9.- Nash Equilibrium

- So far we have assumed that players:

1) Construct beliefs about others' behavior.
2) Maximize their expected utility given these beliefs.
3) These facts are common knowledge

- However, we have not assumed that beliefs are ***correct***, meaning that they are consistent with the strategies actually used by others.
- The potential inability of players to correctly predict others' strategies led to what we called **strategic uncertainty** and the "second strategic tension".

- Suppose there is a mechanism by which players can **coordinate beliefs with actual strategies**.
- If such a coordination is possible, then players' beliefs would be consistent with the actual strategies played by others.
- How can this coordination arise?

1. **If a game is played repeatedly.-** Then players get to learn about the behavior of others, leading to consistency between beliefs and actual strategies.
2. **Precommunication between players.-** In some cases, players could meet before the game is played and agree on the profile of strategies to be played.
3. **Third party.-** Without meeting and communicationg with each other, a third party or "mediator" could recommend a strategy profile to the players.

- Any of the aforementioned mechanisms would eliminate the strategic uncertainty in the game.
- Nash equilibrium arises when players:

1) Construct beliefs about others' behavior and these beliefs are consistent with others' actual strategies played.
2) Maximize their expected utility given these beliefs.

- We will not focus on the specific mechanism that allows players to coordinate beliefs and strategies, we will simply assume such a mechanism exist.

- **Nash Equilibrium (Definition):** A profile of strategies

$$s = (s_1, s_2, \dots, s_n)$$

is a Nash equilibrium if and only if $s_i \in BR_i(s_{-i})$ for each $i = 1, \dots, n$. That is,

$$u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i}) \text{ for every } s'_i \in S_i$$

and for each $i = 1, \dots, n$.

- In words, a profile of strategies is a Nash equilibrium if the strategy prescribed for each player in that profile is the best response to the strategies prescribed for the other players.

- Playing a Nash equilibrium profile $s = (s_1, s_2, \ldots, s_n)$ presupposes that every player i knows that that the rest of the players will actually play s_{-i}.

- If a coordination mechanism between strategies and beliefs exists, then Nash equilibrium behavior can arise.

- **Congruity in a game:** Arises if behavior and beliefs in a game are coordinated or *congruous*.

- Nash equilibrium represents a very strong notion of congruity in which players coordinate on a single strategy profile.

- **Examples:** Two-player normal-form games revisited.
- Let us compare Nash equilibrium and rationalizability. Nash equilibria are represented in circles, non-rationalizable strategies are stricken out.

1 \ 2	H	T
H	1, −1	−1, 1
T	−1, 1	1, −1

Matching Pennies

- Every outcome is rationalizable.
- No Nash equilibrium exists

1 \ 2	~~C~~	D
~~C~~	~~2, 2~~	~~0, 3~~
D	3, 0	(1, 1)

Prisoners' Dilemma

- Only one rationalizable outcome, which is also the unique Nash equilibrium

1 \ 2	Opera	Movie
Opera	(2, 1)	0, 0
Movie	0, 0	(1, 2)

Battle of the Sexes

- Every outcome is rationalizable.
- Only two outcomes are Nash equilibria

- Continued…

1 \ 2	H	D
H	0, 0	3, 1
D	1, 3	2, 2

Hawk-Dove/Chicken

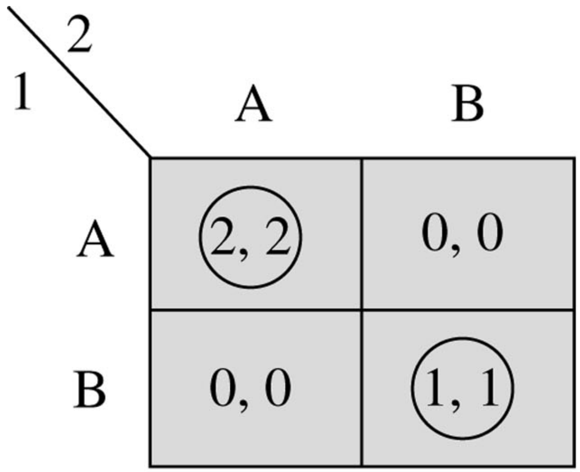
- Every outcome is rationalizable.
- Only two outcomes are Nash equilibria

1 \ 2	A	B
A	1, 1	0, 0
B	0, 0	1, 1

Coordination

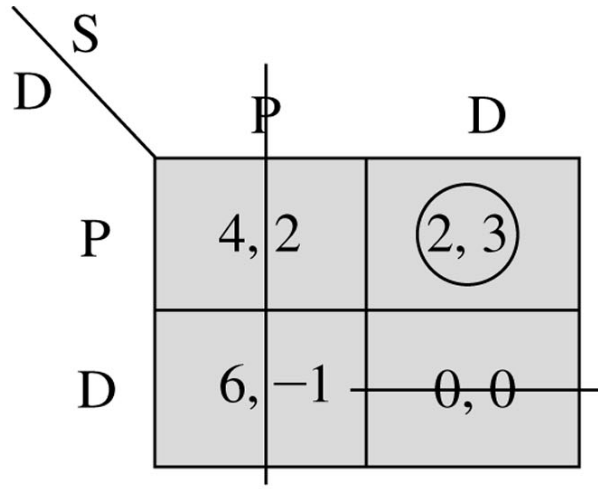
- Every outcome is rationalizable.
- Only two outcomes are Nash equilibria

- Continued…



Pareto Coordination

- Every outcome is rationalizable.
- Only two outcomes are Nash equilibria



Pigs

- Only one rationalizable outcome, which is also the unique Nash equilibrium

- Relationship between rationalizability and Nash equilibrium:

1) **<u>Every Nash equilibrium is always rationalizable</u>. Therefore, we can focus only on rationalizable strategies when looking for Nash equilibria**.
2) Some rationalizable outcomes are not Nash equilibria.
3) Every game has rationalizable strategies, but some games do not have Nash equilibria.
4) Some games have multiple Nash equilibria.

- **Note**: As a consequence of the first statement above, **if a game has a unique rationalizable outcome, then this outcome is also the unique Nash equilibrium of the game**.

