8.- Location, Partnership and Social Unrest (and Cournot revisited)

- Here we will study how iterated dominance produces predictions in some interesting examples of economic models.

  - A Location Game.
  - A Partnership Game of Strategic Complementarities.
  - A Game of Social Unrest.
  - Cournot Duopoly Revisited.

- **A Location Game:** Two soda vendors (competitors) simultaneously decide where to locate on a beach.
- **Description of the game:**

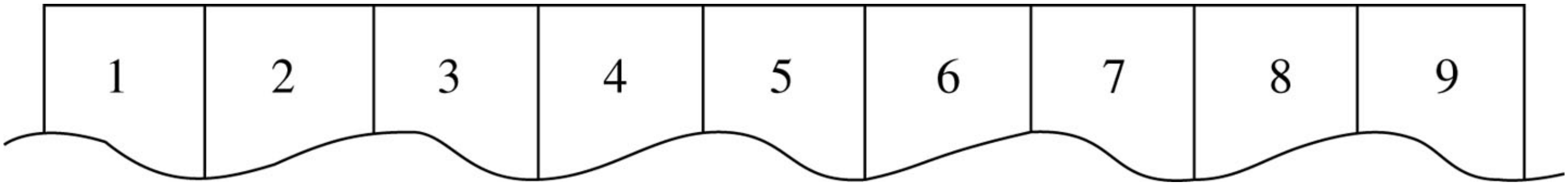
➢ A beach is divided into nine regions or “blocks” of equal size.

➢ Customers are uniformly distributed across each of these nine blocks. There are 50 soda customers per block. Each one will make a purchase.

➢ Customers go to the nearest vendor. If they are equally far, half go to one vendor and half to the other vendor.

➢ Vendors gain a commission of $0.25 per can of soda sold.

- The beach is represented graphically as:



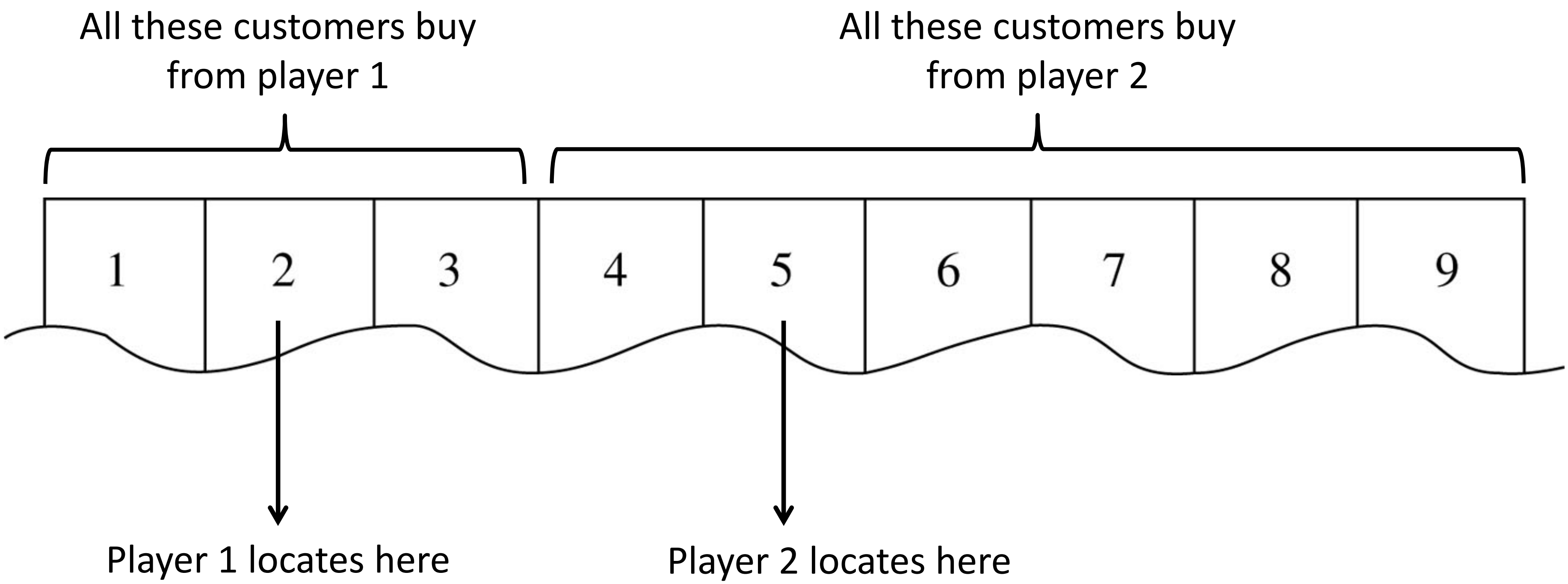
- The strategy space for player $i = 1, 2$ is given by the region where he decides to locate. Therefore,

$$S_i = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

- And a particular strategy profile (s_1, s_2) describes the blocks where both players decided to locate.
- Payoffs are given by the revenue made by each player. That is:

$$Total\ Number\ of\ Cans\ Sold\ \times \$0.25$$

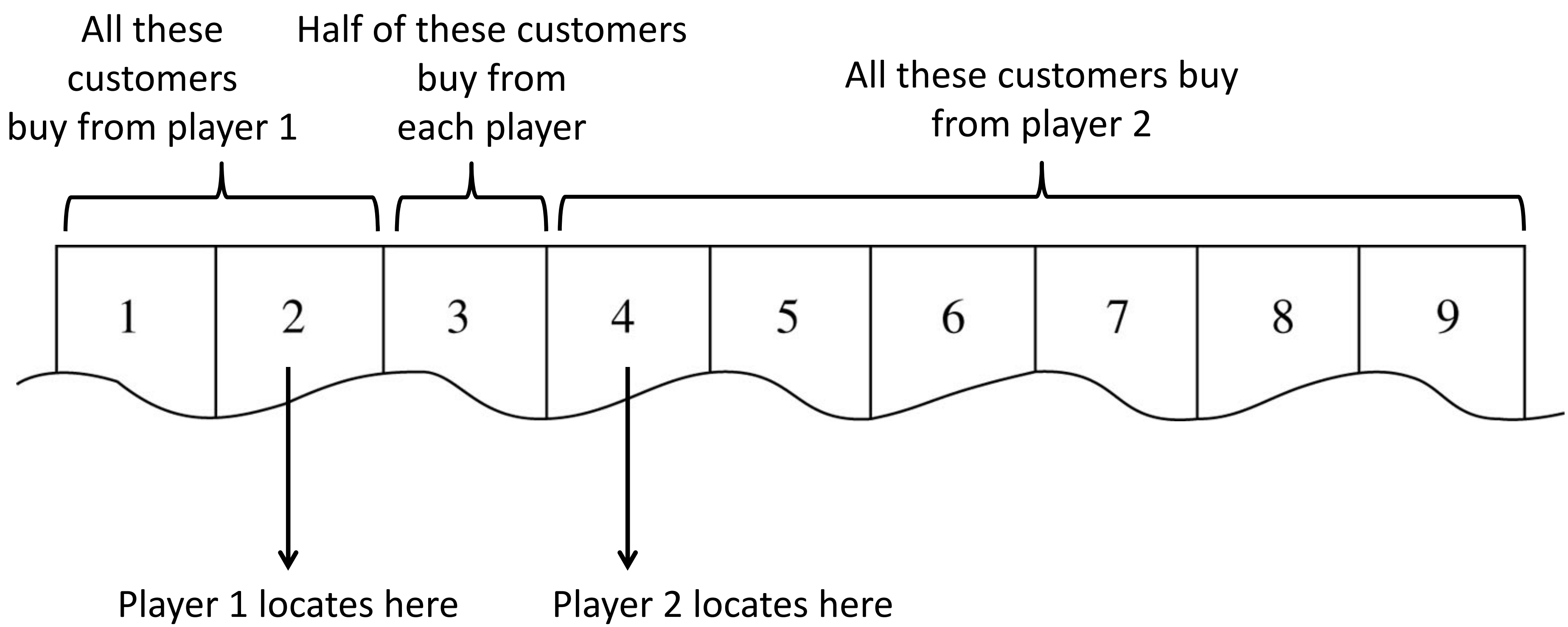
- Suppose $(s_1, s_2) = (2,5)$. Then:



- Then:

$$u_1(2,5) = (3 \cdot 50) \times \$0.25 = \$37.5$$
$$u_2(2,5) = (6 \cdot 50) \times \$0.25 = \$75.0$$

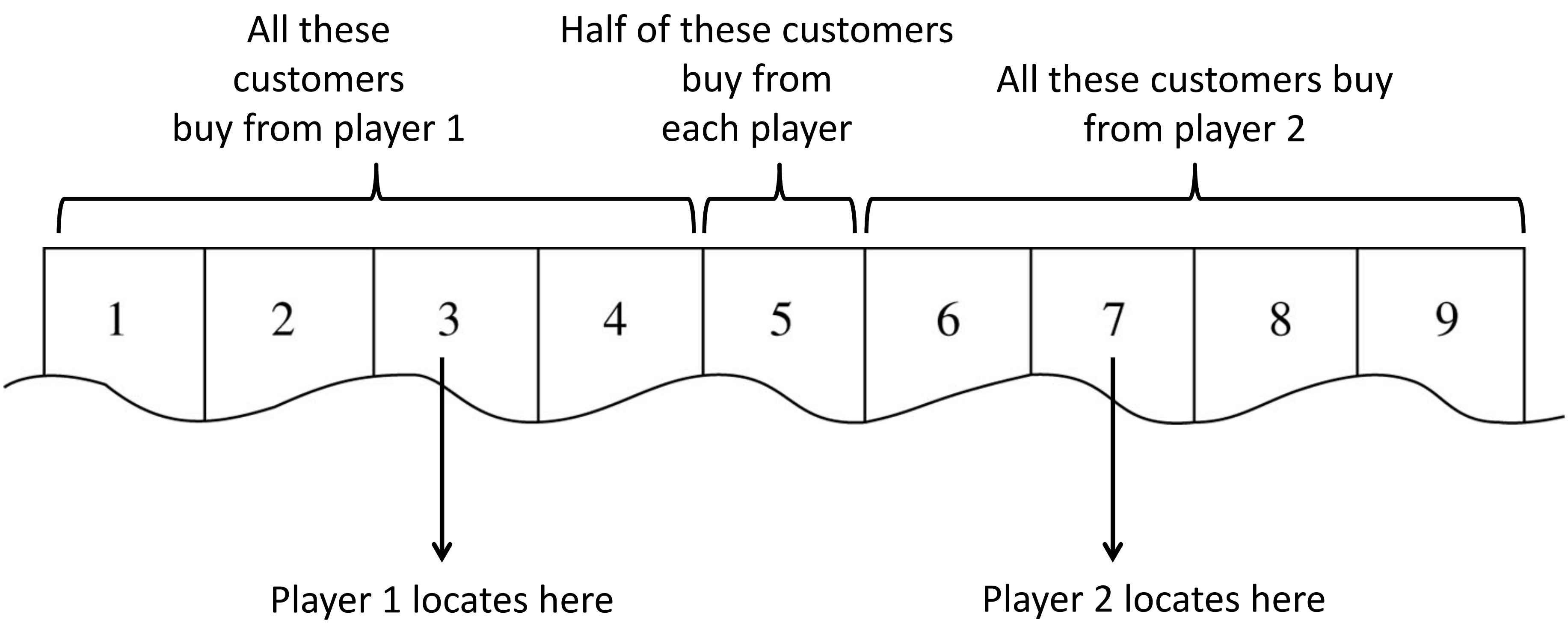
- Suppose $(s_1, s_2) = (2,4)$. Then:



- Then:

$$u_1(2,4) = (2 \cdot 50 + 25) \times \$0.25 = \$31.25$$
$$u_2(2,4) = (6 \cdot 50 + 25) \times \$0.25 = \$81.25$$

