \sigma_1(U)$$

- The payoffs for the pure strategy “C” are:

$$u_1(C, L) = 3, \;\; u_1(C, M) = 1, \;\; u_1(C, R) = 0$$

- Therefore, in order to dominate “C”, the mixed strategy requires that σ_1 be such that:

$$5 + 3 \cdot \sigma_1(U) > 3$$
$$2 - 2 \cdot \sigma_1(U) > 1$$
$$8 - 4 \cdot \sigma_1(U) > 0$$

- Simple algebraic manipulation shows that these restrictions reduce to:

$$\sigma_1(U) > -\frac{2}{3}$$
$$\frac{1}{2} > \sigma_1(U)$$
$$2 > \sigma_1(U)$$

- Since σ_1 is a well-defined probability (therefore bounded between zero and one), the first and last restrictions are redundant. **The only relevant restriction is** $\boldsymbol{\frac{1}{2} > \sigma_1(U)}$

- We conclude that mixed strategies where player 1 randomizes only between “U” and “D” dominate the strategy “C” if and only if “U” is chosen with probability strictly less than ½ .

- If a strategy is not dominated, we say that it is an **undominated strategy.** Rationality assumes that players only choose undominated strategies.

- We will let $\boldsymbol{UD_i}$ denote the **set of all undominated strategies for player $\boldsymbol{i}$**.

- In the previous example, the only dominated strategies were:
  - “C” for player 1.
  - “R” for player 2.

- Therefore:

$$UD_1 = \{U, D\} \quad \text{and} \quad UD_2 = \{L, M\}$$

- **A general two-step procedure to check if a strategy s_i is dominated:**

1. Check if s_i is dominated by another pure strategy.

1. If s_i is not dominated by a pure-strategy, check if it is dominated by a mixed strategy where I randomizes across strategies other than s_i.

- **Example:** Consider the matrix-form game:

1 \ 2	W	X	Y	Z
U	3, 6	4, 10	5, 0	0, 8
M	2, 6	3, 3	4, 10	1, 1
D	1, 5	2, 9	3, 0	4, 6

(c)

1) Find all strategies that are dominated by a <u>pure</u> strategy.
2) Is “W” a dominated strategy?
3) Is “M” a dominated strategy?

- **Finding strategies that are dominated by another pure strategy:**

1 \ 2	W	X	Y	Z
U	3, 6	4, 10	5, 0	0, 8
M	2, 6	3, 3	4, 10	1, 1
D	1, 5	2, 9	3, 0	4, 6

“X” dominates “Z” for Player 2

1) “Z” is the <u>only</u> strategy for either player that is dominated by another pure strategy (“X”).

2) Is “W” a dominated strategy?

- Since “W” is not dominated by a pure strategy, we must see if there exists a mixed strategy where player 2 chooses (X,Y,Z) that dominates the pure strategy “W”.
- Take a mixed strategy σ_2 that includes only X,Y,Z. We need to see if we can find a σ_2 such that:

$$u_2(U, \sigma_2) > u_2(U, W)$$
$$u_2(M, \sigma_2) > u_2(M, W)$$
$$u_2(D, \sigma_2) > u_2(D, W)$$

σ_2 dominates “W”

- Since σ_2 only includes "X", "Y" and "Z", we have:

$$\Pr(X) = \sigma_2(X), \qquad \Pr(Y) = \sigma_2(Y)$$
$$\Pr(Z) = 1 - \sigma_2(X) - \sigma_2(Y)$$

- Now we will compute $u_2(U, \sigma_2)$, $u_2(M, \sigma_2)$ and $u_2(D, \sigma_2)$.

- We have:

$$\begin{aligned} u_2(U, \sigma_2) & \\ &= 10 \cdot \sigma_2(X) + 0 \cdot \sigma_2(Y) + 8 \cdot \big(1 - \sigma_2(X) - \sigma_2(Y)\big) \\ &= 8 + 2 \cdot \sigma_2(X) - 8 \cdot \sigma_2(Y) \end{aligned}$$

- Also:

$$u_2(M, \sigma_2) \\ = 3 \cdot \sigma_2(X) + 10 \cdot \sigma_2(Y) + 1 \cdot \big(1 - \sigma_2(X) - \sigma_2(Y)\big) \\ = 1 + 2 \cdot \sigma_2(X) + 9 \cdot \sigma_2(Y)$$

- And:

$$u_2(D, \sigma_2) = 9 \cdot \sigma_2(X) + 0 \cdot \sigma_2(Y) + 6 \cdot \big(1 - \sigma_2(X) - \sigma_2(Y)\big) \\ = 6 + 3 \cdot \sigma_2(X) - 6 \cdot \sigma_2(Y)$$

- On the other hand, we have:

$$u_2(U, W) = 6, \;\; u_2(M, W) = 6, \;\; u_2(D, W) = 5$$

- The mixed strategy σ_2 will dominate "W" if and only if the following three conditions are satisfied by $\sigma_2(Y)$ and $\sigma_2(X)$:

$$\underbrace{8 + 2 \cdot \sigma_2(X) - 8 \cdot \sigma_2(Y)}_{u_2(U,\sigma_2)} > \underbrace{6}_{u_2(U,W)}$$

$$\underbrace{1 + 2 \cdot \sigma_2(X) + 9 \cdot \sigma_2(Y)}_{u_2(M,\sigma_2)} > \underbrace{6}_{u_2(M,W)}$$

$$\underbrace{6 + 3 \cdot \sigma_2(X) - 6 \cdot \sigma_2(Y)}_{u_2(D,\sigma_2)} > \underbrace{5}_{u_2(D,W)}$$

- Using simple algebra, these three inequalities can be simplified to:

$$2 \cdot \sigma_2(X) - 8 \cdot \sigma_2(Y) > -2$$
$$2 \cdot \sigma_2(X) + 9 \cdot \sigma_2(Y) > 5$$
$$3 \cdot \sigma_2(X) - 6 \cdot \sigma_2(Y) > -1$$

- **"W" is a dominated strategy <u>if and only if</u> we can find well-defined probabilities $\boldsymbol{\sigma_2(X)}$ and $\boldsymbol{\sigma_2(Y)}$ that satisfy all of the three inequalities above.**

- To verify this, using a graph is convenient.