- **Example:** Find the Nash equilibria (if any) in this game:

1 \ 2	X	Y	Z
J	5, 6	3, 7	0, 4
K	8, 3	3, 1	5, 2
L	7, 5	4, 4	5, 6
M	3, 4	7, 5	3, 3

(a)

- We proceed in two steps:

1. For each player, identify all the strategies that are best responses.
2. From this set, look for if there exists a profile that are best responses to each other.

- Let us first identify the best responses for player 1:

1 \ 2	X	Y	Z
J	5, 6	3, 7	0, 4
K	8, 3	3, 1	5, 2
L	7, 5	4, 4	5, 6
M	3, 4	7, 5	3, 3

(a)

- Next, let us first identify the best responses for player 2:

1 \ 2	X	Y	Z
J	5, 6	3, 7 •	0, 4
K	8, 3 •	3, 1	5, 2
L	7, 5	4, 4	5, 6 •
M	3, 4	7, 5 •	3, 3

(a)

- Finally, let us compare the best-responses for both players:

1 \ 2	X	Y	Z
J	5, 6	3, 7	0, 4
K	8, 3	3, 1	5, 2
L	7, 5	4, 4	5, 6
M	3, 4	7, 5	3, 3

(a)

All the circled outcomes are the Nash equilibria in this game: (K, X), (L, Z) and (M, Y)

- Remember: To check if a strategy profile is a Nash equilibrium, we only have to check if the strategy prescribed to each player is a best response to the strategies prescribed to the other players.
- **Example: Partnership/Coordination game (continued).-** Recall that in this game, player 1 chose effort level x and player 2 chose effort level y, and best-response functions were given by:

$$BR_1(y) = 1 + c \cdot y$$
$$BR_2(x) = 1 + c \cdot x$$

- Strategies are continuous in this game, so there is no matrix form representation.
- How do we look for Nash equilibria? By <u>applying the definition</u>.
- We need to look for a profile of strategies (x^*, y^*) such that:

$$x^* \in BR_1(y^*)$$
$$y^* \in BR_2(x^*)$$

- Best-responses are given by the functions described above, so we just need to find a profile of strategies (x^*, y^*) such that…

- (cont…)

$$x^* = 1 + c \cdot y^*$$
$$y^* = 1 + c \cdot x^*$$

- Plugging the expression for x^* from the first equation into the second equation yields:

$$y^* = 1 + c \cdot (1 + c \cdot y^*)$$

- This will be satisfied if:

$$y^* = \frac{1+c}{1-c^2} = \frac{1+c}{(1+c)\cdot(1-c)} = \frac{1}{1-c}$$

- Plugging this expression into the first equation:

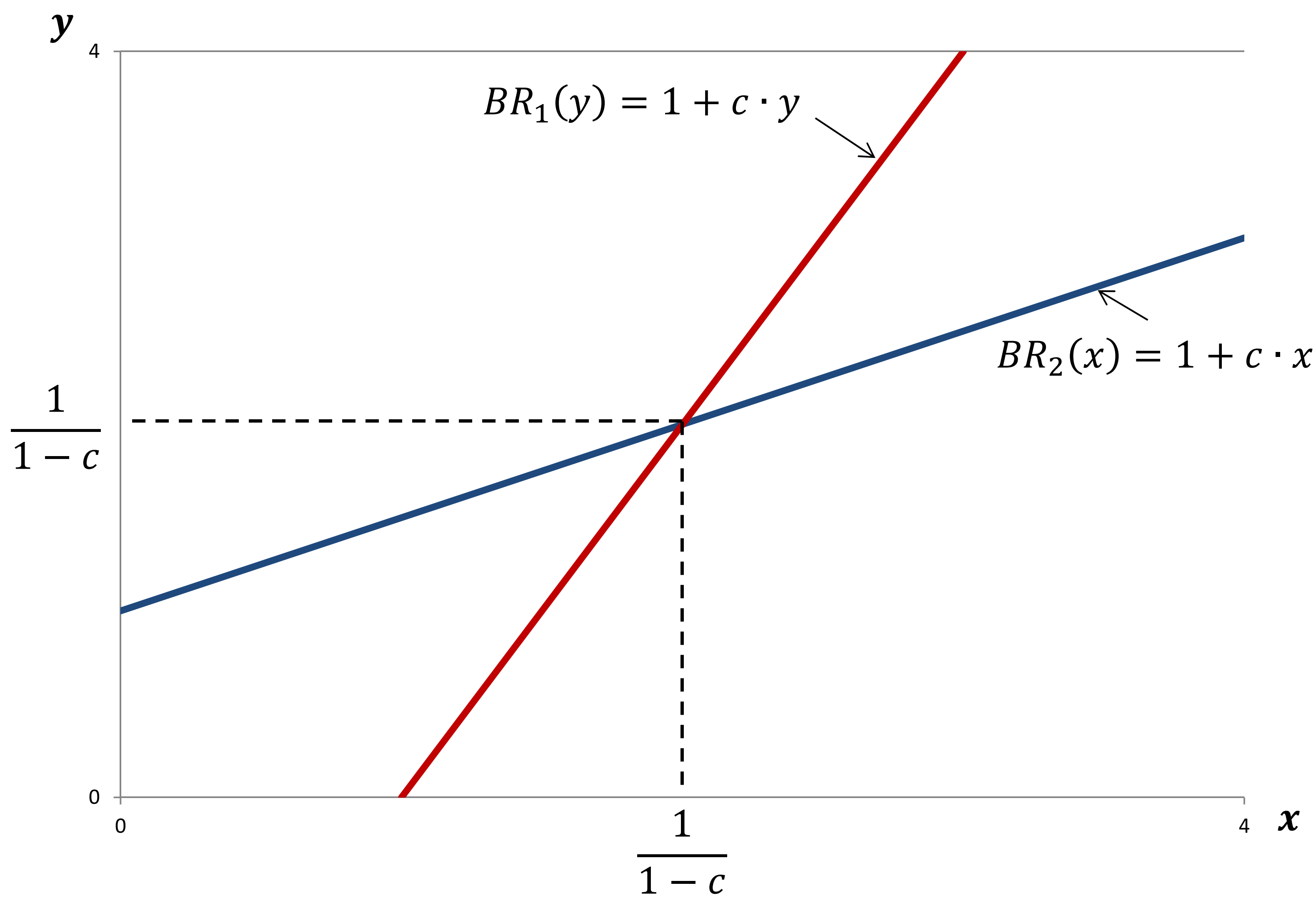
$$x^* = 1 + c \cdot \left(\frac{1}{1-c}\right) = \frac{1-c+c}{1-c} = \frac{1}{1-c}$$

- Therefore, this game has a unique Nash equilibrium, given by the profile of strategies

$$(x^*, y^*) = \left(\frac{1}{1-c}, \frac{1}{1-c}\right)$$

- In Chapter 8 we had already shown that this was the unique rationalizable profile of strategies.