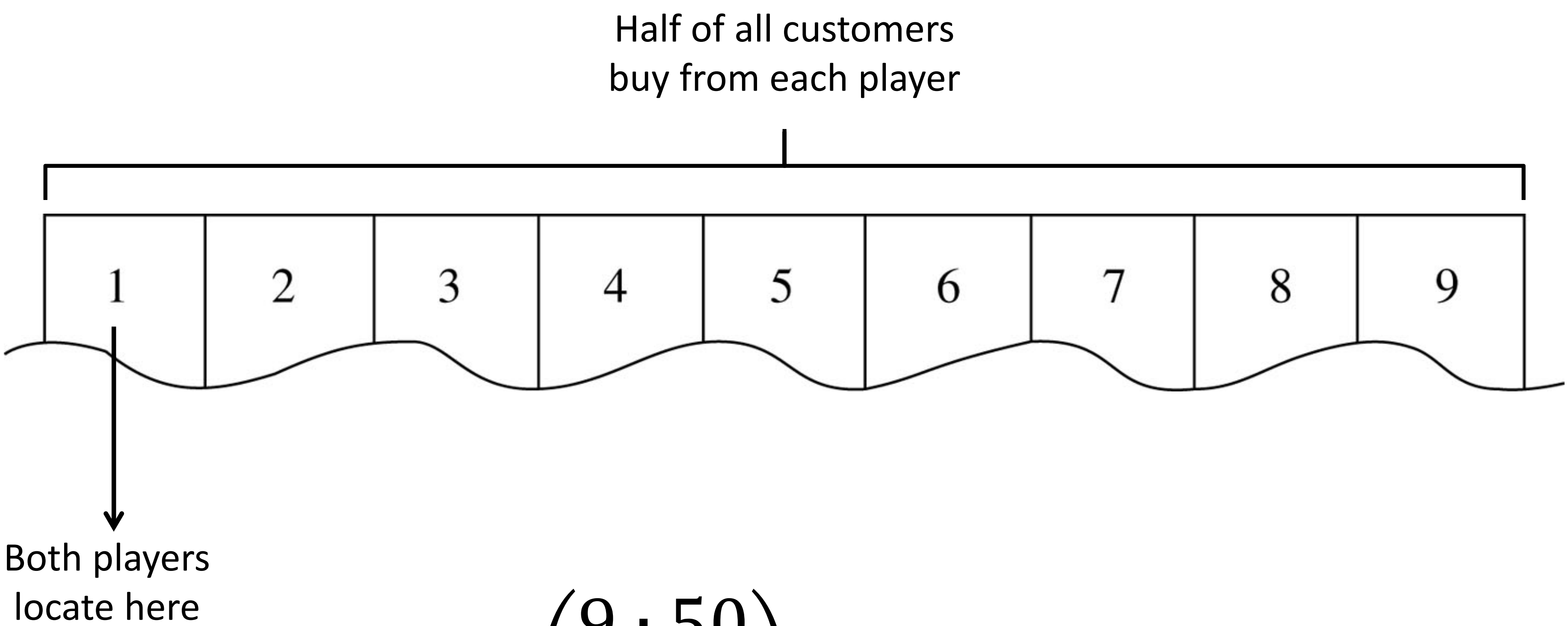
- Suppose $(s_1, s_2) = (3,7)$. Then:



- Then:

$$u_1(3,7) = (4 \cdot 50 + 25) \times \$0.25 = \$56.25$$
$$u_2(3,7) = (4 \cdot 50 + 25) \times \$0.25 = \$56.25$$

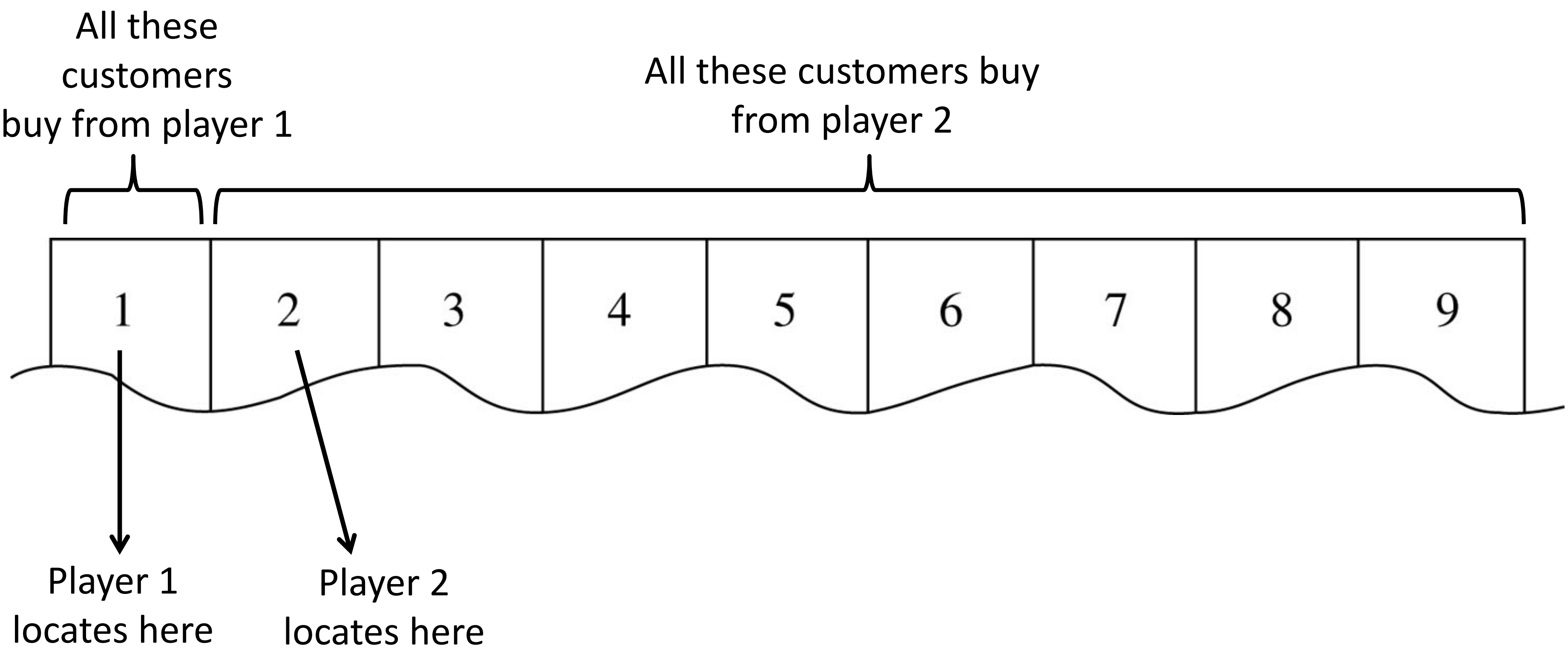
- Suppose $(s_1, s_2) = (1,1)$. Then:



$$u_1(1,1) = \left(\frac{9 \cdot 50}{2}\right) \times \$0.25 = \$56.25$$

$$u_2(1,1) = \left(\frac{9 \cdot 50}{2}\right) \times \$0.25 = \$56.25$$

- Suppose $(s_1, s_2) = (1,2)$. Then:



- Then:

$$u_1(1,2) = (1 \cdot 50) \times \$0.25 = \$12.5$$
$$u_2(1,2) = (8 \cdot 50) \times \$0.25 = \$100.0$$

- A general expression for payoffs is the following:

a) If $s_1 < s_2$:

$$u_1(s_1, s_2) = 50 \cdot \left(s_1 + \frac{s_2 - s_1 - 1}{2}\right) \times \$0.25$$

$$u_2(s_1, s_2) = 50 \cdot \left(10 - s_2 + \frac{s_2 - s_1 - 1}{2}\right) \times \$0.25$$

b) If $s_1 > s_2$:

$$u_1(s_1, s_2) = 50 \cdot \left(10 - s_1 + \frac{s_1 - s_2 - 1}{2}\right) \times \$0.25$$

$$u_2(s_1, s_2) = 50 \cdot \left(s_2 + \frac{s_1 - s_2 - 1}{2}\right) \times \$0.25$$

a) If $s_1 = s_2$:

$$u_1(s_1, s_2) = u_2(s_1, s_2) = 50 \cdot \left(\frac{9}{2}\right) \times \$0.25 = \$56.25$$

- Use iterated dominance to find the set of rationalizable strategies, $\boldsymbol{R}$.

- To do this, **we first write down the normal (matrix) form representation of this game**.

- Then we proceed with iterated dominance.

- Matrix form representation for the location game:

P1 \ P2	1	2	3	4	5	6	7	8	9
1	56.25, 56.25	12.5, 100	18.75, 93.75	25, 87.5	31.25, 81.25	37.5, 75	43.75, 68.75	50, 62.5	56.25, 56.25
2	100, 12.5	56.25, 56.25	25, 87.5	31.25, 81.25	37.5, 75	43.75, 68.75	50, 62.5	56.25, 56.25	62.5, 50
3	93.75, 18.75	87.5, 25	56.25, 56.25	37.5, 75	43.75, 68.75	50, 62.5	56.25, 56.25	62.5, 50	68.75, 43.75
4	87.5, 25	81.25, 31.25	75, 37.5	56.25, 56.25	50, 62.5	56.25, 56.25	62.5, 50	68.75, 43.75	75, 37.5
5	81.25, 31.25	75, 37.5	68.75, 43.75	62.5, 50	56.25, 56.25	62.5, 50	68.75, 43.75	75, 37.5	81.25, 31.25
6	75, 37.5	68.75, 43.75	62.5, 50	56.25, 56.25	50, 62.5	56.25, 56.25	75, 37.5	81.25, 31.25	87.5, 25
7	68.75, 43.75	62.5, 50	56.25, 56.25	50, 62.5	43.75, 68.75	37.5, 75	56.25, 56.25	87.5, 25	93.75, 18.75
8	62.5, 50	56.25, 56.25	50, 62.5	43.75, 68.75	37.5, 75	31.25, 81.25	25, 87.5	56.25, 56.25	100, 12.5
9	56.25, 56.25	50, 62.5	43.75, 68.75	37.5, 75	31.25, 81.25	25, 87.5	18.75, 93.75	12.5, 100	56.25, 56.25

- Dominated strategies are:

$$s_1 = \{1,9\} \text{ and } s_2 = \{1,9\}$$

- Reduced game $\boldsymbol{R^1}$ looks as follows:

P1 \ P2	2	3	4	5	6	7	8
2	56.25, 56.25	25, 87.5	31.25, 81.25	37.5, 75	43.75, 68.75	50, 62.5	56.25, 56.25
3	87.5, 25	56.25, 56.25	37.5, 75	43.75, 68.75	50, 62.5	56.25, 56.25	62.5, 50
4	81.25, 31.25	75, 37.5	56.25, 56.25	50, 62.5	56.25, 56.25	62.5, 50	68.75, 43.75
5	75, 37.5	68.75, 43.75	62.5, 50	56.25, 56.25	62.5, 50	68.75, 43.75	75, 37.5
6	68.75, 43.75	62.5, 50	56.25, 56.25	50, 62.5	56.25, 56.25	75, 37.5	81.25, 31.25
7	62.5, 50	56.25, 56.25	50, 62.5	43.75, 68.75	37.5, 75	56.25, 56.25	87.5, 25
8	56.25, 56.25	50, 62.5	43.75, 68.75	37.5, 75	31.25, 81.25	25, 87.5	56.25, 56.25

- Dominated strategies in the reduced game $\boldsymbol{R}^{\mathbf{1}}$ are:

$$s_1 = \{2,8\} \quad \text{and} \quad s_2 = \{2,8\}$$

- Reduced game $\boldsymbol{R}^{\mathbf{2}}$ looks as follows:

P1 \ P2	**3**	**4**	**5**	**6**	**7**
3	56.25, 56.25	37.5, 75	43.75, 68.75	50, 62.5	56.25, 56.25
4	75, 37.5	56.25, 56.25	50, 62.5	56.25, 56.25	62.5, 50
5	68.75, 43.75	62.5, 50	56.25, 56.25	62.5, 50	68.75, 43.75
6	62.5, 50	56.25, 56.25	50, 62.5	56.25, 56.25	75, 37.5
7	56.25, 56.25	50, 62.5	43.75, 68.75	37.5, 75	56.25, 56.25

- Dominated strategies in the reduced game $\boldsymbol{R^2}$ are:

$$s_1 = \{3,7\} \quad \text{and} \quad s_2 = \{3,7\}$$

- Reduced game $\boldsymbol{R^3}$ looks as follows:

P1 \ P2	4	5	6
4	56.25, 56.25	50, 62.5	56.25, 56.25
5	62.5, 50	56.25, 56.25	62.5, 50
6	56.25, 56.25	50, 62.5	56.25, 56.25

- Dominated strategies in the reduced game $\boldsymbol{R^3}$ are:

$$s_1 = \{4,6\} \quad \text{and} \quad s_2 = \{4,6\}$$

- Reduced game $\boldsymbol{R^4}$ consists of the unique profile of strategies $(s_1, s_2) = (5,5)$:

P1 \ P2	5
5	56.25, 56.25

- Since this set cannot be reduced any further, we conclude that the set of rationalizable strategies is

$$R = R^4 = (5,5)$$

- Rationalizability yields a unique prediction in this game: **Both players locating in the center of the beach is the only rationalizable strategy profile**.