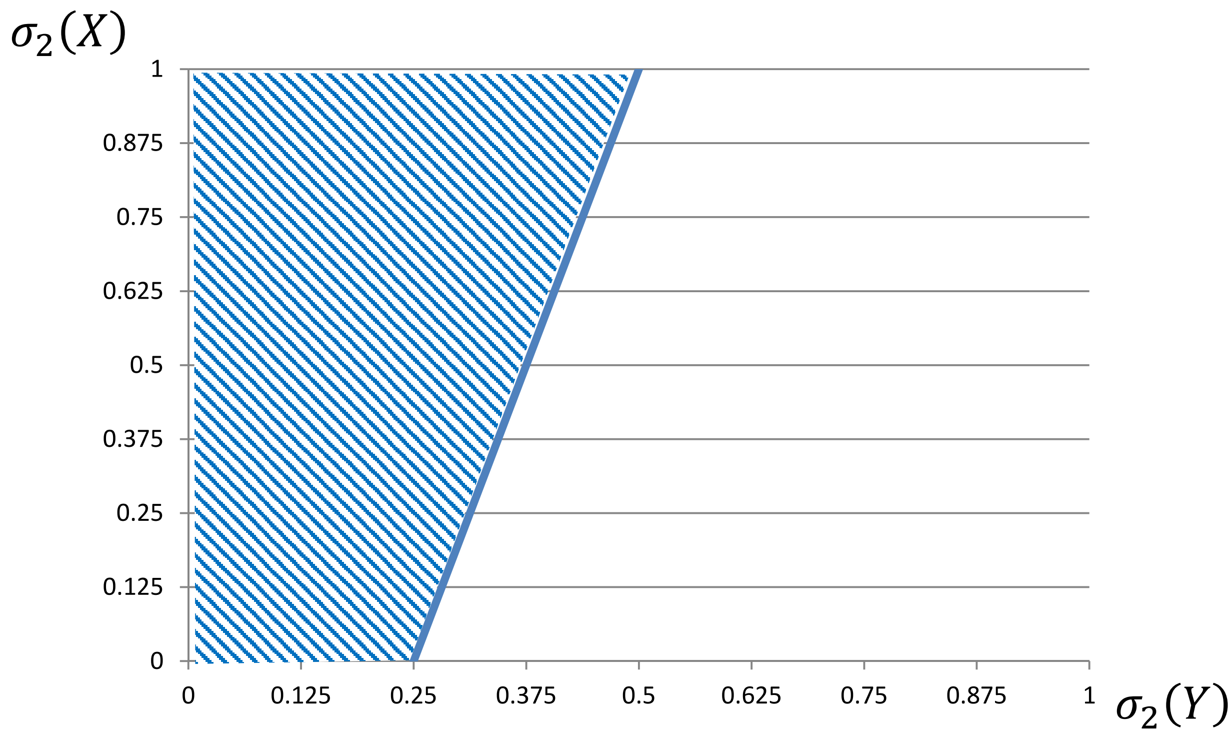
- The first inequality is

$$2 \cdot \sigma_2(X) - 8 \cdot \sigma_2(Y) > -2$$

- Dividing both sides by 2, this inequality can be re-expressed as:

$$\sigma_2(X) > -1 + 4 \cdot \sigma_2(Y)$$

- Graphically:



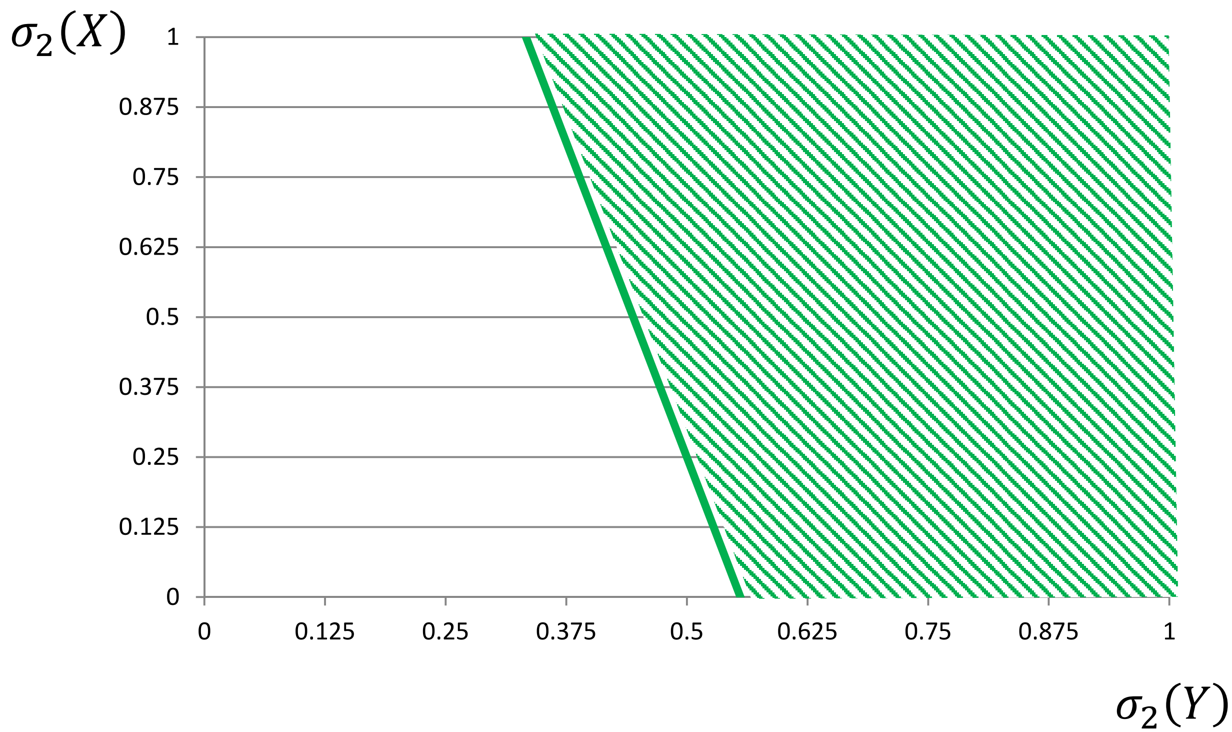
- The second inequality is

$$2 \cdot \sigma_2(X) + 9 \cdot \sigma_2(Y) > 5$$

- Dividing both sides by 2, this inequality can be re-expressed as:

$$\sigma_2(X) > 2.5 - 4.5 \cdot \sigma_2(Y)$$

- Graphically:



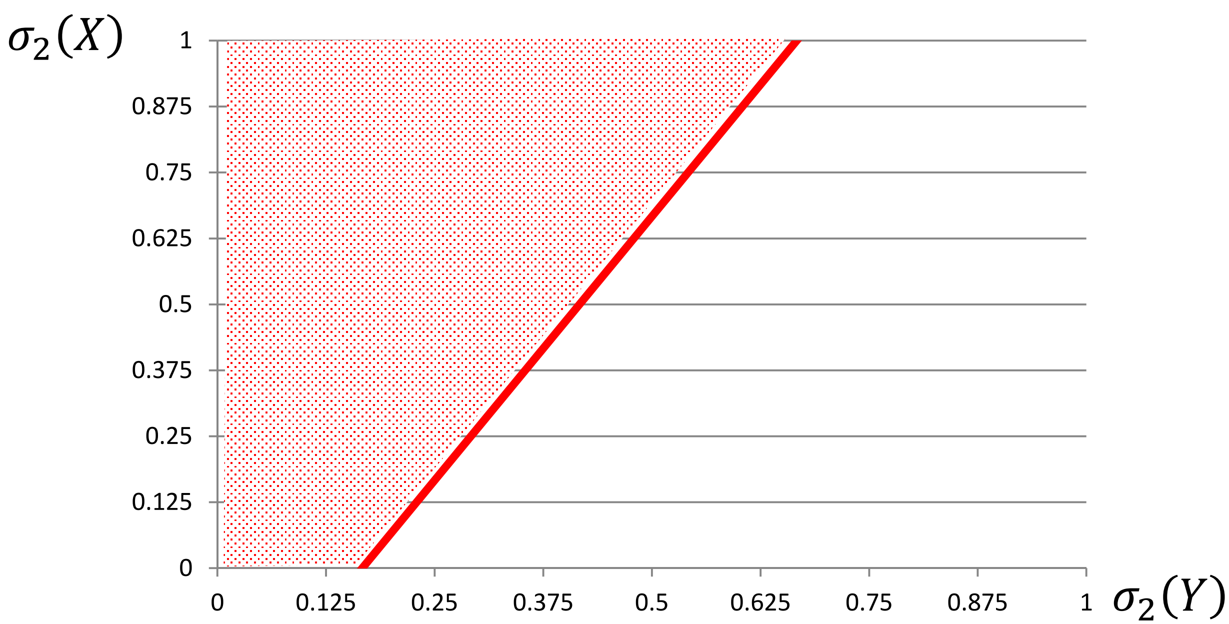
- The third inequality is

$$3 \cdot \sigma_2(X) - 6 \cdot \sigma_2(Y) > -1$$

- Dividing both sides by 3, this inequality can be re-expressed as:

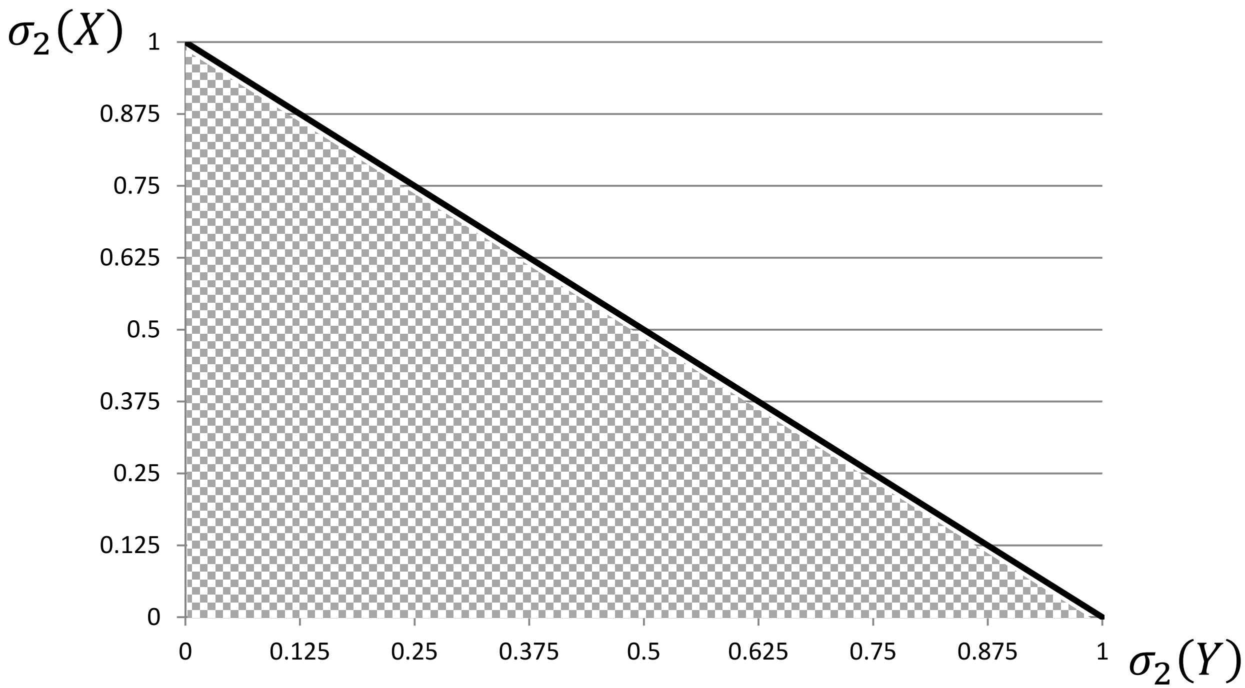
$$\sigma_2(X) > -\frac{1}{3} + 2 \cdot \sigma_2(Y)$$

- Graphically:

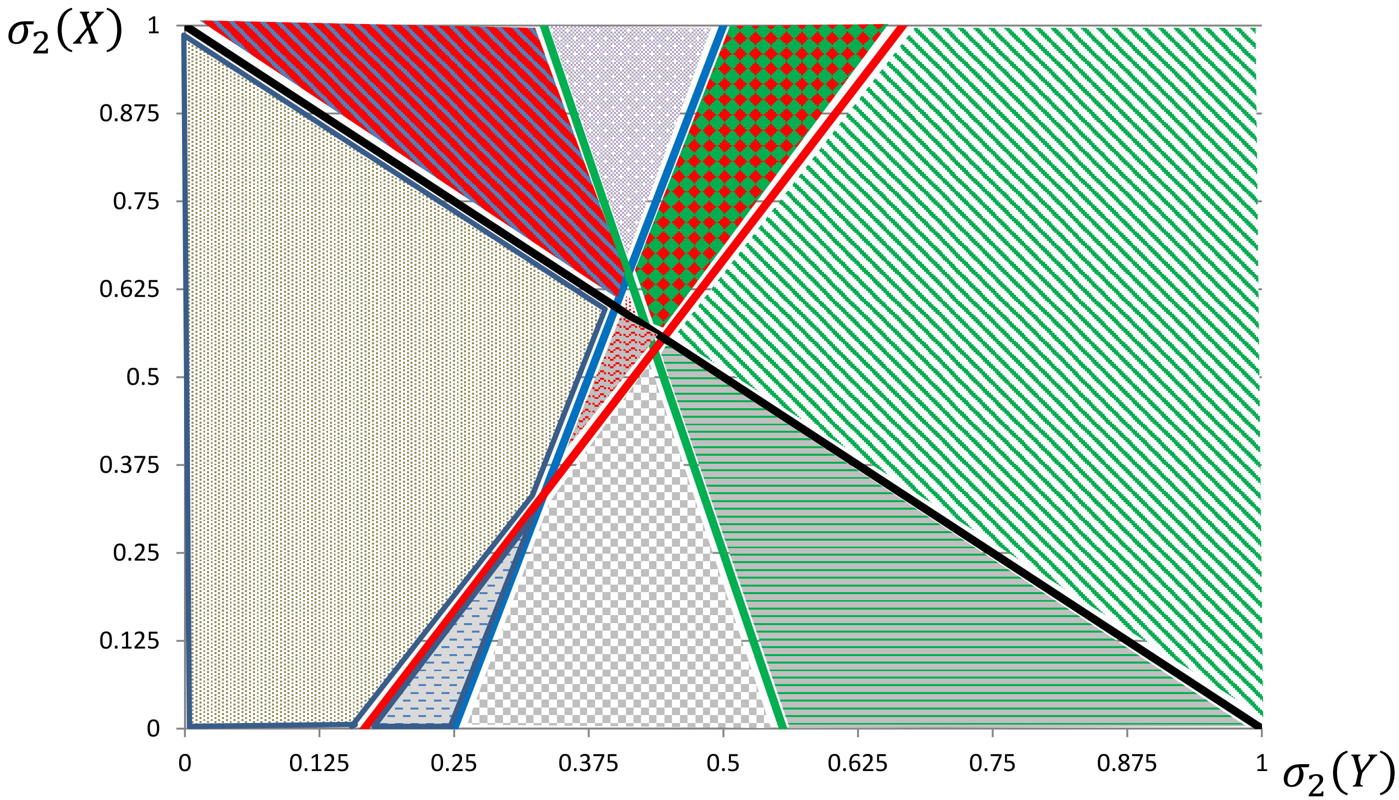


- Lastly, we need to make sure that $\sigma_2(X)$ and $\sigma_2(Y)$ constitute a **well-defined probability distribution**. That is, they must be nonnegative, smaller than 1, and also: $\boldsymbol{\sigma_2(X) + \sigma_2(Y) \leq 1}$.
- This can be re-expressed as:

$$\sigma_2(X) \leq 1 - \sigma_2(Y)$$



- The last step is to combine the four regions and see if they intersect with each other. If they do, then "W" is a dominated strategy. If their intersection is empty, then "W" is NOT a dominated strategy:



There is no region where all four restrictions are satisfied. Therefore, "W" is <u>NOT</u> <u>dominated</u>.

3) Is “M” a dominated strategy?