- The Nash equilibrium in this game is given by the point where both players' best response functions cross:



- **Example: Game of social unrest.-** In chapter 8, we showed that the only rationalizable outcome was **“everybody participates in the protest”**.
- As we stated above, since this is the only rationalizable outcome, it must also be a Nash equilibrium.
- Let’s prove that it is indeed a Nash equilibrium: Recall, to check this we need to verify if “Participating” is a best response if everybody else participates in the protest.
- If everybody else participates in the protest, then $\boldsymbol{x = 1}$ and the payoffs to each player i are given by:

$$\boldsymbol{u_i(H, 1) = 4 \cdot 1 - 2 = 2}$$

$$\boldsymbol{u_i(P, 1) = 8 \cdot 1 - 4 + 3 \cdot i = 4 + 3 \cdot i}$$

- Participating in the protest is a best response if and only if

$$u_i(P, 1) \geq u_i(H, 1)$$

- Note that $u_i(P, 1) \geq u_i(H, 1)$ will occur if and only if

$$4 + 3 \cdot i \geq 2$$

- This, in turn will occur if and only if $i \geq -\frac{2}{3}$
- Therefore, if everybody else participates, then participating is a best response if $i \geq -\frac{2}{3}$.
- But recall that $i \in [0, 1]$. Therefore, $i \geq -\frac{2}{3}$ is true for **everybody**. Therefore, participating is a best response for everybody. **"Everybody participating in the protest" <u>is</u> a Nash equilibrium**.

- **Nash Equilibrium in the Cournot Example:** Recall that best response functions are given by:

$$\boldsymbol{BR_1(q_2) = 20 - \frac{1}{2} \cdot q_2}$$
$$\boldsymbol{BR_2(q_1) = 20 - \frac{1}{2} \cdot q_1}$$

- A Nash equilibrium in this game is a profile $\left(q^*{}_1, q^*{}_2\right)$ such that:

$$q^*{}_1 = BR_1\left(q^*{}_2\right)$$
$$q^*{}_2 = BR_2\left(q^*{}_1\right)$$

- That is,

$$q^*{}_1 = 20 - \frac{1}{2} \cdot q^*{}_2$$
$$q^*{}_2 = 20 - \frac{1}{2} \cdot q^*{}_1$$

- Plugging the first equation into the second one we have

$$q^*{}_2 = 20 - \frac{1}{2} \cdot \left(20 - \frac{1}{2} \cdot q^*{}_2\right)$$

- The solution to this equation is:

$$q^*{}_2 = \frac{4}{3} \cdot 10 = \frac{40}{3}$$

- Plugging this into the first equation we have:

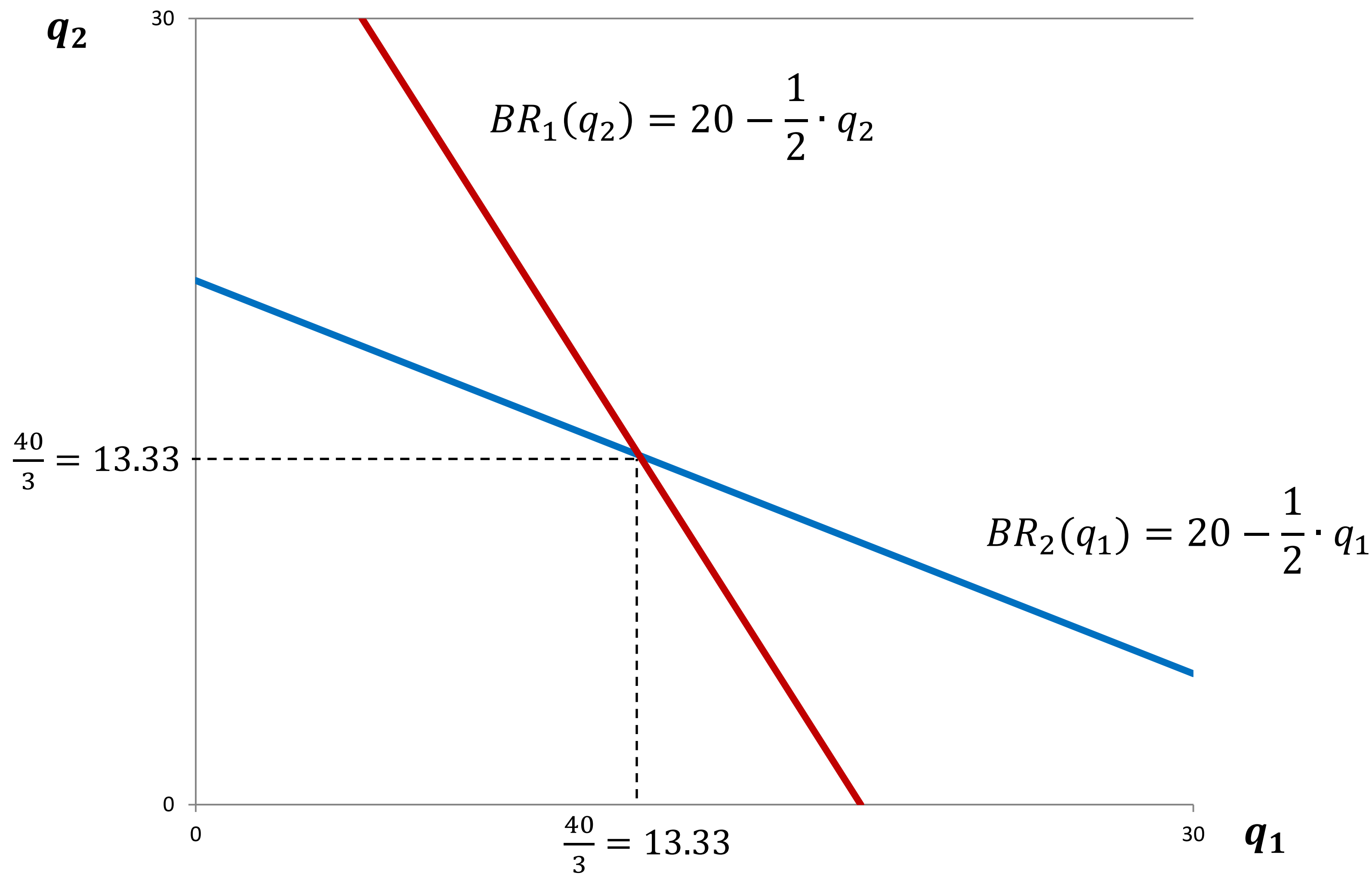
$$q^*{}_1 = 20 - \frac{1}{2} \cdot \frac{40}{3} = \frac{80}{6} = \frac{40}{3}$$