- The previous game is based on **Hotelling's location model.**
- It can be used to explain competitor's behavior beyond physical location decisions. For example, it can be used to analyze:

a) Where politicians choose to "locate" within a political spectrum (from liberal to conservative).
b) Where firms choose to "locate" their products within a spectrum of tastes (they choose the degree of product differentiation). For example, where they choose to locate within the spectrum of "sweetness" of cereals.

- The model helps explain why it is rational for firms (or politicians) to make their products as similar as possible (why they choose to locate at the center of the beach).

- **A Partnership Game of Strategic Complementarities.-** Suppose two people enter into a partnership and form a firm. The firm's profit depends on the effort exerted by both.
- Let x denote the effort exerted by the first partner and let y denote the effort by the second partner.
- Suppose the profits for the firm are given by:

$$\boldsymbol{p = 4 \cdot (x + y + c \cdot xy), \text{ where } 0 \leq c \leq \frac{1}{4}}$$

- The **coefficient "$\boldsymbol{c}$" measures the degree of complementarity** between the effort of both partners.
- Effort is costly. Suppose the cost for player 1 of exerting effort level x is given by $\boldsymbol{x^2}$, and the cost for player 2 of exerting effort level y is $\boldsymbol{y^2}$.

- Suppose profits are split 50/50 between both partners. Then the payoffs to each for a strategy profile (x, y) are given by:

$$u_1(x, y) = \frac{1}{2} \times 4 \cdot (x + y + c \cdot xy) - x^2$$

$$u_2(x, y) = \frac{1}{2} \times 4 \cdot (x + y + c \cdot xy) - y^2$$

- Suppose any binding contract for “effort” cannot be enforced because it is impossible to verify effort accurately.
- Therefore suppose we model this as a **noncooperative game** where both partners **simultaneously decide how much effort to exert**.

- Suppose effort by each player can take values in the range $[0,4]$. Then the strategy spaces for both players are:

$$S_1 = [0,4] \quad \text{and} \quad S_2 = [0,4]$$

- **Question: Using iterared dominance, characterize the space of rationalizable strategies "R" in this game.**

- **Note first of all, that this is a game where the strategy space is <u>continuous</u> and therefore does not have a matrix-form representation.**

- **First Step:** As always in iterated dominance, the first step consists of identifying all the strategies that are dominated.

- This means, identify all strategies that are not best-responses for any set of beliefs.

- Thus, we first need to characterize how best responses look like in this game.

- We already showed that payoff functions are:

$$u_1(x, y) = \frac{1}{2} \times 4 \cdot (x + y + c \cdot xy) - x^2$$
$$u_2(x, y) = \frac{1}{2} \times 4 \cdot (x + y + c \cdot xy) - y^2$$

- **Best responses have to do with beliefs. In this case, with the expected effort by the other player.** Let:

$$\bar{y} = Beliefs\ by\ player\ 1\ about\ the\ expected$$
$$effort\ \ exerted\ by\ player\ 2.$$
$$\bar{x} = Beliefs\ by\ player\ 2\ about\ the\ expected$$
$$effort\ \ exerted\ by\ player\ 1.$$

- Therefore, given beliefs $\bar{y}$ by player 1, his expected payoff function of exerting effort x is given by:

$$u_1(x, \bar{y}) = \frac{1}{2} \times 4 \cdot (x + \bar{y} + c \cdot x\bar{y}) - x^2$$

- And given beliefs $\bar{x}$ by player 2, his expected payoff function of exerting effort y is given by:

$$u_2(\bar{x}, y) = \frac{1}{2} \times 4 \cdot (\bar{x} + y + c \cdot \bar{x}y) - y^2$$

- **Players' best responses are the effort values that maximize their expected payoffs given their beliefs.**
- **<u>In games with continuous strategies</u>**, we characterize best-responses by solving the **<u>first-order conditions</u>** for the maximization of players' expected payoff functions.

- The first order conditions for player 1 require finding the solution (in x) to:

$$\frac{du_1(x,\bar{y})}{dx} = 0$$

- We have:

$$\frac{du_1(x,\bar{y})}{dx} = 2 \cdot (1 + c \cdot \bar{y}) - 2 \cdot x$$

- Therefore, solving $\frac{du_1(x,\bar{y})}{dx} = 0$ in x yields:

$$\boldsymbol{x = 1 + c \cdot \bar{y}}$$

- This is the effort level that maximizes player 1's expected payoff given beliefs $\bar{y}$. Therefore,

$$\boldsymbol{BR_1(\bar{y}) = 1 + c \cdot \bar{y}}$$

- Similarly, the first order conditions for player 2 require finding the solution (in y) to:

$$\frac{du_2(\bar{x}, y)}{dy} = 0$$

- We have:

$$\frac{du_2(\bar{x}, y)}{dy} = 2 \cdot (1 + c \cdot \bar{x}) - 2 \cdot y$$

- Therefore, solving $\frac{du_2(\bar{x},y)}{dy} = 0$ in y yields:

$$\boldsymbol{y = 1 + c \cdot \overline{x}}$$

- This is the effort level that maximizes player 2's expected payoff given beliefs $\bar{x}$. Therefore,

$$\boldsymbol{BR_2(\overline{x}) = 1 + c \cdot \overline{x}}$$

- Thus, best response functions are given by:

$$\boldsymbol{BR_1(\overline{y}) = 1 + c \cdot \overline{y}}$$

$$\boldsymbol{BR_2(\overline{x}) = 1 + c \cdot \overline{x}}$$

- This is a game of <u>strategic complements</u> because each player's best response is an <u>increasing function</u> of the other player's own strategy.
- Higher effort by the other player makes it optimal for each player to exert more effort themselves.

- Characterizing the set of dominated strategies for player 1 requires us to find the range of effort values "x" that cannot be best-responses to any set of beliefs $\bar{y}$.

- That is, the set of dominated strategies for player 1 is:

$$\{x \in [0,4]\ such\ that\ \nexists\ \bar{y} \in [0,4]\ such\ that\ x = 1 + c \cdot \bar{y}\}$$

- Similarly, the set of dominated strategies for player 2 is:

$$\{y \in [0,4]\ such\ that\ \nexists\ \bar{x} \in [0,4]\ such\ that\ y = 1 + c \cdot \bar{x}\}$$

- What is the <u>smallest possible value</u> of $BR_1(\bar{y})$?
- Remember that effort can only take values in the interval [0,4]. Therefore, $\bar{y} \in [0,4]$.
- Next recall that

$$BR_1(\bar{y}) = 1 + c \cdot \bar{y}$$

- Therefore, since $c \geq 0$, the smallest possible value of $BR_1(\bar{y})$ is:

$$1 + c \cdot 0 = 1$$

- What is the <u>largest possible value</u> of $BR_1(\bar{y})$?

$$1 + c \cdot 4$$

- Therefore, best responses for player 1 must satisfy:

$$1 \leq BR_1(\bar{y}) \leq 1 + c \cdot 4$$

- Player 2 has the analogous best-response function:

$$BR_2(\bar{x}) = 1 + c \cdot \bar{x}$$

- Since $\bar{x}$ also ranges in the [0,4] interval, we also have:

$$1 \leq BR_2(\bar{x}) \leq 1 + c \cdot 4$$

- Therefore, the reduced game R^1 is given by:

$$R^1 = [1, 1 + c \cdot 4] \times [1, 1 + c \cdot 4]$$

- In the next step of iterated dominance, we once again identify the range of best-responses but this time in the reduced game R^1.

- What is the <u>smallest possible value</u> of $BR_1(\bar{y})$ in the reduced game R^1?
- In R^1, effort can only take values in the interval $[1, 1 + c \cdot 4]$. Therefore, $\bar{y} \in [1, 1 + c \cdot 4]$.
- Therefore, the smallest possible value of $BR_1(\bar{y})$ in the reduced game R^1 is:

$$1 + c \cdot 1 = 1 + c$$

- What is the <u>largest possible value</u> of $BR_1(\bar{y})$ in the reduced game R^1?

$$1 + c \cdot (1 + c \cdot 4) = 1 + c + c^2 \cdot 4$$

- Therefore, best responses for player 1 in the reduced game R^1 must satisfy:

$$1 + c \leq BR_1(\bar{y}) \leq 1 + c + c^2 \cdot 4$$

- By symmetry, we obtain the analogous result for player 2. Namely:

$$1 + c \leq BR_2(\bar{x}) \leq 1 + c + c^2 \cdot 4$$

- And therefore the reduced game R^2 is given by:

$$R^2 = [1 + c, 1 + c + c^2 \cdot 4] \times [1 + c, 1 + c + c^2 \cdot 4]$$

- In the next step of iterated dominance, we once again identify the range of best-responses but this time in the reduced game R^2.

- What is the smallest possible value of $BR_1(\bar{y})$ in the reduced game R^2?
- In R^2, effort can only take values in the interval $[1 + c, 1 + c + c^2 \cdot 4]$. Therefore,
$$\bar{y} \in [1 + c, 1 + c + c^2 \cdot 4].$$
- Therefore, the smallest possible value of $BR_1(\bar{y})$ in the reduced game R^1 is:
$$1 + c \cdot (1 + c) = 1 + c + c^2$$
- What is the largest possible value of $BR_1(\bar{y})$ in the reduced game R^2?
$$1 + c \cdot (1 + c + c^2 \cdot 4) = 1 + c + c^2 + c^3 \cdot 4$$
- Therefore, best responses for player 1 in the reduced game R^2 must satisfy:
$$1 + c + c^2 \leq BR_1(\bar{y}) \leq 1 + c + c^2 + c^3 \cdot 4$$

- By symmetry, we obtain the analogous result for player 2. Namely:

$$1 + c + c^2 \leq BR_2(\bar{x}) \leq 1 + c + c^2 + c^3 \cdot 4$$

- And therefore the reduced game R^3 is given by:

$$R^3 = [1 + c + c^2, 1 + c + c^2 + c^3 \cdot 4] \times [1 + c + c^2, 1 + c + c^2 + c^3 \cdot 4]$$

- In the next step of iterated dominance, we once again identify the range of best-responses but this time in the reduced game R^3…

- From here, **a pattern clearly emerges**: After k steps of deletion of dominated strategies, the best-responses by players 1 and 2 must lie in the interval:

$$1 + c + c^2 + \cdots + c^{k-1} \leq BR_1\ (\bar{y}) \\ \leq 1 + c + c^2 + \cdots + c^{k-1} + 4 \cdot c^k$$

and

$$1 + c + c^2 + \cdots + c^{k-1} \leq BR_2\ (\bar{x}) \\ \leq 1 + c + c^2 + \cdots + c^{k-1} + 4 \cdot c^k$$

- Both of these bounds have the form of **geometric series**. Using the properties of geometric series, we can simplify:

$$1 + c + c^2 + \cdots + c^{k-1} = \frac{1 - c^k}{1 - c}$$

- Therefore, after k rounds of deletion of dominated strategies, we have:

$$\frac{1 - c^k}{1 - c} \leq BR_1\left(\bar{y}\right) \leq \frac{1 - c^k}{1 - c} + 4 \cdot c^k$$

$$\frac{1 - c^k}{1 - c} \leq BR_2\left(\bar{x}\right) \leq \frac{1 - c^k}{1 - c} + 4 \cdot c^k$$

- In this particular game, finding the set of rationalizable strategies R requires that we continue with the process of deleting dominated strategies indefinitely. That is, we must let $\boldsymbol{k \to \infty}$.
- Since strategic complementarity coefficient "c" is assumed to satisfy $c < 1$ (we assumed $c \leq \frac{1}{4}$ at the beginning), we have

$$\lim_{k\to\infty} c^k = 0$$

- And therefore:

$$\lim_{k\to\infty} \frac{1 - c^k}{1 - c} = \frac{1}{1 - c}$$

and

$$\lim_{k\to\infty} \left(\frac{1 - c^k}{1 - c} + 4 \cdot c^k \right) = \frac{1}{1 - c}$$