- Since “M” is not dominated by a pure strategy, we must see if there exists a mixed strategy where player 1 chooses (U,D) that dominates the pure strategy “M”.
- That is, look for a mixed strategy σ_1 such that:

$$u_1(\sigma_1, \mathrm{W}) > u_1(M, \mathrm{W})$$
$$u_1(\sigma_1, \mathrm{X}) > u_1(M, \mathrm{X})$$
$$u_1(\sigma_1, \mathrm{Y}) > u_1(M, \mathrm{Y})$$
$$u_1(\sigma_1, \mathrm{Z}) > u_1(M, \mathrm{Z})$$

- Since σ_1 only includes "U" and "D", we have:

$$\Pr(U) = \sigma_1(U) \quad \text{and} \quad \Pr(D) = 1 - \sigma_1(U)$$

- We have:

$$u_1(\sigma_1, \mathrm{W}) = 3 \cdot \sigma_1(U) + 1 \cdot \left(1 - \sigma_1(U)\right) = 1 + 2 \cdot \sigma_1(U)$$
$$u_1(\sigma_1, \mathrm{X}) = 4 \cdot \sigma_1(U) + 2 \cdot \left(1 - \sigma_1(U)\right) = 2 + 2 \cdot \sigma_1(U)$$
$$u_1(\sigma_1, \mathrm{Y}) = 5 \cdot \sigma_1(U) + 3 \cdot \left(1 - \sigma_1(U)\right) = 3 + 2 \cdot \sigma_1(U)$$
$$u_1(\sigma_1, \mathrm{Z}) = 0 \cdot \sigma_1(U) + 4 \cdot \left(1 - \sigma_1(U)\right) = 4 - 4 \cdot \sigma_1(U)$$

- And,

$$u_1(M, \mathrm{W}) = 2, u_1(M, \mathrm{X}) = 3,$$
$$u_1(M, \mathrm{Y}) = 4, u_1(M, \mathrm{Z}) = 1$$

- Therefore, “M” is a dominated strategy if and only if there exists a $0 \leq \sigma_1(U) \leq 1$ such that:

$$1 + 2 \cdot \sigma_1(U) > 2$$
$$2 + 2 \cdot \sigma_1(U) > 3$$
$$3 + 2 \cdot \sigma_1(U) > 4$$
$$4 - 4 \cdot \sigma_1(U) > 1$$

- Using simple algebra, we can see that the first three inequalities will be satisfied if and only if:

$$\sigma_1(U) > \frac{1}{2}$$

- And the fourth inequality will be satisfied if and only if:

$$\sigma_1(U) < \frac{3}{4}$$

- We conclude from here that **"M" is a dominated strategy**, since it is dominated by any mixed strategy where

$$\Pr(U) = \sigma_1(U) \quad \text{and} \quad \Pr(D) = 1 - \sigma_1(U)$$

- As long as:

$$\boldsymbol{\frac{1}{2} < \sigma_1(U) < \frac{3}{4}}$$

- For example, suppose $\sigma_1(U) = \frac{2}{3}$. Then,

$$u_1(\sigma_1, W) = 1 + 2 \cdot \frac{2}{3} = \frac{7}{3} > 2 = u_1(M, W)$$

$$u_1(\sigma_1, X) = 2 + 2 \cdot \frac{2}{3} = \frac{10}{3} > 3 = u_1(M, X)$$

$$u_1(\sigma_1, Y) = 3 + 2 \cdot \frac{2}{3} = \frac{13}{3} > 4 = u_1(M, Y)$$

$$u_1(\sigma_1, Z) = 4 - 4 \cdot \frac{2}{3} = \frac{4}{3} > 1 = u_1(M, Z)$$

- **Best Response:** The notion of <u>beliefs</u> did not play a direct role in the definition of dominance. We say that (pure) strategy s_i is a best response for i given beliefs $\boldsymbol{\theta_{-i}}$ if

$$u_i(s_i, \theta_{-i}) \geq u_i(s'_i, \theta_{-i}) \text{ for every } s'_i \in S_i$$

- Key to the definition is the **weak inequality**. As a result of this weak inequality, we have the following two facts:
  - Every set of beliefs θ_{-i} always has a best response (at least in finite games).
  - There can exist **multiple best responses** to a given set of beliefs θ_{-i}.

- Therefore, the best responses to any set of beliefs θ_{-i} is always (at least in finite games) a nonempty set of strategies. We denote this set as:

$$\boldsymbol{BR_i(\theta_{-i})}$$

- Intuitively (we can show this formally), by definition of dominance, a **dominated strategy can never be a best response for <u>any</u> set of beliefs**.

- **Example:**

1 \ 2	L	C	R
U	8, 3	0, 4	4, 4
M	4, 2	1, 5	5, 3
D	3, 7	0, 1	2, 0

(b)

- We have:

$$BR_2(U) = \{C, R\}$$
$$BR_2(M) = \{C\}$$
$$BR_2(D) = \{L\}$$

- Now consider the set of beliefs for player 2 given by:

$$\theta_1(U) = \theta_1(M) = \frac{3}{8},\ \theta_1(D) = \frac{1}{4}$$

- To find the set of best-responses for θ_1 we first need to compute the expected payoff for player 2 for each of his three possible actions: L, C and R.

- We have:

$$u_2(\theta_1, L\ ) = 3 \cdot \frac{3}{8} + 2 \cdot \frac{3}{8} + 7 \cdot \frac{1}{4} = \frac{29}{8}$$

$$u_2(\theta_1, C) = 4 \cdot \frac{3}{8} + 5 \cdot \frac{3}{8} + 1 \cdot \frac{1}{4} = \frac{29}{8}$$

$$u_2(\theta_1, R) = 4 \cdot \frac{3}{8} + 3 \cdot \frac{3}{8} + 0 \cdot \frac{1}{4} = \frac{21}{8}$$

- Thus, the best responses for this set of beliefs are “C” and “L”. That is,

$$\boldsymbol{BR_2(\theta_1) = \{C, L\}}$$

- Are there any set of beliefs θ_1 such that
$$BR_2 = \{L, C, R\} \text{ ?}$$
- That is, are there any set of beliefs such that all three of player 2's possible actions are best-responses to θ_1?
- This would occur if and only if θ_1 is such that:
$$u_2(L, \theta_1) = u_2(C, \theta_1) = u_2(R, \theta_1)$$
- Naturally this would be satisfied if we have
$$u_2(L, \theta_1) = u_2(C, \theta_1)$$
$$u_2(L, \theta_1) = u_2(R, \theta_1)$$
- The question then becomes whether we can find a set of <u>**well-defined beliefs**</u> that solve the above system of two equations.

- Note that, since probabilities must add up to one, we have:

$$\theta_1(D) = 1 - \theta_1(U) - \theta_1(M)$$

- Using this expression for $\theta_1(D)$, straightforward algebraic manipulation yields:

$$u_2(L, \theta_1) = 7 - 4 \cdot \theta_1(U) - 5 \cdot \theta_1(M)$$
$$u_2(C, \theta_1) = 1 + 3 \cdot \theta_1(U) + 4 \cdot \theta_1(M)$$
$$u_2(R, \theta_1) = 4 \cdot \theta_1(U) + 3 \cdot \theta_1(M)$$

- And so the system of two equations described above simplifies to:

$$7 \cdot \theta_1(U) + 9 \cdot \theta_1(M) = 6$$
$$8 \cdot \theta_1(U) + 8 \cdot \theta_1(M) = 7$$

- This is a system of two linear equations, with two unknowns ($\theta_1(U)$ and $\theta_1(M)$). The solution is given by:

$\boldsymbol{\theta_1(U) = 0.9375}$, and $\boldsymbol{\theta_1(L) = -0.0625}$

- Notice that $\theta_1(L) = -0.0625$ is <u>not a well-defined probability</u>.