- Therefore, the Cournot game has a unique Nash equilibrium given by the outcome

$$(q^*{}_1, q^*{}_2) = \left(\frac{40}{3}, \frac{40}{3}\right)$$

- We had already shown in Chapter 8 that this was the unique rationalizable outcome of this game.

- Nash equilibrium is given by the point where both players' best response functions cross each other:

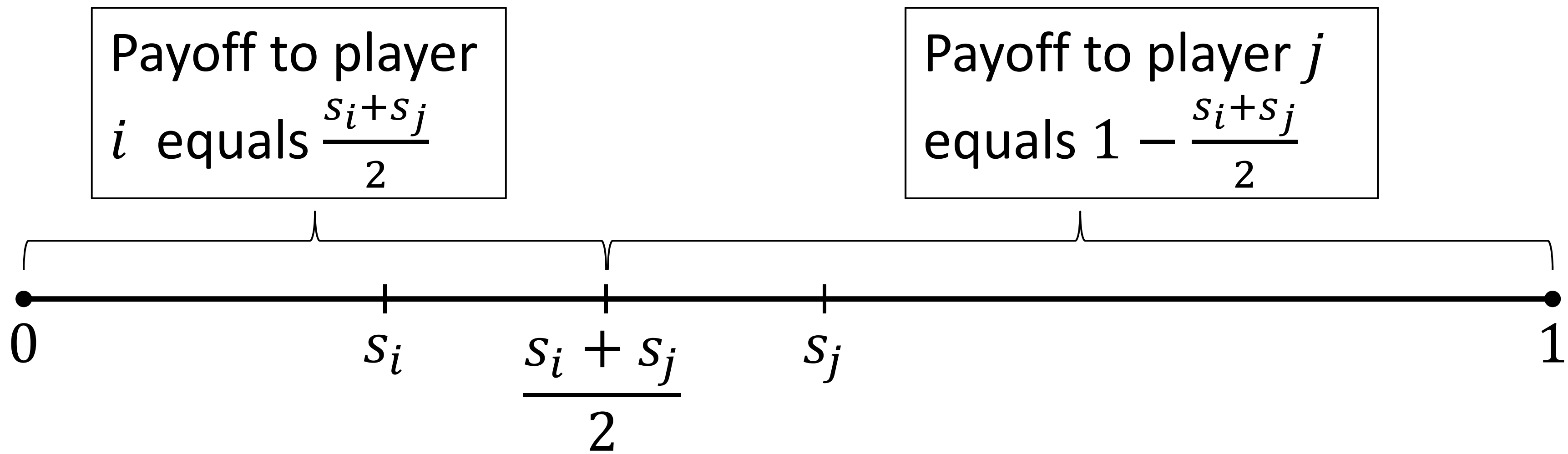


- **Example: Location game from Chapter 8.-** We had already shown there that this game has a unique rationalizable outcome, consisting of the profile of strategies $(s_1, s_2) = (5, 5)$ (both vendors locate in the middle of the beach).

- Since this is the only rationalizable outcome of the game, it is also the unique Nash equilibrium.

- **Example:** Consider a game where two players have to choose a number between zero and one. Let s_1 and s_2 denote the numbers chosen by players 1 and 2.
- Payoffs are as follows:
- If $s_i < s_j$, then the payoff to i is $\frac{s_i+s_j}{2}$ and the payoff to j is $1 - \frac{s_i+s_j}{2}$.
- If $s_i = s_j$, then both players get a payoff of $\frac{1}{2}$.
- Find the Nash equilibria of this game.

- Graphically, payoffs look like this:



- This is effectively a continuous version of the location game in Chapter 8.

- Once again, this amounts to finding a profile $(s^*{}_1, s^*{}_2)$ such that:

$$s^*{}_1 \in BR_1(s^*{}_2)$$
$$s^*{}_2 \in BR_2(s^*{}_1)$$

- Therefore, the first step is to characterize the best response functions by both players.

- Take any $i = 1{,}2$ and let j denote the other player. Let s_j denote the strategy played by *j*. What is $BR_i(s_j)$?

- We have three relevant cases:

a) **Suppose $s_j < \frac{1}{2}$:**

1) **Suppose s_i< s_j :** Then the payoff to i would be

$$\frac{s_i + s_j}{2} < \frac{s_j + s_j}{2} = s_j < \frac{1}{2}$$

2) **Suppose $s_i = s_j$:** Then the payoff to i would be ½.