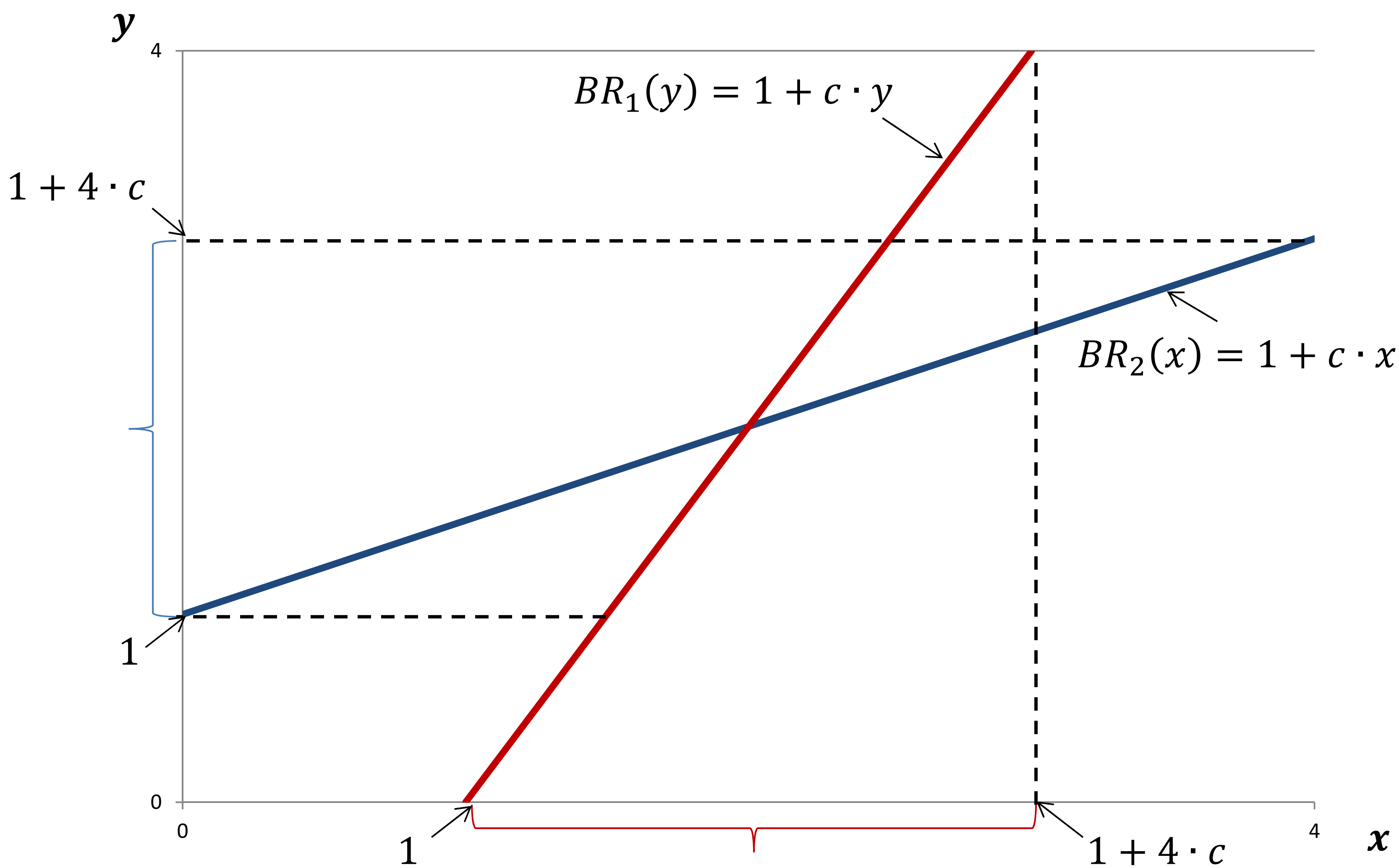
- Therefore, as $k \to \infty$, the only best response for either player is

$$\frac{1}{1-c}$$

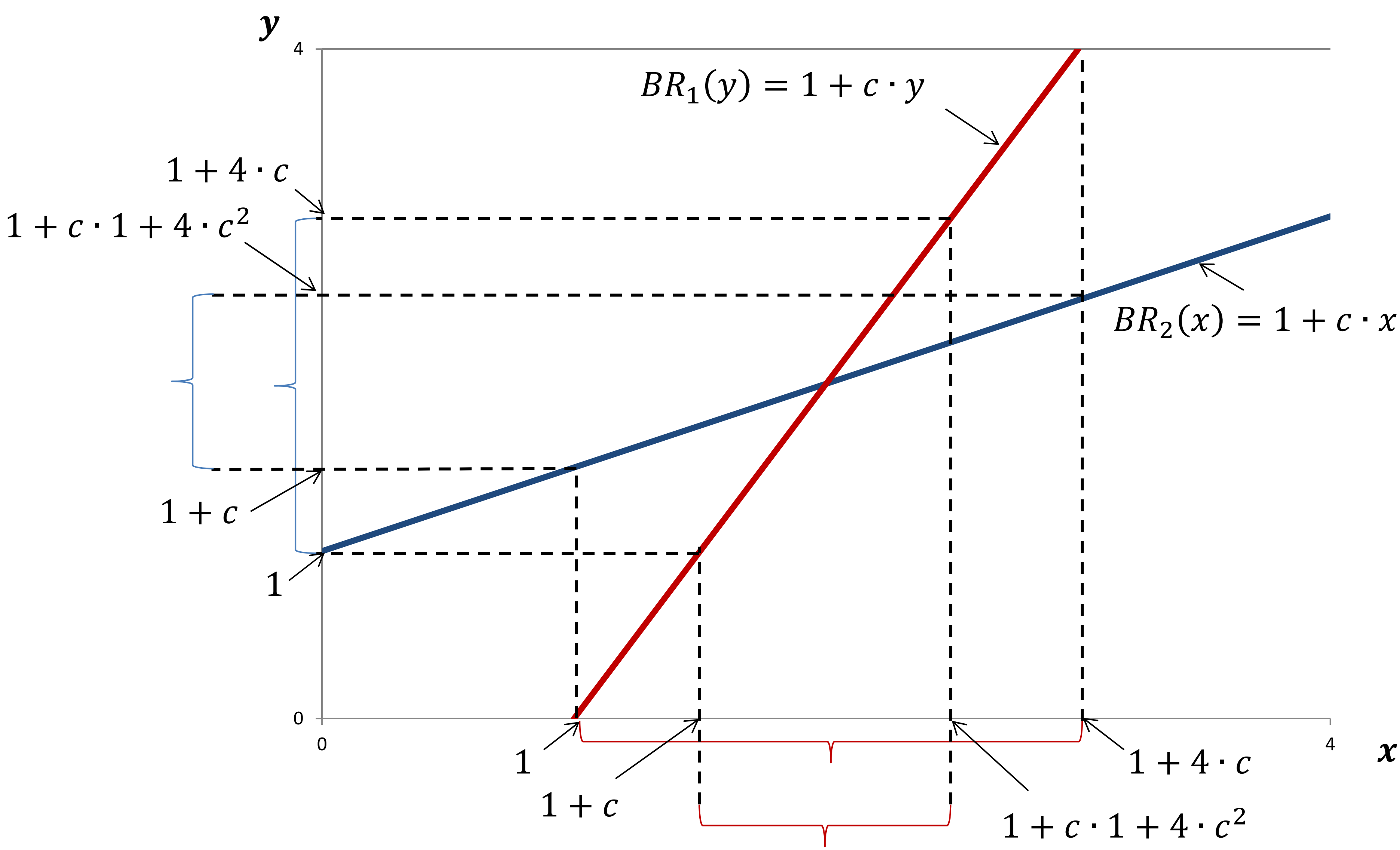
- Consequently, the set of rationalizable strategies *R* is given simply by the unique strategy profile:

$$(\boldsymbol{x}, \boldsymbol{y}) = \left\{\left(\frac{\mathbf{1}}{\mathbf{1}-\boldsymbol{c}}, \frac{\mathbf{1}}{\mathbf{1}-\boldsymbol{c}}\right)\right\}$$

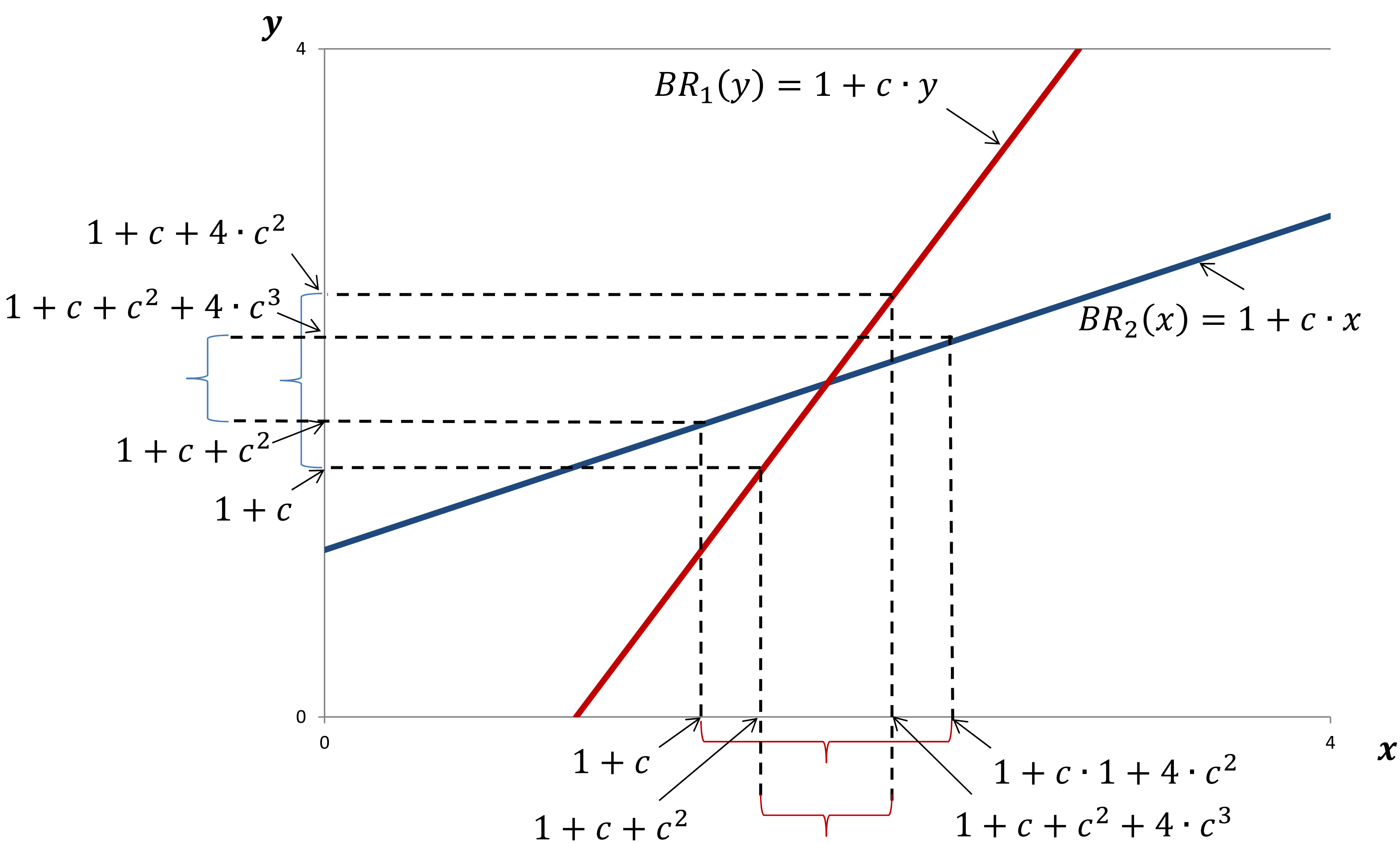
- Graphically: **Step k=1** of deleting dominated strategies:



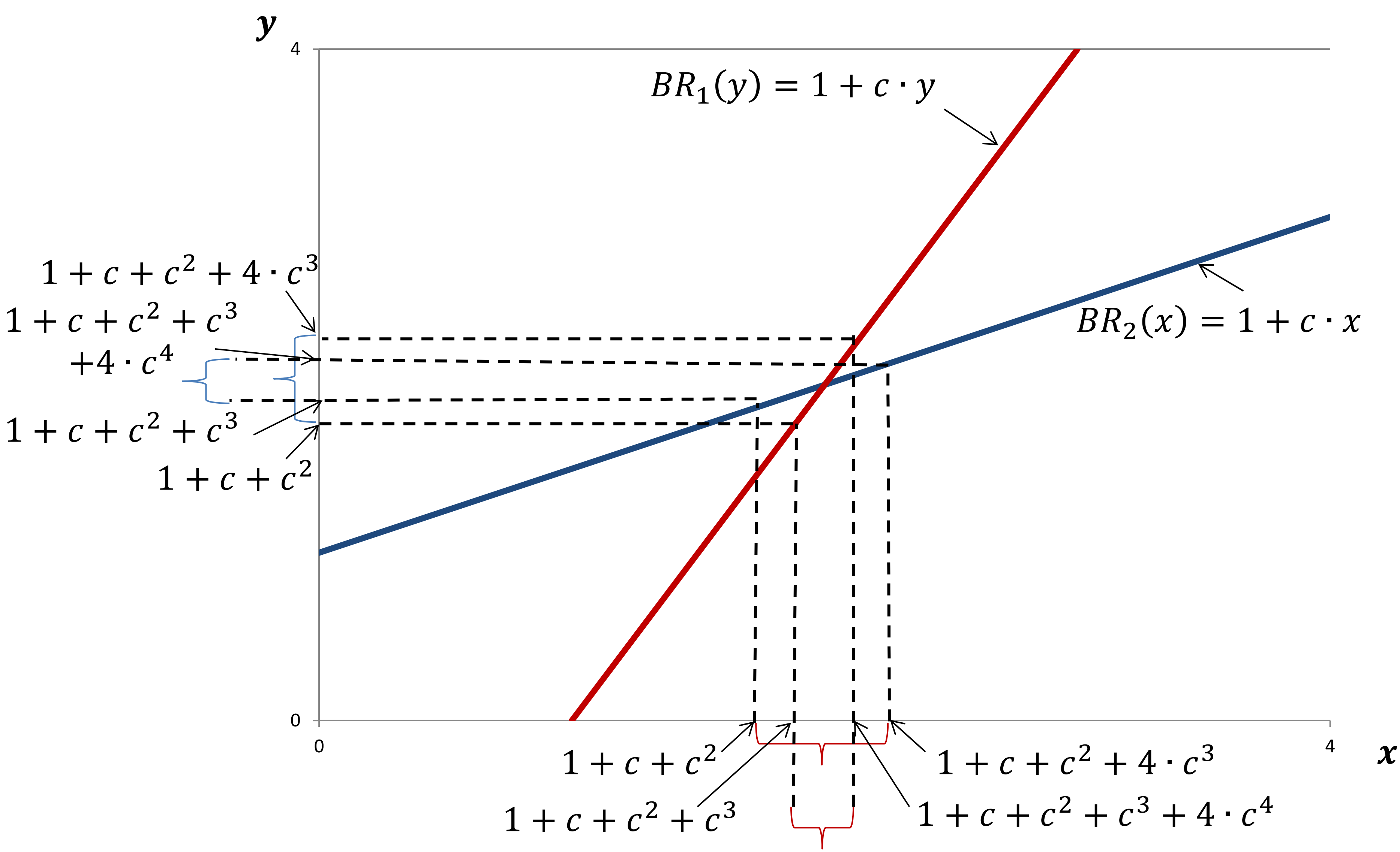
- **Step k=2** of deleting dominated strategies:



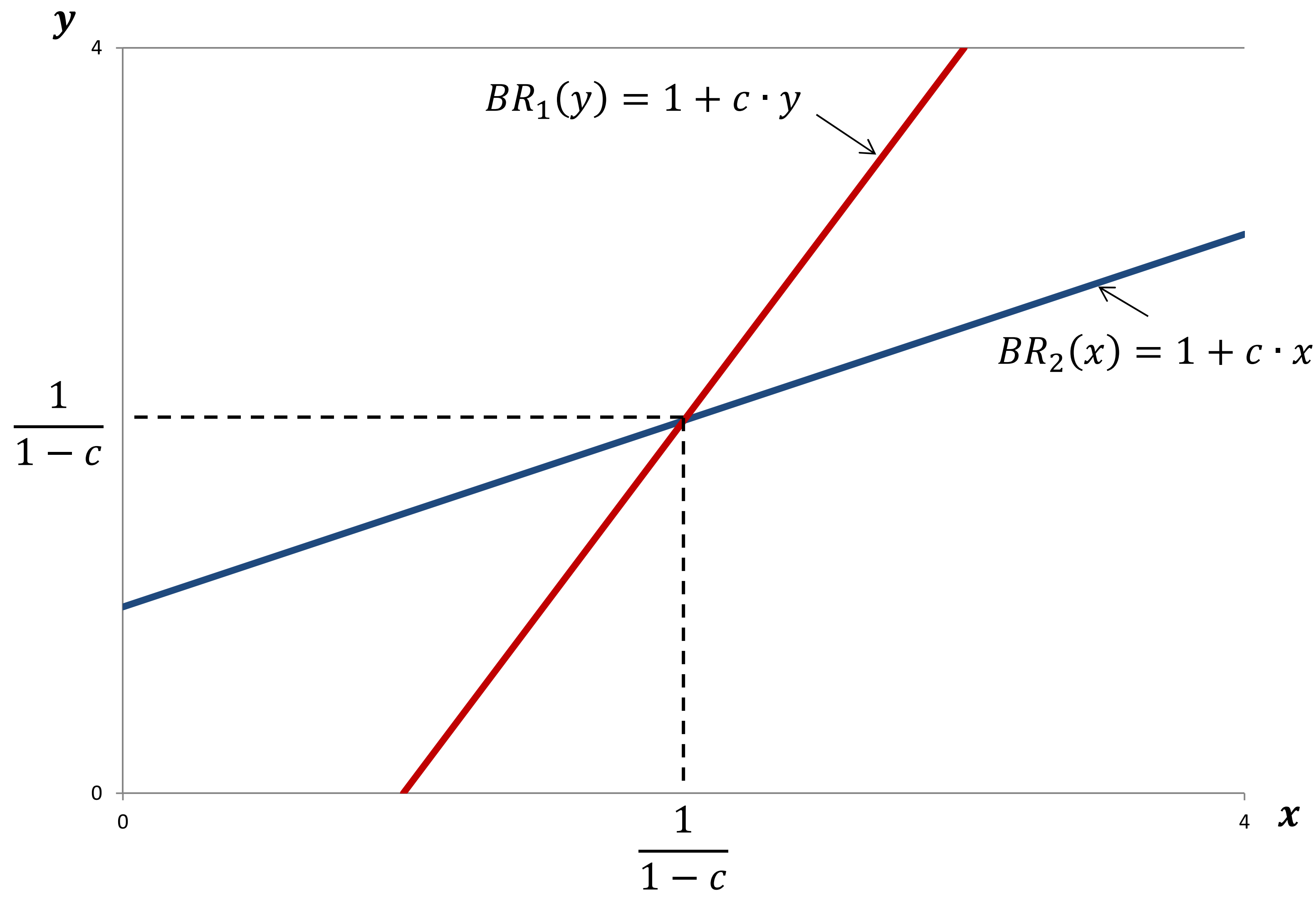
- **Step k=3** of deleting dominated strategies:



- **Step k=4** of deleting dominated strategies:



- **As we let $\boldsymbol{k} \to \infty$**, this iterative process converges to the single point where these two curves cross each other:



- The previous example highlights what the textbook calls the "**first strategic tension**", which (recall) is identified as ***the clash between individual and group interests***.
- **"First strategic tension":** Individually rational behavior may lead to **inefficient outcomes**.
- Consider the previous example, where the only rationalizable strategy profile is

$$(x^*, y^*) = \left\{\left(\frac{1}{1-c}, \frac{1}{1-c}\right)\right\}$$

- The corresponding payoff earned by each player is given by

$$u_1(x^*, y^*) = u_2(x^*, y^*) = \frac{3 - 2 \cdot c}{(1 - c)^2}$$

- As it turns out, $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ **is an inefficient outcome**.
- To see this, consider the profile of efforts (x, y) that maximizes the total payoffs of both players. That is, the one that **maximizes the total net profits of the firm**.
- The total net profit of the firm is given by:

$$\boldsymbol{4 \cdot (x + y + c \cdot xy) - x^2 - y^2}$$

- Taking partial derivatives with respect to x and with respect to y and making both equal to zero (i.e, solving the first order conditions), we obtain

$$\hat{x} = \frac{2}{1-2\cdot c} \quad \text{and} \quad \hat{y} = \frac{2}{1-2\cdot c}$$

- The profile $(\hat{x}, \hat{y})$ maximizes the total net profit of the firm. This profile constitutes the **jointly optimal effort level** in this game.

- Suppose players' effort corresponds to $(\hat{x}, \hat{y})$. Their net payoff corresponds to ½ of the profits, minus the cost of exerting the effort. That is,

$$u_1(\hat{x}, \hat{y}) = \left(\frac{1}{2}\right) \cdot 4 \cdot (\hat{x} + \hat{y} + c \cdot \hat{x}\hat{y}) - \hat{x}^2$$

$$u_2(\hat{x}, \hat{y}) = \left(\frac{1}{2}\right) \cdot 4 \cdot (\hat{x} + \hat{y} + c \cdot \hat{x}\hat{y}) - \hat{y}^2$$

- Both of these simplify to:

$$u_1(\hat{x}, \hat{y}) = u_2(\hat{x}, \hat{y}) = \frac{4}{1 - 2 \cdot c}$$

- It is not hard to verify that

$u_1(x^*, y^*) < u_1(\hat{x}, \hat{y})$ and $u_2(x^*, y^*) < u_2(\hat{x}, \hat{y})$

Therefore, the outcome (x^*, y^*) is **inefficient**.

- Note that

$$\hat{x} > x^* \quad \text{and} \quad \hat{y} > y^*$$

- So the jointly optimal effort levels are higher than the ones produced by individual rationality.

- Because each player cares only about their own cost and benefit, they fail to internalize the effect that their effort has on the payoff of the other player.

- **Cournot Example (continued):** Characterize the set of rationalizable strategies *R* in the Cournot example of previous chapters.

- Payoff functions are

$$u_1(q_1, q_2) = (80 - 2 \cdot q_1 - 2 \cdot q_2) \cdot q_1$$
$$u_2(q_1, q_2) = (80 - 2 \cdot q_1 - 2 \cdot q_2) \cdot q_2$$

- Like the previous example, this is a game with **continuous strategies**. Therefore, to identify dominated strategies we need to characterize the **best response functions** for each player.

- **Best response functions in Cournot game:** We obtain them once again through the first-order conditions:

1) **For player 1:** Fix q_2 and solve the first order conditions with respect to q_1. That is, solve:

$$\frac{\partial u_1(q_1, q_2)}{\partial q_1} = 0$$

2) **For player 2:** Fix q_1 and solve the first order conditions with respect to q_2. That is, solve:

$$\frac{\partial u_2(q_1, q_2)}{\partial q_2} = 0$$

- We have:

$$\frac{\partial u_1(q_1, q_2)}{\partial q_1} = 80 - 4 \cdot q_1 - 2 \cdot q_2$$

- Therefore, first order conditions for player 1 are:

$$80 - 4 \cdot q_1 - 2 \cdot q_2 = 0$$

- Solving with respect to q_1, we obtain:

$$q_1 = 20 - \frac{1}{2} \cdot q_2$$

- That is, the best response for player 1 if player 2 produces q_2 is given by:

$$\boldsymbol{BR_1(q_2) = 20 - \frac{1}{2} \cdot q_2}$$

- Similarly, for player 2 we have:

$$\frac{\partial u_2(q_1, q_2)}{\partial q_2} = 80 - 2 \cdot q_1 - 4 \cdot q_2$$

- Therefore, first order conditions for player 2 are:

$$80 - 2 \cdot q_1 - 4 \cdot q_2 = 0$$

- Solving with respect to q_2, we obtain:

$$q_2 = 20 - \frac{1}{2} \cdot q_1$$

- That is, the best response for player 2 if player 1 produces q_1 is given by:

$$\boldsymbol{BR_2(q_1) = 20 - \frac{1}{2} \cdot q_1}$$

- So, players' best response functions are:

$$\boldsymbol{BR_1(q_2) = 20 - \frac{1}{2} \cdot q_2}$$
$$\boldsymbol{BR_2(q_1) = 20 - \frac{1}{2} \cdot q_1}$$

- This is a game of strategic substitutes, which are those where players' best responses are decreasing functions of the strategy of the other player.