- We conclude that **there does not exist any set of beliefs $\boldsymbol{\theta_1}$ such that $\boldsymbol{BR_2 = \{L, C, R\}}$.**

- **Cournot Model (continued):** The notion of best response applies to games with discrete strategies as well as games with continuous strategies.

- Let us revisit the Cournot model we have studied previously. Payoff functions were given by:

$$u_1(q_1, q_2) = (80 - 2 \cdot q_1 - 2 \cdot q_2) \cdot q_1$$
$$u_2(q_1, q_2) = (80 - 2 \cdot q_1 - 2 \cdot q_2) \cdot q_2$$

- Suppose player 1 has **beliefs about player 2** given by the probability distribution:

$$\theta_2(10) = \frac{1}{4}, \theta_2(12) = \frac{1}{2}, \theta_2(15) = \frac{1}{8}, \theta_2(20) = \frac{1}{8}$$

and

$$\theta_2(q_2) = 0 \text{ for all } q_2 \neq 10,12,15,20$$

- In Chapter 4 we derived the expected payoff function for player 1 of producing q_1 for the beliefs described above. The expected payoff function was given by:

$$u_1(q_1, \theta_2) = (54.25 - 2 \cdot q_1) \cdot q_1$$

- **What is $\boldsymbol{BR_1(\theta_2)}$?**

- By definition, it would be the collection of all **values of $\boldsymbol{q_1}$ that maximize $\boldsymbol{u_1(q_1, \theta_2)}$**.

- Formally, to find the value of q_1 that maximizes $u_1(q_1, \theta_2)$ we need to solve the **first order conditions**:

$$\frac{du_1(q_1, \theta_2)}{dq_1} = 0$$

- We have:

$$\frac{du_1(q_1, \theta_2)}{dq_1} = 54.25 - 4 \cdot q_1$$

- The first order conditions are satisfied if

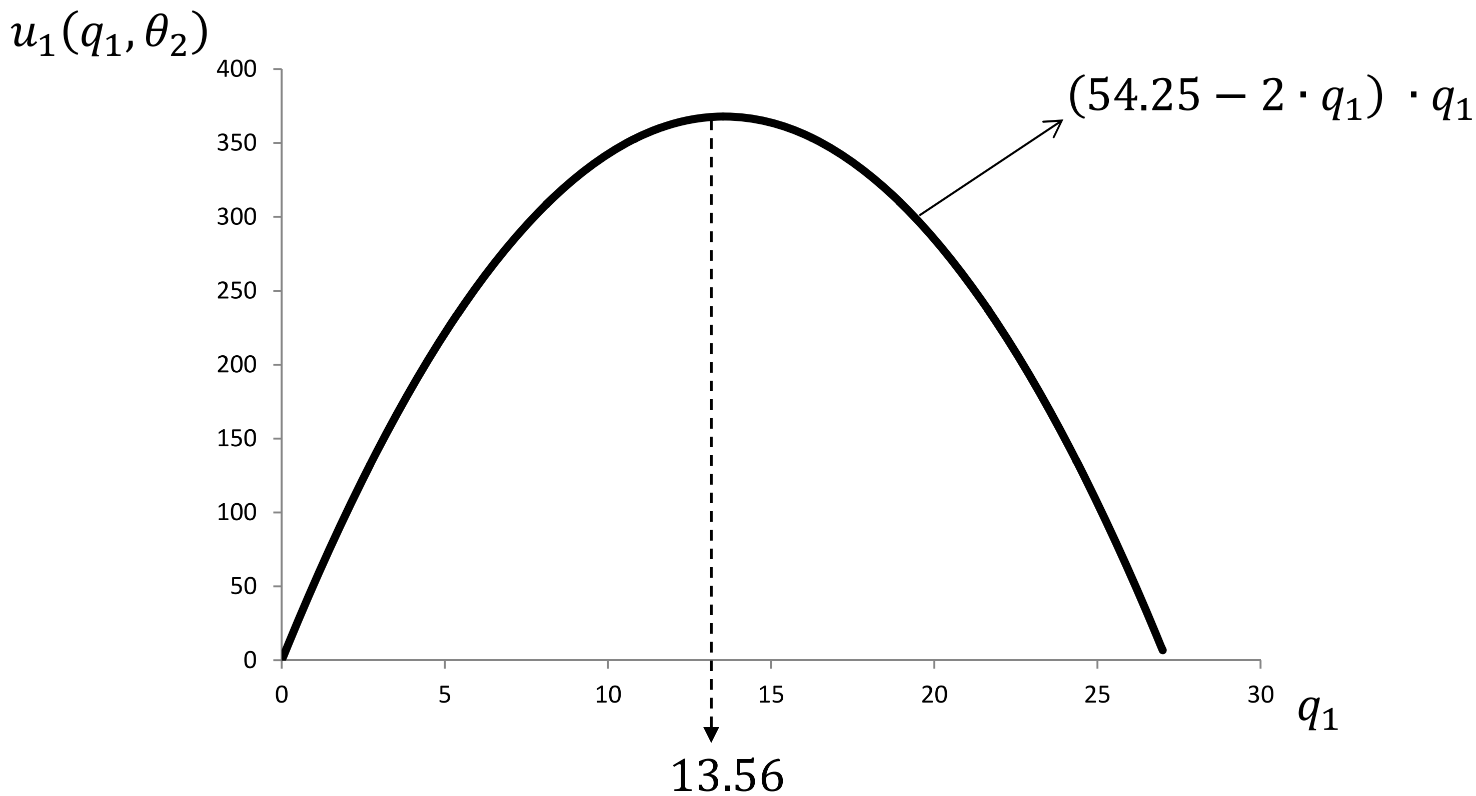
$$54.25 - 4 \cdot q_1 = 0$$

- That is, if $\boldsymbol{q_1 = 13.56}$**. This is the unique best-response for the set of beliefs** $\boldsymbol{\theta_2}$**.**

- That is,

$$BR_1(\theta_2) = \{13.56\}$$

- Graphically:



- **Relationship between Dominance and Best Responses**: By definition of dominated strategy, it is clear that dominated strategies can never be best-responses for any set of beliefs. But what is the precise relationship between these two concepts?
- First define the set of **all possible best responses** for player i as:

$$\boldsymbol{B_i} = \{\boldsymbol{s_i} | \textbf{\textit{there exists a belief }} \boldsymbol{\theta_{-i}} \textbf{\textit{ such that }} \boldsymbol{s_i \epsilon BR_i(\theta_{-i})}\}$$

- As before, let $\boldsymbol{UD_i}$ denote the **set of all undominated strategies for player $\boldsymbol{i}$**.
- Since only undominated strategies can be best-responses, we must have $\underline{\boldsymbol{B_i} \subset \boldsymbol{UD_i}}$ <u>(B_i is a **subset** of UD_i)</u>.

- Can we strengthen this result into $\boldsymbol{B_i = UD_i}$?
- That is, if a strategy is undominated, is it necessarily a best-response for some beliefs?
- For this, we first introduce the concept of **correlated conjectures (beliefs)**.
- Recall from basic probability theory that two events, “A” and “B” are independent if and only if $\mathbf{Pr}(\boldsymbol{A\ and\ B}) = \mathbf{Pr}(\boldsymbol{A}) \cdot \mathbf{Pr}(\boldsymbol{B})$.

- Consider a game with three players, $i = 1,2,3$ where:
  - Player 1 has two possible strategies: A and B.
  - Player 2 has two possible strategies: M and N.
  - Player 3 has two possible strategies: X and Y.