3) **Suppose s_i> s_j :** Then the payoff to i would be

$$1 - \frac{s_i + s_j}{2} > 1 - \frac{s_i + \frac{1}{2}}{2} = \frac{3}{4} - \frac{s_i}{2}$$

this is maximized by letting s_i be "infinitesimally larger" than s_j. **The payoff to player i would be strictly larger than $\frac{1}{2}$.**

Therefore, if $s_j < \frac{1}{2}$, the best response by player i is to let s_i be infinitesimally larger than s_j.

b) Suppose $s_j > \frac{1}{2}$:

1) **Suppose s_i< s_j :** Then the payoff to i would be

$$\frac{s_i + s_j}{2} > \frac{s_i + \frac{1}{2}}{2} = \frac{s_i}{2} + \frac{1}{4}$$

this is maximized by letting s_i be "infinitesimally smaller" than s_j. **The payoff to player i would be strictly larger than $\frac{1}{2}$.**

2) **Suppose $s_i = s_j$:** Then the payoff to i would be ½.

3) **Suppose s_i> s_j :** Then the payoff to i would be

$$1 - \frac{s_i + s_j}{2} < 1 - \frac{s_j + s_j}{2} = 1 - s_j < 1 - \frac{1}{2} = \frac{1}{2}$$

Therefore, if $s_j > \frac{1}{2}$, the best response by player i is to let s_i be infinitesimally smaller than s_j.

c) Suppose $s_j = \frac{1}{2}$:

1) **Suppose $s_i < s_j$:** Then the payoff to i would be

$$\frac{s_i + \frac{1}{2}}{2} = \frac{s_i}{2} + \frac{1}{4} < \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{4} = \frac{1}{2}$$

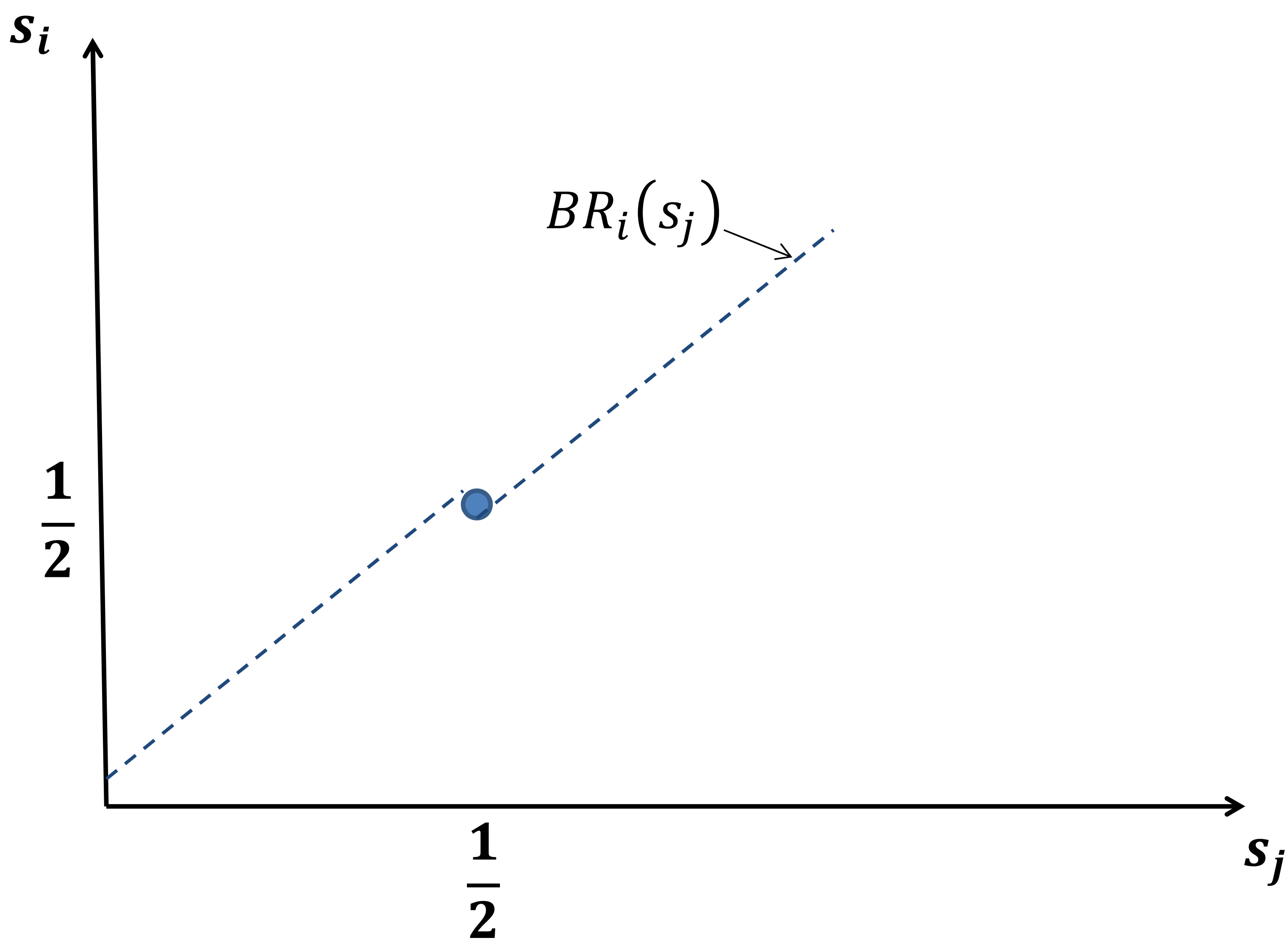
2) **Suppose $s_i = s_j$:** Then the payoff to i would be ½.

3) **Suppose $s_i > s_j$:** Then the payoff to i would be

$$1 - \frac{s_i + \frac{1}{2}}{2} = 1 - \frac{s_i}{2} - \frac{1}{4} = \frac{3}{4} - \frac{s_i}{2} < \frac{3}{4} - \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2}$$

Therefore, if $s_j = \frac{1}{2}$, the best response by player i is to let $s_i = s_j = \frac{1}{2}$.