- A higher production level by player 2 drives down market price, making it optimal to decrease the quantity produced by player 1, and viceversa.

- **Iterated dominance:** To characterize the set of rationalizable strategies, we proceed as always by first identifying the set of dominated strategies.
- **Suppose we add the assumption that production can only take values in the interval $[\mathbf{0}, \mathbf{30}]$.**
- What are the smallest and largest possible values of $BR_1(q_2)$?
- Since it is a decreasing function of q_2, the smallest possible value of $BR_1(q_2)$ is:

$$20 - \frac{1}{2} \cdot 30 = 5$$

- And the largest possible value of $BR_1(q_2)$ is:

$$20 - \frac{1}{2} \cdot 0 = 20$$

- Therefore,

$$BR_1(q_2) \in [5, 20]$$

- Any production value outside of this interval is a dominated strategy.
- By symmetry, the same is true for player 2:

$$BR_2(q_1) \in [5, 20]$$

- Therefore, the reduced game R^1 is given by:

$$R^1 = [5, 20] \times [5, 20]$$

- In the next step, we look for dominated strategies in the reduced game R^1.

- The smallest possible value of $BR_1(q_2)$ in the reduced game R^1 is:

$$20 - \frac{1}{2} \cdot 20 = 10$$

- And the largest possible value of $BR_1(q_2)$ in the reduced game R^1 is:

$$20 - \frac{1}{2} \cdot 5 = \frac{35}{2} = 17.5$$

- Therefore, in the reduced game R^1 we have:

$$BR_1(q_2) \in [10, 17.5]$$

- And by symmetry we have the same for player 2:

$$BR_2(q_1) \in [10, 17.5]$$

- Therefore, the reduced game R^2 is given by:
$$R^2 = [10, 17.5] \times [10, 17.5]$$
- We continue by looking for the smallest and largest possible best responses in R^2 and so on…
- As in the previous example of the strategic complements game, this iterative process can continue indefinitely… **as we let $\boldsymbol{k \to \infty}$, the reduced game $\boldsymbol{R^k}$converges to a unique strategy profile, which is the profile $\boldsymbol{\left(q^*{}_1, q^*{}_2\right)}$ where $\boldsymbol{BR_1(q_2)}$ and $\boldsymbol{BR_2(q_1)}$ cross each other. This is given by the profile$\boldsymbol{\left(q^*{}_1, q^*{}_2\right)}$such that:**
$$\boldsymbol{BR_1\left(q^*{}_2\right) = q^*{}_1} \text{ and } \boldsymbol{BR_2\left(q^*{}_1\right) = q^*{}_2}$$

- That is, the profile $\left(q^*{}_1, q^*{}_2\right)$ such that:

$$20 - \frac{1}{2} \cdot q^*{}_2 = q^*{}_1$$

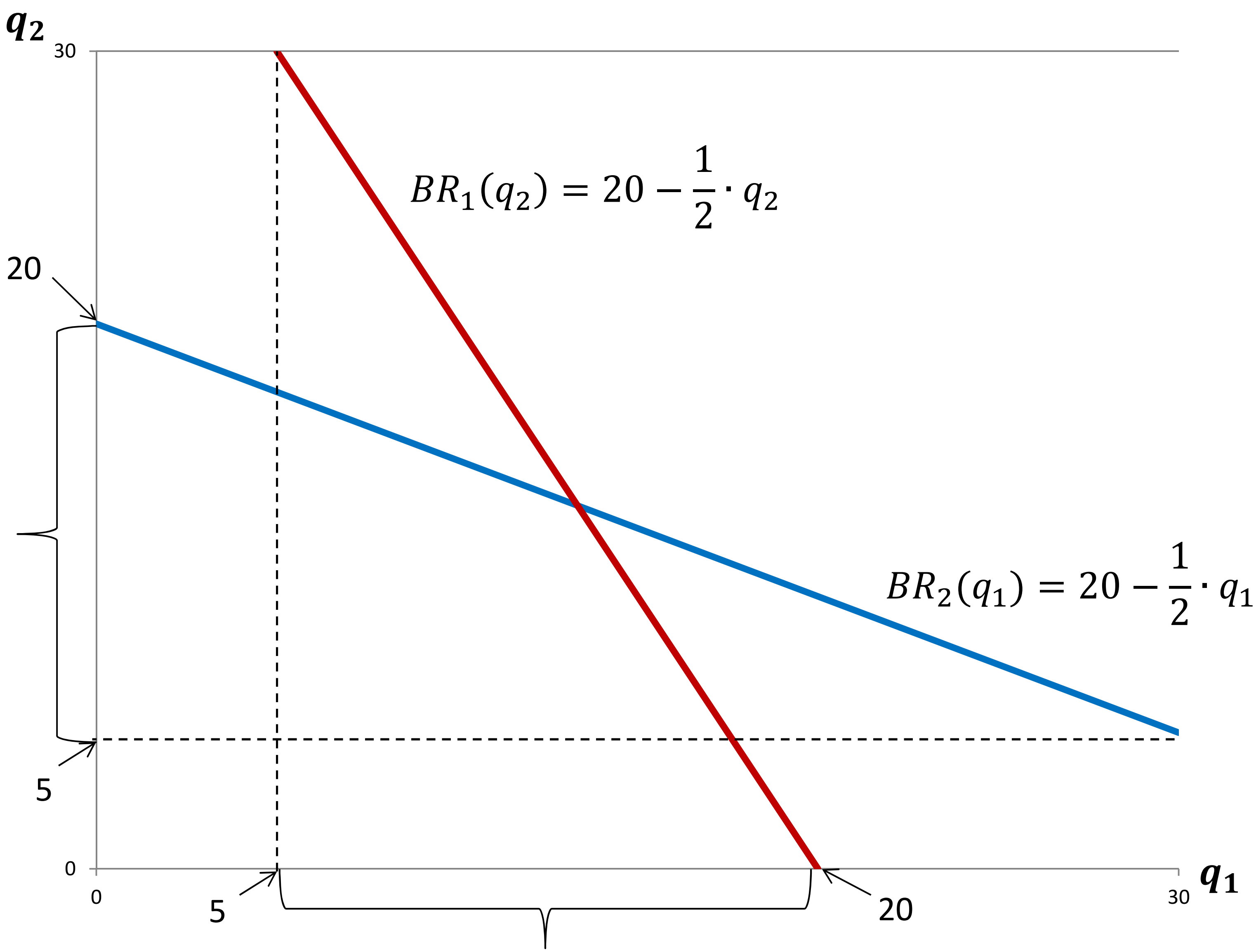
and

$$20 - \frac{1}{2} \cdot q^*{}_1 = q^*{}_2$$

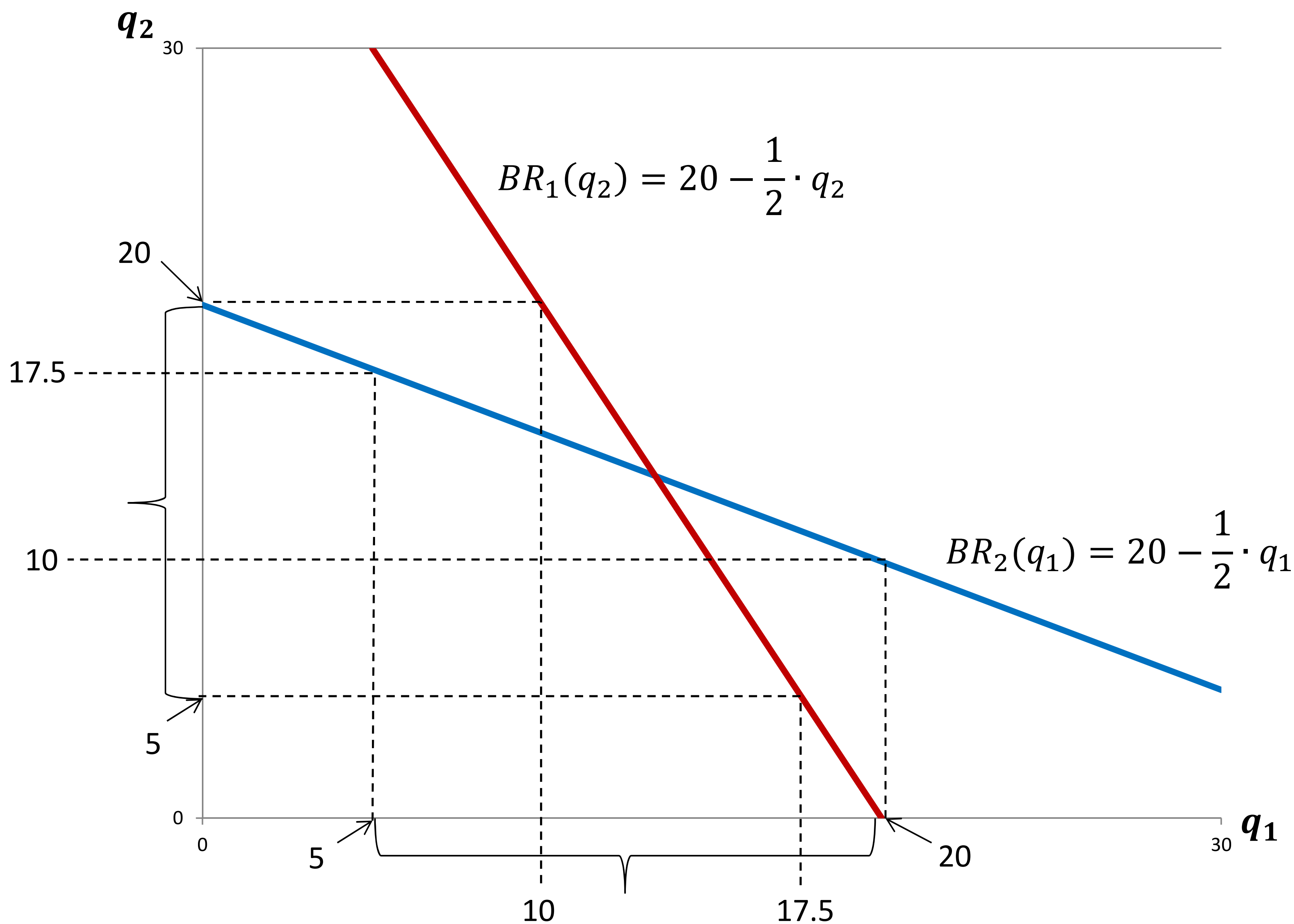
- This yields $\left(q^*{}_1, q^*{}_2\right) = \left(\frac{40}{3}, \frac{40}{3}\right)$.
- The set of rationalizable strategies in this game is given by the unique strategy profile:

$$R = \left\{\left(\frac{40}{3}, \frac{40}{3}\right)\right\}$$

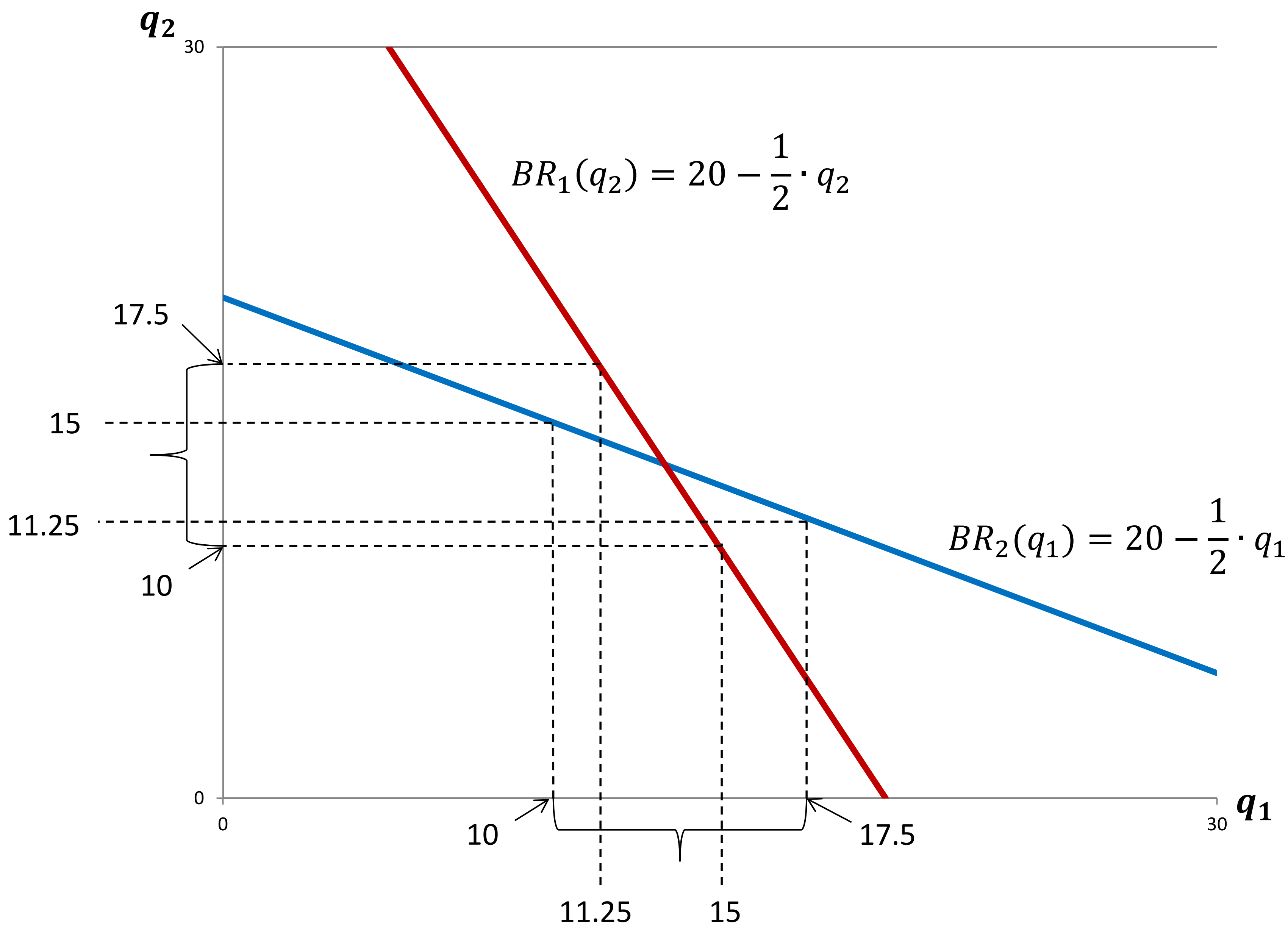
- Graphically: Step 1, obtaining R^1



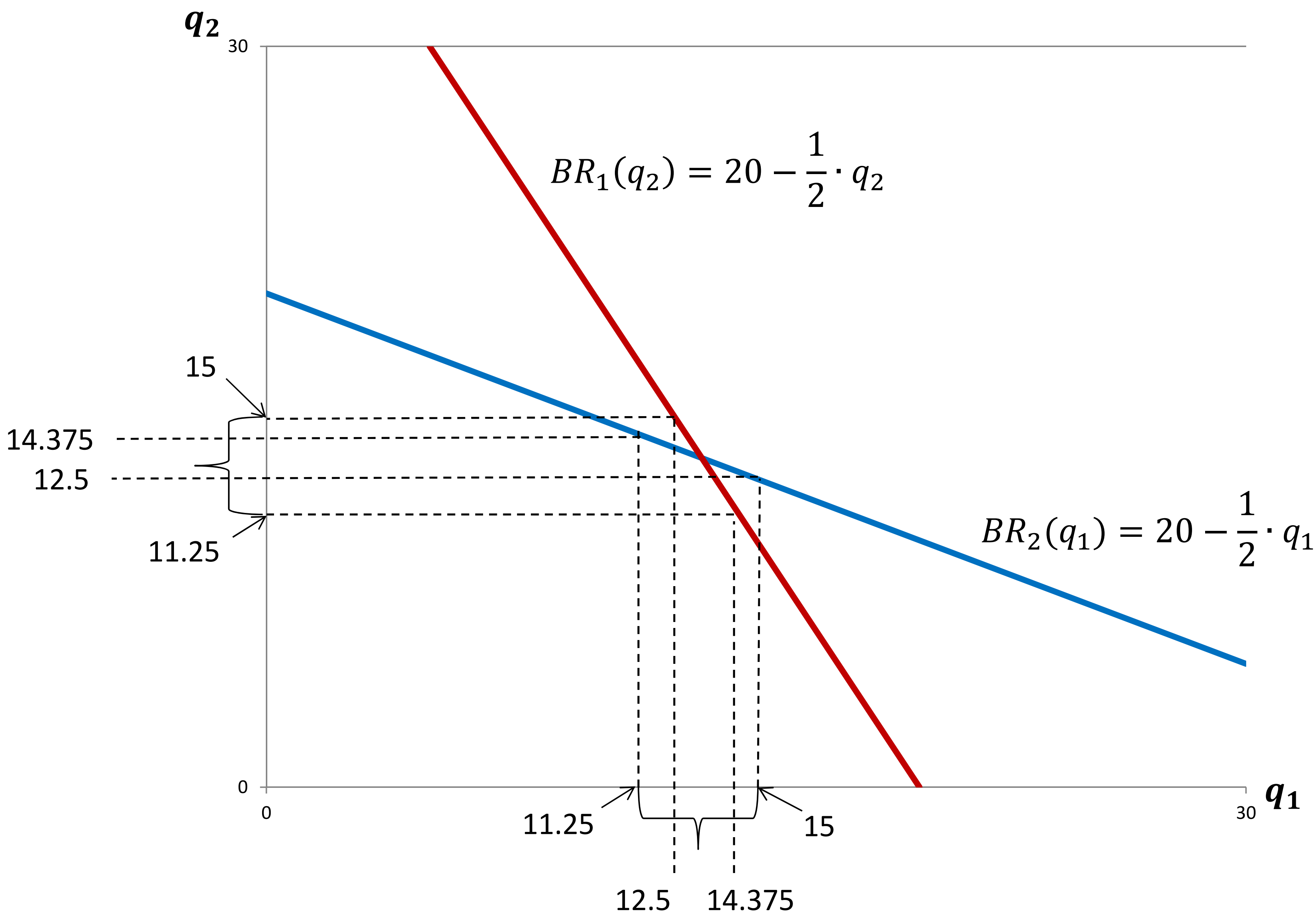
- Step 2, obtaining R^2



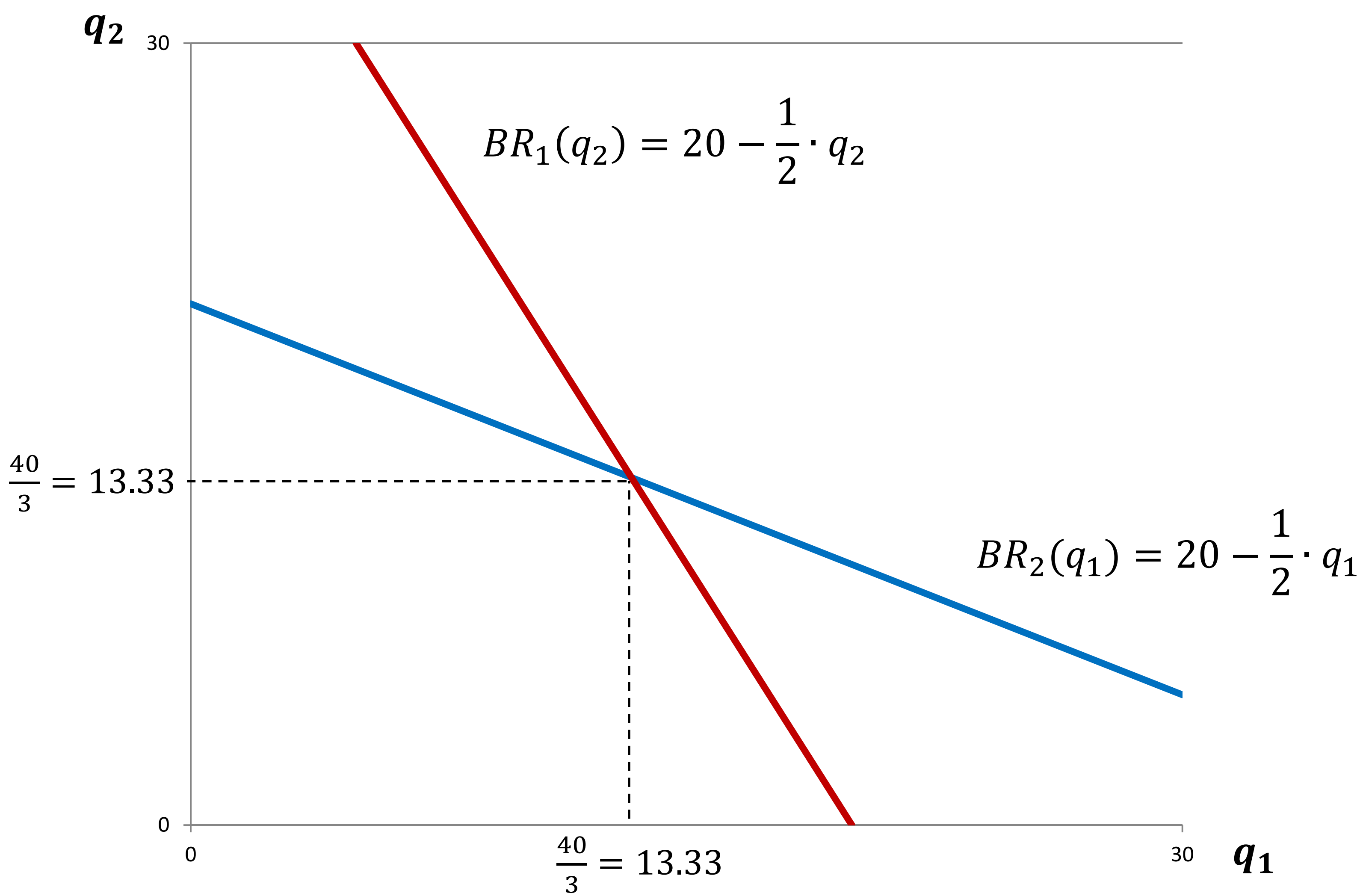
- Step 3, obtaining R^3



- Step 4, obtaining R^4



- As we continue and let $k \to \infty$, the set R^k collapses to a the point where both best-response functions cross each other:



- **A Game of Social Unrest:** In recent years we have seen governments being toppled in various parts of the world as a result of massive revolts.

- In many cases, the people revolting had been living under poor conditions for a long time. So, why did these revolts occur suddenly only recently? Why did it happen in so many countries at the same time?

- We can gain some insight through a simple game.

- Participating in a revolt carries a **risk.** The amount of risk incurred depends on the number of other people who also participate.
- Why? Because unsuccessful revolts carry the risk of jail and execution to those who participated.
- The likelihood of having a successful revolt depends on the number of people who participate.

- Consider a game where there is a large population. For simplicity, suppose the population is a continuum, and let’s represent this as the interval $[0, 1]$.

- Suppose individuals in the population are distributed uniformly in the interval $[0, 1]$.

- Therefore, each player i belongs in the interval $[0, 1]$ which is denoted as $i \in [0, 1]$.

- Each player $i \in [0, 1]$ has two possible strategies:
  - Protest (denoted as “P”)
  - Stay home (denoted as “H”)
- Suppose the degree of “commitment to the cause” of player i is measured by where i belongs in the interval $[0, 1]$.
- More precisely, suppose that if i is located close to zero, he has a weak dedication to the cause, while if he is located closer to 1, he is very dedicated (a zealot).