- Suppose that player 1 believes that:
  - Player 2 will choose M with probability ½
  - Player 3 will choose X with probability ½
  - The actions of players 2 and 3 are independent.

- The assumption of <u>independence</u> means that player 1’s beliefs over strategy profiles can be summarized as:

2 \ 3	X	Y	
M	1/4	1/4	1/2
N	1/4	1/4	1/2
	1/2	1/2	

(a)

- Now suppose that player 1 believes that:
  - Player 2 will choose M with probability 1/3
  - Player 3 will choose X with probability 1/2
  - The actions of players 2 and 3 are independent.
- The assumption of <u>independence</u> means that player 1's beliefs over strategy profiles can be now be summarized as:

2 \ 3	X	Y	
M	1/6	1/6	1/3
N	1/3	1/3	2/3
	1/2	1/2	

(b)

- Now suppose player 1’s beliefs can be summarized in the following way:

2 \ 3	X	Y	
M	1/2	0	1/2
N	0	1/2	1/2
	1/2	1/2	

(c)

- These are perfectly well-defined beliefs, but they are <u>not consistent with independence</u> (the joint probabilities do not equal the products of the marginal probabilities). These are called ***<u>correlated conjectures</u> or correlated beliefs***.

- Note that **uncorrelated beliefs are a special case of correlated beliefs**, so the latter is a richer class.
- Let us make the distinction:
- $B_i =$ Set of best responses over **uncorrelated** beliefs.
- ${B_i}^c$= Set of best responses over **correlated** beliefs.
- The following result summarizes the relation between dominance and best response:
- **Result:** For a finite game with $i = 1, \ldots, n$ players, we have $B_i \subset UD_i$ and ${B_i}^c = UD_i$ for each player i.

- In two-player games, the issue of correlation is irrelevant since each player has only one opponent. Therefore, ${B_i}^c = B_i$. As a consequence of the previous result we have the following corollary:

- **Corollary:** In any finite **two-player game**, we have $B_i = UD_i$ for each player i=1,2.

- **Proving of the main result:** First note that if a strategy $s_i \in S_i$ is dominated, then it cannot be a best-response for **any** set of beliefs. Therefore, any best response must be undominated. In set notation, this means that:

$$B_i{}^c \subset UD_i \text{ (and therefore, } B_i \subset UD_i \text{)}$$

- **Since $\boldsymbol{B_i{}^c \subset UD_i}$, to show that $\boldsymbol{UD_i = B_i{}^c}$ it suffices to show that $\boldsymbol{UD_i \subset B_i{}^c}$** (if a set "A" is a subset of "B" and "B" is also a subset of "A", then both sets must be equal to each other).

- A formal proof of $UD_i \subset B_i{}^c$ in general requires **mathematical tools beyond the scope of our course**.
- Appendix B illustrates it for a simple example, but does not provide a general proof.

- In the game described below, is Player 1's strategy "M" dominated? If so, describe a strategy that dominates it. If not, describe a belief to which M is a best response.

1 \ 2	X	Y
K	9, 2	1, 0
L	1, 0	6, 1
M	3, 2	4, 2

- First, note that “M” is not dominated by any pure strategy (“K” or “L”).

- So, we have to see if “M” is dominated by a **mixed** strategy where Player 1 randomizes between “K” and “L”. Let

  $\sigma_1\,(K) = \Pr(K)$ and $1 - \sigma_1(K) = \Pr(L)$

- We need to verify if there exists a $\sigma_1(K)$ such that

$$u_1(\sigma_1, X) > u_1(M, X)$$

AND

$$u_1(\sigma_1, Y) > u_1(M, Y)$$

- We have:

$$u_1(\sigma_1, X) = 9 \cdot \sigma_1(K) + 1 \cdot \big(1 - \sigma_1(K)\big) = 1 + 8 \cdot \sigma_1(\mathrm{K})$$
$$u_1(\sigma_1, Y) = 1 \cdot \sigma_1(K) + 6 \cdot \big(1 - \sigma_1(K)\big) = 6 - 5 \cdot \sigma_1(\mathrm{K})$$

- And:

$$u_1(M, X) = 3, \quad \text{and} \quad u_1(M, Y) = 4$$

- Therefore we need to find a mixing probability σ that satisfies:

$$1 + 8 \cdot \sigma_1(\mathrm{K}) > 3$$
$$6 - 5 \cdot \sigma_1(\mathrm{K}) > 4$$

- Simple algebraic manipulation simplifies these inequalities to:

$$\sigma_1(\mathrm{K}) > \frac{1}{4} = 0.25 \quad \text{and} \quad \sigma_1(\mathrm{K}) < \frac{2}{5} = 0.40$$

- Therefore, “M” is dominated by any mixed strategy where

$$\sigma_1(\mathrm{K}) = \Pr(K), 1 - \sigma_1(\mathrm{K}) = \Pr(L)$$

as long as:

$$\frac{1}{4} < \sigma_1(\mathrm{K}) < \frac{2}{5}$$

- For example, take $\sigma_1(\mathrm{K}) = 0.3$. Then,

$$u_1(\sigma_1, X) = 1 + 8 \cdot 0.3 = 3.4 > 3 = u_1(M, X)$$

and

$$u_1(\sigma_1, Y) = 6 - 5 \cdot \sigma = 4.5 > 4 = u_1(M, Y)$$

- Therefore, this particular mixed strategy dominates “M”.

- **A step-by-step procedure to compute B_i in two-player games:** Characterizing the set of undominated strategies will be an important step to predict rational behavior in games.
- In two-player matrix-form games this can be done in the following steps:

1. Find the best responses to pure strategies. These will automatically belong in B_i.
2. Look for strategies that are dominated by other pure strategies. These will automatically be ruled out from B_i.
3. Test each <u>remaining</u> strategy to see if it is dominated by a mixed strategy.
4. All strategies that remain will constitute B_i.

- **Example:** Consider the following matrix-form game:

1 \ 2	L	R
W	2, 4	2, 5
X	2, 0	7, 1
Y	6, 5	1, 2
Z	5, 6	3, 0

- Find B_i for i=1,2.

- Step 1: Find the best responses to pure strategies.-
  - For player 2: “L” is a best-response to “Y” and “Z”, and “R” is a best-response to “W” and “X”. Therefore, both “L” and “R” belong in B_2. No further steps are needed for player 2.
  - For player 1: “Y” is a best response to “L” and “X” is a best response to “R”. Therefore, “Y” and “X” belong in B_1.

- Step 2: Look for strategies that are dominated by other pure strategies. (We only need to do this for Player 1).
  - "W" is dominated by "Z". Therefore, "W" cannot belong in B_1.
- Step 3: Test each remaining strategy to see if it is dominated by a mixed strategy. Here we need to see if "Z" is dominated by a mixed strategy where player 1 mixes "Y" and "X".