- Best response function for each player i=1,2 looks like this:



- From our previous analysis, we can deduce that:

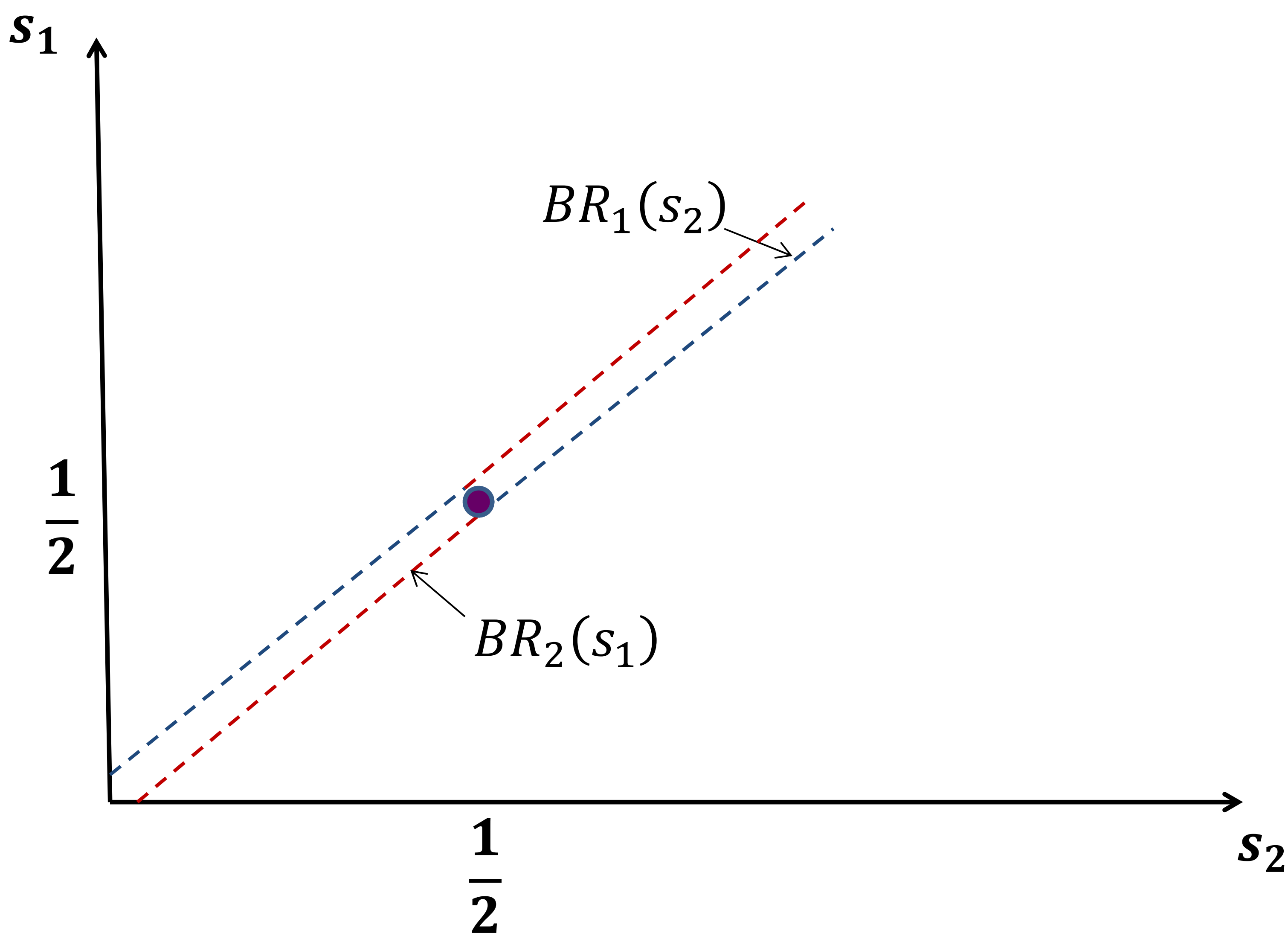
a) **No player will choose $s_i > \frac{1}{2}$ in a Nash equilibrium:** To see why, note that if $s_i > \frac{1}{2}$, the best response by *j* is to let s_j be infinitesimally smaller than s_i. But if this is the case, the best response by i is to let s_i be infinitesimally smaller than s_j, which would in turn be undercut by j, then undercut by i, and so on, as long as $s_i > \frac{1}{2}$.

b) No player will choose $s_i < \frac{1}{2}$ in a Nash equilibrium: To see why, note that if $s_i < \frac{1}{2}$, the best response by *j* is to let s_j be infinitesimally larger than s_i. But if this is the case, the best response by i is to let s_i be infinitesimally larger than s_j, which would in turn be topped by j, then topped by i, and so on, as long as $s_i < \frac{1}{2}$.

c) **The only Nash equilibrium is the profile $\boldsymbol{(s_i, s_j) = \left(\frac{1}{2}, \frac{1}{2}\right)}$:** We showed previously that $s_i = \frac{1}{2}$ is a best response to $s_j = \frac{1}{2}$ and therefore this is a Nash equilibrium.

- It is the unique Nash equilibrium because we also showed that we cannot have $s_i > \frac{1}{2}$ or $s_i < \frac{1}{2}$ in any Nash equilibrium.

- Graphically:



- **Example: A partnership game.-** Suppose two players are partners in a business. The effort of each partner will determine its success. Suppose there will be no revenues unless both partners exert at least 1 unit of effort**.**
- In particular, suppose payoffs are given as follows:

$$u_i(e_i, e_j) = \begin{cases} -e_i & \text{if } e_j < 1 \\ e_i\,(e_j - 1)^2 + e_i - \frac{1}{2} \cdot e_i^2 & \text{if } e_j \geq 1 \end{cases}$$