- Player i's payoff of protesting depends on:
  - His degree of dedication to the cause (i.e, the value of i).
  - The proportion of the population who participates in the revolt.
  - Let x denote the proportion of the population who participate in the revolt.
- Suppose we assume the following about player i's payoffs:
  1. Whether he protests or not, each i gets a payoff of $\mathbf{4 \cdot x - 2}$ from the current protest movement.
  2. If player i decides to participate, he gets an additional payoff of $\mathbf{4 \cdot x - 2 + \alpha \cdot i}$, where $\boldsymbol{\alpha > 0}$

- The term $\alpha \cdot i$ represents the satisfaction obtained from protesting which depends on i's dedication to the cause.

- For simplicity, let us assume that $\boldsymbol{\alpha = 3}$ (the main features we will describe will hold for any $\boldsymbol{\alpha > 2}$)

- Therefore, we have:

$$\boldsymbol{u_i(H, x) = 4 \cdot x - 2}$$
$$\boldsymbol{u_i(P, x) = 8 \cdot x - 4 + 3 \cdot i}$$

- Let $\bar{x}$ denote player i's belief about the proportion of people who will participate in the revolt. Player i will decide to participate if and only if

$$u_i(P, \bar{x}) \geq u_i(H, \bar{x})$$

- That is, player i will decide to participate if and only if

$$8 \cdot \bar{x} - 4 + 3 \cdot i \geq 4 \cdot \bar{x} - 2$$

- Rearranging, this can be expressed as:

$$i \geq \frac{2 - 4 \cdot \bar{x}}{3}$$

- Therefore, player i will participate if and only if

$$i \geq \frac{2 - 4 \cdot \bar{x}}{3}$$

- **Characterize the set of rationalizable strategies $\boldsymbol{R}$ in this game…**
- As before, we will use iterated dominance.
- Therefore, the first step is to identify the set of dominated strategies.
- How do we do it in this game?
  - This game has only two strategies: P and H.
  - Beliefs must be well-defined. Therefore: $\bar{x} \in [0,1]$.
  - P will be a dominated strategy if:

$$u_i(P,\bar{x}) \leq u_i(H,\bar{x}) \text{ for all } \bar{x} \in [0,1].$$

  - H will be a dominated strategy if:

$$u_i(P,\bar{x}) > u_i(H,\bar{x}) \text{ for all } \bar{x} \in [0,1].$$

- Therefore:
- P will be a dominated strategy for player i if the value of i is such that:

$$i \leq \frac{2-4\cdot\bar{x}}{3} \quad \text{for all} \quad \bar{x} \in [0,1].$$

- H will be a dominated strategy for player i if the value of i is such that:

$$i > \frac{2-4\cdot\bar{x}}{3} \quad \text{for all} \quad \bar{x} \in [0,1].$$

- So, we need to look for:
  - The smallest value of $\frac{2-4\cdot\bar{x}}{3}$ such that $\bar{x} \in [0,1]$.
  - The largest value of $\frac{2-4\cdot\bar{x}}{3}$ such that $\bar{x} \in [0,1]$.

- The smallest value of $\frac{2-4\cdot\bar{x}}{3}$ such that $\bar{x} \in [0, 1]$ is given by $-\frac{2}{3}$

- The largest value of $\frac{2-4\cdot\bar{x}}{3}$ such that $\bar{x} \in [0, 1]$ is given by $\frac{2}{3}$

- Therefore:
- P will be a dominated strategy for player i if the value of i is such that:

$$i < -\frac{2}{3}$$

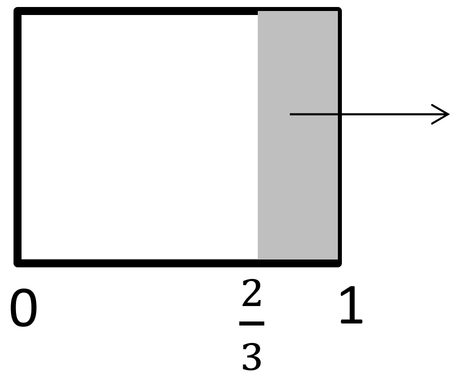
- H will be a dominated strategy for player i if the value of i is such that:

$$i \geq \frac{2}{3}$$

- Recall that $i \in [0,1]$. Therefore, we cannot have $i < -\frac{2}{3}$. Consequently, **P cannot be a dominated strategy for any player.**

- **Can H be a dominated strategy?** Recall that this will occur for player i if $i \geq \frac{2}{3}$.

- Therefore: H (staying at home) is a dominated strategy for every player $i \geq \frac{2}{3}$. On the other hand, P is not a dominated strategy for any player.



Participating is a dominant strategy for all these players. These are the "zealots".

- **Step 2 of deletion of dominated strategies:** Rational players know that every player i such that $i \geq \frac{2}{3}$ will participate in the revolt.

- Therefore, beliefs must reflect this.

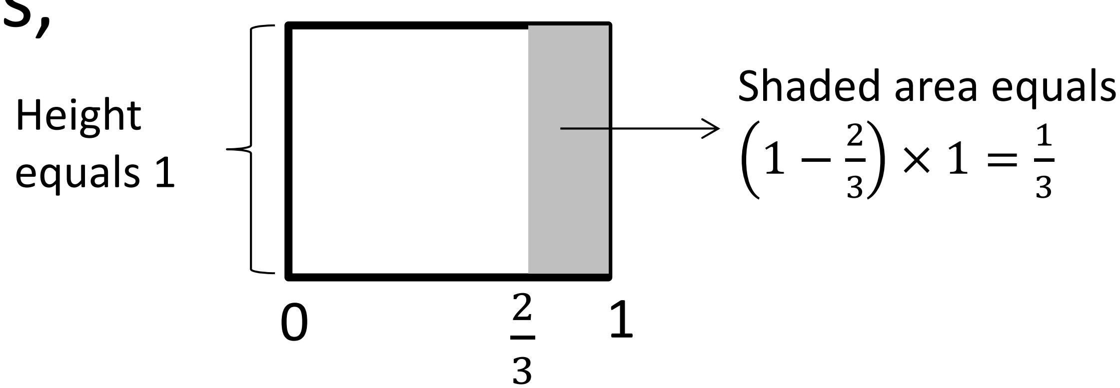
- Therefore, in the second step of iterated dominance we must have:

$$\bar{x} \geq \textbf{Proportion of players such that } \boldsymbol{i} \geq \frac{\mathbf{2}}{\mathbf{3}}$$

- Since we assumed a uniform distribution of i along the interval $[0, 1]$, we have:

Proportion of players such that *i*= Shaded area in the previous figure

- That is,



- Therefore, after the first step of deleting dominated strategies we must have

$$\boldsymbol{\bar{x} \geq \frac{1}{3}}$$

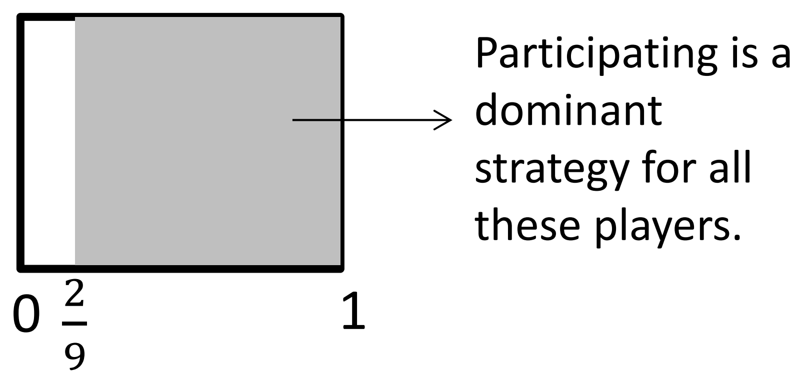
- The smallest value of $\frac{2-4\cdot\bar{x}}{3}$ such that $\bar{x} \in \left[\frac{1}{3}, 1\right]$ is given by $-\frac{2}{3}$

- The largest value of $\frac{2-4\cdot\bar{x}}{3}$ such that $\bar{x} \in \left[\frac{1}{3}, 1\right]$is given by $\frac{2-4\cdot\left(\frac{1}{3}\right)}{3} = \frac{2}{9}$

- As in the first step of deletion, “P” is not a dominated strategy for any player.

- However, in the second step we now have that H (staying at home) is a dominated strategy for every player $i \geq \frac{2}{9}$

- We now have:



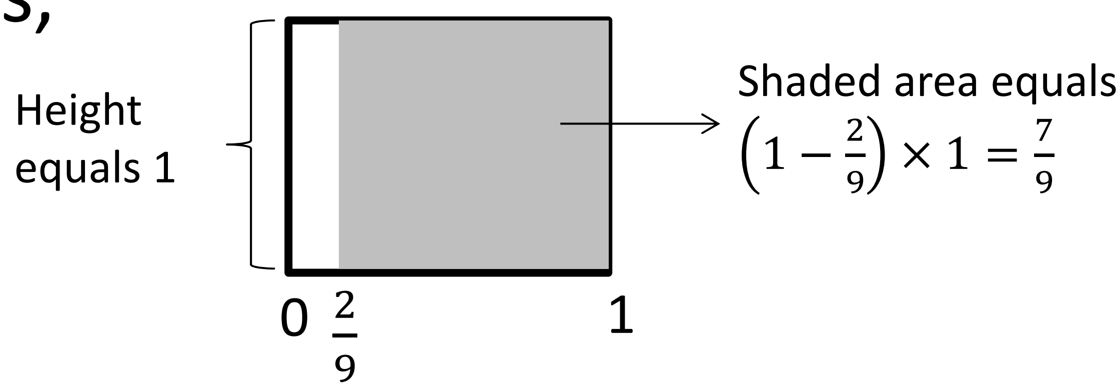
- Therefore, in the third step of iterated dominance we must have:

$$\bar{x} \geq \textbf{Proportion of players such that } \boldsymbol{i} \geq \frac{\mathbf{2}}{\mathbf{9}}$$

- Since we assumed a uniform distribution of i along the interval $[0, 1]$, we have:

Proportion of players such that $\boldsymbol{i}$= Shaded area in the previous figure

- That is,



- Therefore, after the second step of deleting dominated strategies we must have

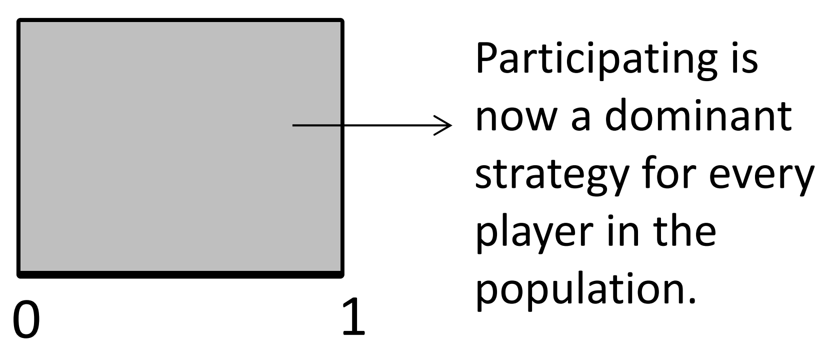
$$\bar{\boldsymbol{x}} \geq \frac{\mathbf{7}}{\mathbf{9}}$$

- The smallest value of $\frac{2-4\cdot\bar{x}}{3}$ such that $\bar{x} \in \left[\frac{7}{9}, 1\right]$ is given by $-\frac{2}{3}$

- The largest value of $\frac{2-4\cdot\bar{x}}{3}$ such that $\bar{x} \in \left[\frac{7}{9}, 1\right]$is given by $\frac{2-4\cdot\left(\frac{7}{9}\right)}{3} = -\frac{10}{27}$

- As in the first and second steps of deletion, "P" is not a dominated strategy for any player.
- However, in the second step we now have that H (staying at home) is a dominated strategy for every player $i \geq -\frac{10}{27}$. Note that **this now includes every player in the population.**

- We now have:



- Therefore, after three steps of deletion of dominated strategies, participating in the revolt is **a dominant strategy for every player in the population**.

- Participating in the revolt is the only rationalizable strategy for everyone in the population.

- We can show that, as long as the coefficient α is greater than 2, participating in the revolt is the only rationalizable strategy for everyone.

- <u>**Remember**</u>**: A strategy is dominated if there do not exist any set of beliefs for which that strategy is a best response**. This is the principle we used in all the examples of Chapter 8. It is true for discrete games as well as continuous games.