- Let $\sigma_1(X) = \Pr(X)$ and $1 - \sigma_1(X) = \Pr(Y)$. We want to see if there exists $p \in [0,1]$ such that

$$u_1(\sigma_1, L) > u_1(Z, L)$$

AND

$$u_1(\sigma_1, R) > u_1(Z, R)$$

- We have

$$u_1(\sigma_1, L) = \sigma_1(X) \cdot 2 + (1 - \sigma_1(X)) \cdot 6$$
$$u_1(\sigma_1, R) = \sigma_1(X) \cdot 7 + (1 - \sigma_1(X)) \cdot 1$$

- Therefore we need to verify if $\exists p \in [0,1]$ such that:

$$\sigma_1(X) \cdot 2 + (1 - \sigma_1(X)) \cdot 6 > 5$$
$$\sigma_1(X) \cdot 7 + (1 - \sigma_1(X)) \cdot 1 > 3$$

- The first equation requires:

$$\frac{1}{4} > \sigma_1(X)$$

- The second equation requires:

$$\sigma_1(X) > \frac{1}{3}$$

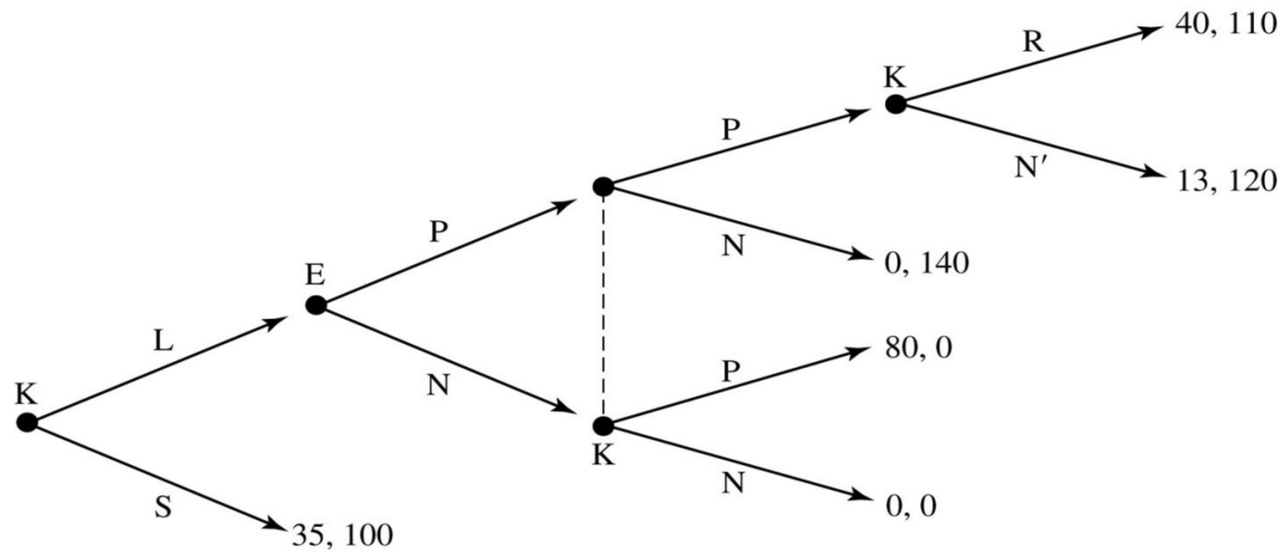
- Therefore, in order for "Z" to be dominated by this mixed strategy, the mixing distribution must satisfy $\frac{1}{4} > \sigma_1(X)$ AND $\sigma_1(X) > \frac{1}{3}$.

- But such a "$\sigma_1(X)$" cannot exist.

- We conclude that **"Z" cannot be dominated by a mixed strategy**. Therefore, $Z \in B_1$.

- Therefore: $\boldsymbol{B_1} = \{\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}\}$ and $\boldsymbol{B_2} = \{\boldsymbol{L}, \boldsymbol{R}\}$.

- **Example (cont): Katzenberg-Eisner game.-** Let us revisit the extensive-form of this game,



- Previously we derived its normal-form representation.

- The full normal-form matrix is:

K \ E	P	N
LPR	40, 110	80, 0
LPN′	13, 120	80, 0
LNR	0, 140	0, 0
LNN′	0, 140	0, 0
SPR	35, 100	35, 100
SPN′	35, 100	35, 100
SNR	35, 100	35, 100
SNN′	35, 100	35, 100

- Immediate inspection shows that the following strategies are dominated:

- Staying at Disney is a dominated strategy for Katzenberg.
- Not producing the movie after leaving Disney is ALSO a dominated strategy for Katzenberg.

K \ E	P	N
LPR	40, 110	80, 0
LPN′	13, 120	80, 0
~~LNR~~	~~0, 140~~	~~0, 0~~
~~LNN′~~	~~0, 140~~	~~0, 0~~
~~SPR~~	~~35, 100~~	~~35, 100~~
~~SPN′~~	~~35, 100~~	~~35, 100~~
~~SNR~~	~~35, 100~~	~~35, 100~~
~~SNN~~	~~35, 100~~	~~35, 100~~

- **Weak Dominance:** Some games do not have dominated strategies because this concept requires a strict inequality in payoffs.
- Weak dominance relaxes this requirement, and it allows payoffs to be the same in some cases.
- **Definition (Weak Dominance):** A (mixed or pure) strategy σ_i **weakly dominates** a pure strategy s_i if $u_i(\sigma_i, s_{-i}) \geq u_i(s_i, s_{-i})$ **for all** $s_{-i} \in S_{-i}$ **and** $u_i(\sigma_i, s'_{-i}) > u_i(s_i, s'_{-i})$ **for at least one** $s'_{-i} \in S_{-i}$.

- Clearly, any strategy that is dominated is also automatically weakly dominated.

- Let WUD_i denote the set of all strategies of player i that are **not weakly dominated**.

- Note that if a strategy is not weakly dominated, then automatically it is not dominated either. Therefore,

$$\boldsymbol{WUD_i \subset UD_i}$$

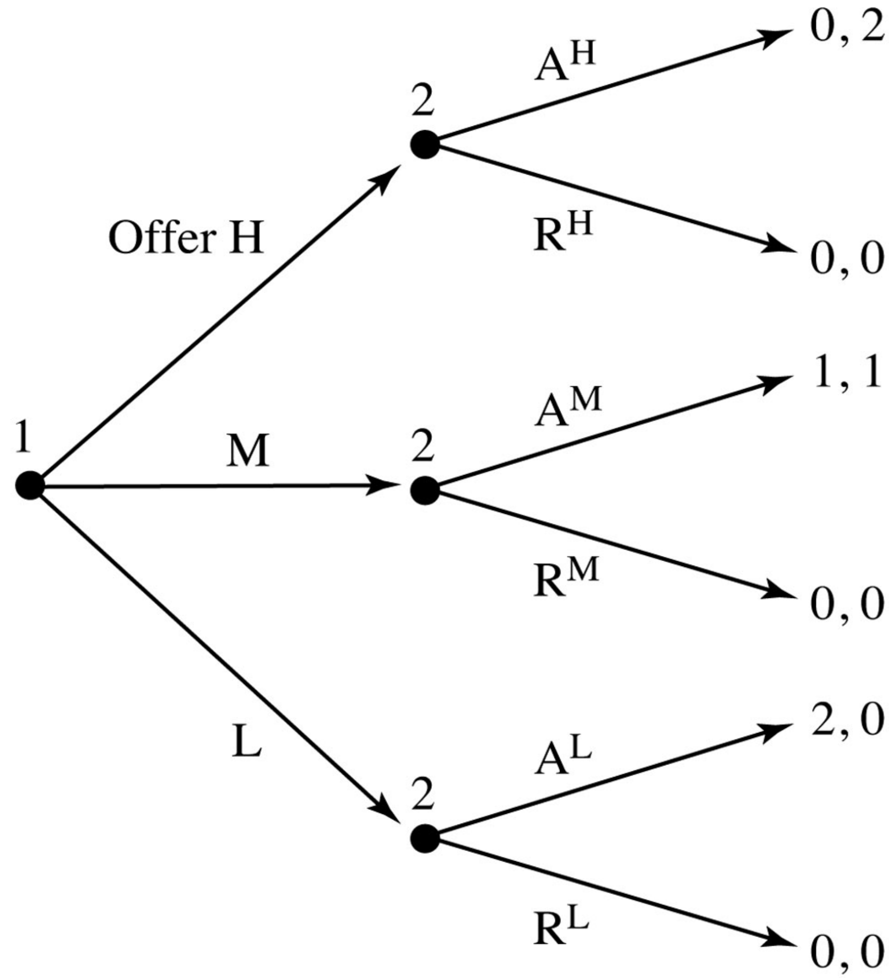
- **Example:** Consider the following game:

1 \ 2	L	M
X	3, 5	3, 5
Y	7, 0	1, 1

- Note that player 2 has no dominated strategies. However, “L” is weakly dominated by “M”. Therefore,

$$UD_2 = \{L, M\} \text{ and } WUD_2 = \{M\}$$

- **Example:** Ultimatum bargaining game (from before).