- **Characterize the best response functions in this game**
- Note first that if $e_j < 1$, then the best response by player i is always to choose $e_i = 0$ (since the payoff to player i in this case is given by $-e_i$).
- If $e_j \geq 1$, then the payoff to player i is given by the function $u_i(e_i, e_j) = e_i(e_j - 1)^2 + e_i - \frac{1}{2} \cdot e_i{}^2$
- In this case, the best response by player i will be given by the first order conditions:

$$\frac{\partial u_i(e_i, e_j)}{\partial e_i} = 0$$

- We have:

$$\frac{\partial u_i(e_i, e_j)}{\partial e_i} = (e_j - 1)^2 + 1 - e_i$$

- Therefore, the first order conditions will be satisfied if

$$(e_j - 1)^2 + 1 - e_i = 0$$

- That is, if

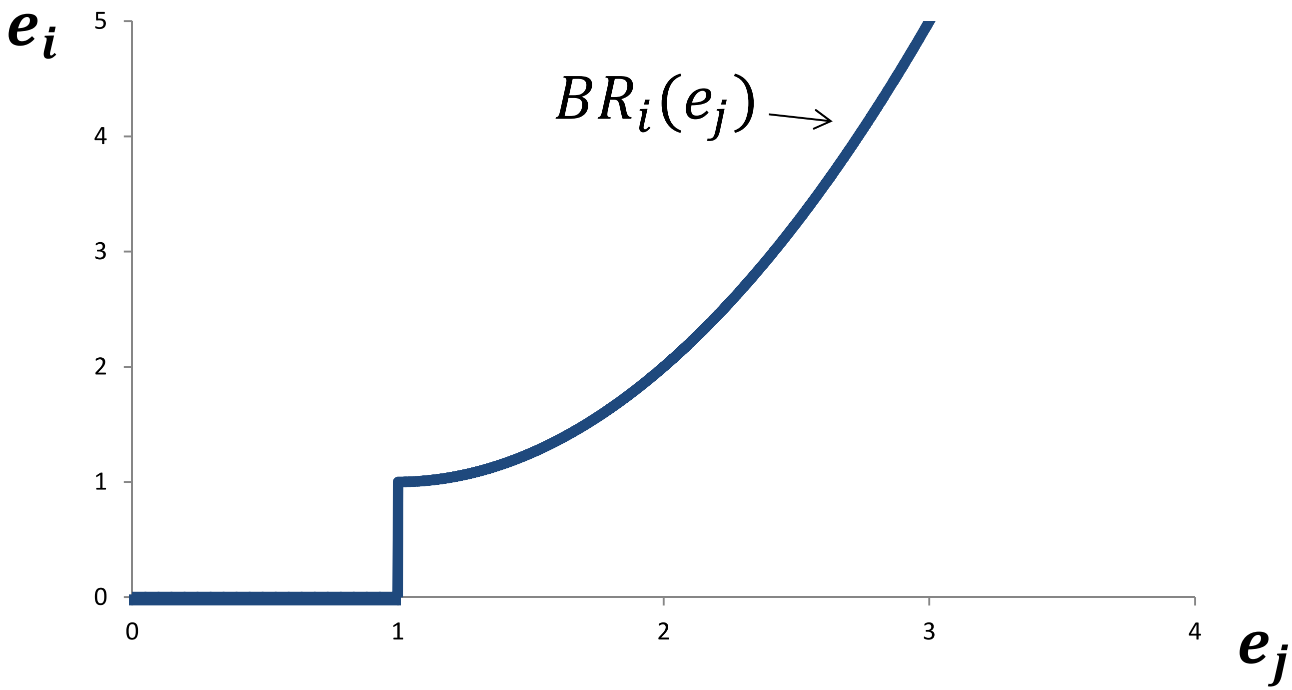
$$e_i = (e_j - 1)^2 + 1$$

- This is player i's best response function if $e_j \geq 1$.

- Therefore, the best response function for player $i = 1,2$ is given by:

$$BR_i(e_j) = \begin{cases} 0 & \text{if } e_j < 1 \\ (e_j - 1)^2 + 1 & \text{if } e_j \geq 1 \end{cases}$$

- Graphically:



- **Prove that $(\mathbf{0}, \mathbf{0})$ is a Nash equilibrium:**
  – This is easy once we have specified the best response functions. We have:

$$BR_1(0) = 0 \quad \text{and} \quad BR_2(0) = 0$$

therefore "nobody puts any effort" is a Nash equilibrium.

- **Find all other Nash equilibria in this game:**
  - This requires finding every pair $({e^*}_1, {e^*}_2)$ such that:

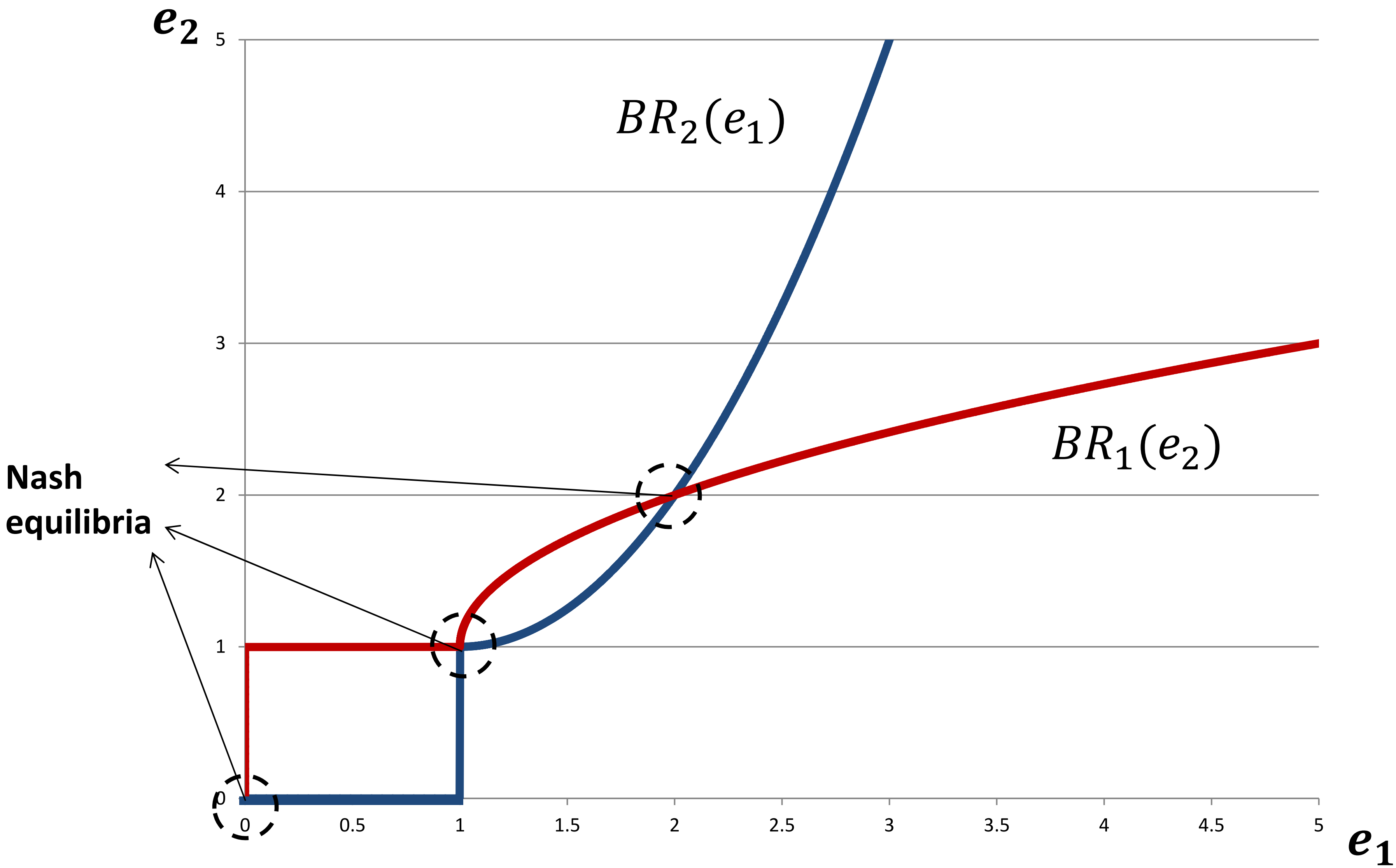
$$BR_1({e^*}_2) = {e^*}_1 \quad \text{and} \quad BR_2({e^*}_1) = {e^*}_2$$

There are **three** Nash equilibrium profiles in this game:

$$(\mathbf{0}, \mathbf{0}) \quad (\mathbf{1}, \mathbf{1}) \quad \boldsymbol{and} \quad (\mathbf{2}, \mathbf{2})$$

This can be verified graphically by showing that these are the three points where both players' best response functions cross each other.

- Checking Nash equilibria graphically in the partnership game:



- **Rank the Nash equilibria in terms of the social payoff they produce:**

$$\boldsymbol{u_1(e_1, e_2) + u_2(e_1, e_2)}$$

- For the Nash equilibrium (0,0):

$$u_1(0,0) + u_2(0,0) = 0$$

- For the Nash equilibrium (1,1):

$$u_1(1,1) + u_2(1,1) = \frac{1}{2} + \frac{1}{2} = 1$$

- For the Nash equilibrium (2,2):

$$u_1(2,2) + u_2(2,2) = 2 + 2 = 4$$

- Note that (2,2) is more efficient than (1,1) which in turn is more efficient than (0,0).

- Therefore, the Nash equilibria (0,0) and (1,1) are **inefficient outcomes. Is (2,2) efficient?**

- A quick way to find out is to increase both players' effort from (2,2) and see if they are both better off.

- Suppose $(e_1, e_2) = (3{,}3)$. Then, we have:

$$u_1(3{,}3) = 10.5, \quad \text{and} \quad u_2(3{,}3) = 10.5$$

- This makes both players better off than when $(e_1, e_2) = (2{,}2)$. We conclude that this outcome is inefficient. Therefore, **all Nash equilibria in this game are inefficient outcomes**.

- **Third Strategic Tension:** Whether unique or multiple, there are examples of games where Nash equilibria is **inefficient** (for instance, in the Prisoner's Dilemma example).

- The possibility that players may coordinate to an inefficient equilibrium gives rise to what the book calls the **"third strategic tension".**

- Real-life examples of society coordinating to an inefficient equilibrium include cases where society has decided to adopt **inefficient technologies** in favor of more efficient alternatives (e.g, VHS vs. Beta, QWERTY vs. Dvorak keyboards)

- **Example: Consider the following game**

1 \ 2	a	b	c
w	5, 2	3, 4	8, 4
x	6, 2	2, 3	8, 8
y	1, 1	0, 1	9, 2

a) **What are the Nash equilibria of this game?**

b) **Which of these equilibria are efficient?**

- As mentioned previously, in matrix games we can take the following steps:

1. identify all the strategies that are best responses.
2. From this set, look for if there exists a profile that are best responses to each other.

- **Best responses for player 1:**

1 \ 2	a	b	c
w	5, 2	3, 4	8, 4
x	6, 2	2, 3	8, 8
y	1, 1	0, 1	9, 2

- **Best responses for player 2:**

1 \ 2	a	b	c
w	5, 2	3, 4	8, 4
x	6, 2	2, 3	8, 8
y	1, 1	0, 1	9, 2

- **Combining both:**

1 \ 2	a	b	c
w	5, 2	3, 4	8, 4
x	6, 2	2, 3	8, 8
y	1, 1	0, 1	9, 2

Nash equilibria

- This game has two Nash equilibria:

$$(w, b) \text{ and } (y, c)$$

- Out of these, $(\boldsymbol{w}, \boldsymbol{b})$ **is inefficient** (the outcome (x, c) is more efficient). However, the equilibrium $(\boldsymbol{y}, \boldsymbol{c})$ **is efficient, since no other outcome yields a higher payoff to player 1.**

- **Strict Nash Equilibrium:** The book talks about the special case of equilibrium which arises where each Nash equilibrium strategy is the **unique best response** to the strategies of the others. This is referred to as "Strict Nash Equilibrium".

- Because this concept is too restrictive (many games would fail to have strict Nash equilibria), we will not emphasize it in the course.

- Also, for the time being we will not emphasize the concept of **congruous sets** (pages 104-105) , which is a notion in-between Nash equilibrium and rationalizability, without much behavioral justification.

- The chapter concludes by citing empirical evidence from **experimental economics** which has found that **in the real world, there is variation in the degree of rationality of individuals: In practice some people seem to play according to Nash equilibrium, while others appear to be less rational.**

- **REMARK:** We have defined Nash equilibrium so far as involving **pure strategies** only, and we have seen examples even of simple games that fail to have Nash equilibria (Matching Pennies, for example).

- In Chapter 11 we will extend this notion to **mixed strategy Nash equilibria. Existence of Nash equilibrium in mixed strategies is guaranteed in all the types of games we will study in this course.**