- We derived the Normal form representation before.

- **Example:** Ultimatum bargaining game (from before).

Player 2 \ Player 1	H	M	L
$A^H A^M A^L$	2 , 0	1 , 1	0 , 2
$A^H A^M R^L$	2 , 0	1 , 1	0 , 0
$R^H A^M A^L$	0 , 0	1 , 1	0 , 2
$R^H A^M R^L$	0 , 0	1 , 1	0 , 0
$A^H R^M A^L$	2 , 0	0 , 0	0 , 2
$A^H R^M R^L$	2 , 0	0 , 0	0 , 0
$R^H R^M A^L$	0 , 0	0 , 0	0 , 2
$R^H R^M R^L$	0 , 0	0 , 0	0 , 0

- Does this game have weakly dominated strategies?

- **YES:** An inspection of the matrix of payoffs reveals immediately that the following strategies are weakly dominated:
- For Player 1: "H" is weakly dominated by "M" and also by "L".
- For Player 2:
  - Both $R^H A^M A^L$ and $R^H A^M R^L$ are weakly dominated by $A^H A^M A^L$ and by $A^H A^M R^L$.
  - Both $R^H R^M A^L$ and $R^H R^M R^L$ are weakly dominated by all other strategies of Player 2.

– Striking out the weakly dominated strategies reveals the set WUD_i of weakly undominated strategies for each player…

- WUD_1 and WUD_2 in the ultimatum game:

Player 2 \ Player 1	H	M	L
$A^H A^M A^L$	2 , 0	1 , 1	0 , 2
$A^H A^M R^L$	2 , 0	1 , 1	0 , 0
$R^H A^M A^L$	0 , 0	1 , 1	0 , 2
$R^H A^M R^L$	0 , 0	1 , 1	0 , 0
$A^H R^M A^L$	2 , 0	0 , 0	0 , 2
$A^H R^M R^L$	2 , 0	0 , 0	0 , 0
$R^H R^M A^L$	0 , 0	0 , 0	0 , 2
$R^H R^M R^L$	0 , 0	0 , 0	0 , 0

- Making a high offer (player 1) and rejecting a high offer (player 2) are weakly dominated strategies.

- **Example (cont): Katzenberg-Eisner game.-** We had identified dominated strategies. Combined with weakly dominated strategies, we now have:

- Staying at Disney is a dominated strategy for Katzenberg.
- Not producing the movie after leaving Disney is ALSO a dominated strategy for Katzenberg.
- Releasing the movie late once it is produced is a WEAKLY dominated strategy for Katzenberg.

K \ E	P	N
LPR	40, 110	80, 0
LPN	13, 120	80, 0
LNR	0, 140	0, 0
LNN′	0, 140	0, 0
SPR	35, 100	35, 100
SPN′	35, 100	35, 100
SNR	35, 100	35, 100
SNN	35, 100	35, 100

- Not producing the movie is a WEAKLY dominated strategy for Eisner.

- Previously we described the relationship between the set of best responses B_i and the set of undominated strategies UD_i.

- How about WUD_i? Is there an analogous result for them?

- As it turns out, weakly undominated strategies are best responses to a particular type of beliefs, called **fully mixed beliefs**.

- We say that some beliefs θ_{-i} are **<u>fully mixed beliefs</u> if they assign strictly positive probability to <u>every</u> strategy profile $\boldsymbol{s_{-i}} \in \boldsymbol{S_{-i}}$.**

- Let

$$B_i^{cf} = \text{Set of best responses to fully mixed beliefs that allow correlation.}$$

- **Result:** For any finite game, $B_i^{cf} = WUD_i$ for each player $i = 1,2,\ldots,n$.
- Note that this is analogous to the result

$$\text{“}B_i^{c} = UD_i\text{”}$$

that we described previously.

- **HOWEVER:** The book will focus on dominance and not on weak dominance because fully mixed beliefs can be too restrictive in general.

- **<u>"First Strategic Tension":</u>** The textbook calls refers defines it as ***the clash between individual and group interests***.
- **Efficiency (definition):** We say that an outcome s is **more efficient** than an outcome s' if all of the players prefer the outcome s, and this preference is strict for at least one player. That is:

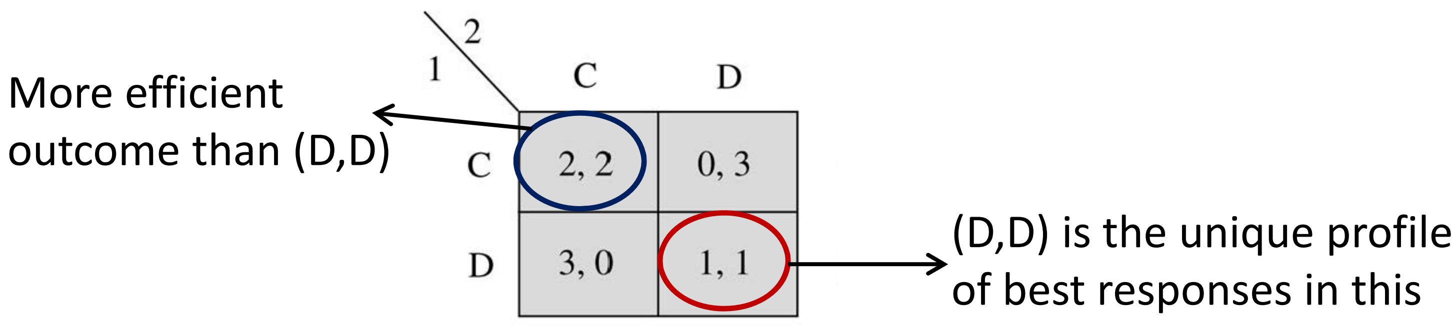
$u_i(s) \geq u_i(s')$ for <u>every</u> player i,

and

$u_j(s) > u_j(s')$ for <u>at least one</u> player j

- An outcome s is efficient if there is no other outcome s' that is more efficient than s. We also call it "**Pareto efficient**".

- **First Strategic Tension:** Individually rational behavior may lead to **inefficient outcomes**.
- Consider the Prisoner’s Dilemma



Prisoners’ Dilemma

- Notice that “Cooperate” is a <u>**dominated strategy**</u> and therefore (“Defect”, “Defect”) is the unique profile of best responses. However, that outcome is <u>**inefficient**</u>, since mutual cooperation would produce a higher payoff to both players.

- **Summary of Chapter 6:** We learned the following:

1. The concept of dominated strategy, and a two-step procedure to identify if a strategy is dominated.
2. The concept of best response.
3. The relationship between the set of undominated strategies UD_i and the set of best responses B_i:
   a) If we allow for correlated beliefs, then $UD_i = B_i$.
   b) Therefore, in any two-player game, $UD_i = B_i$.

c) A three-step procedure to find UD_i (the set of undominated strategies for each player in a game).

d) The notion of weak dominance, and the reason why we will keep focusing on strict dominance.