Powell's "Projection" Theorem for a class of U-Statistics

Theorem 1 (Lemma A.3 in Ahn and Powell (1993), see also Lemma 3.1 in Powell, Stock and Stoker (1989)) *For an iid sample* $\{X_i\}_{i=1}^n$ *of random variables, consider an* m^{th}*-order U-statistic of the form*

$$U_n = \binom{n}{m}^{-1} \sum_c p_n(X_{i_1}, X_{i_2}, \ldots, X_{i_m})$$

where the sum is taken over the set "c" of all $\binom{n}{m}$ *combinations of* m *distinct elements* $\{i_1, i_2, \ldots, i_m\}$ *from the set* $\{1, 2, \ldots, n\}$*. Without loss of generality, the function* $p_n(\cdot)$ *can be taken to be symmetric in its* m *arguments. Define the "projection" of this U-statistic as:*

$$\widehat{U}_n \equiv \theta_n + \frac{m}{n} \sum_{i=1}^{n} [r_n(X_i) - \theta_n],$$

where

$$r_n(X_i) = E\big[p_n(X_i, X_{i_2}, \ldots, X_{i_m})\big|X_i], \quad \theta_n = E[r_n(X_i)]$$

Note that $r_n(X_i)$ *is the conditional expectation of* $p_n(\cdot)$ *with respect to its first-argument, which is irrelevant because* $p_n(\cdot)$ *is assumed symmetric: If* $m = 2$ *for example,* $p_n(X_i, X_j) = p_n(X_j, X_i)$ *and therefore* $r_n(X_i) = E\big[p_n(X_i, X_j)\big|X_i\big] = E\big[p_n(X_j, X_i)\big|X_i\big]$.
If $p_n(\cdot)$ *satisfies:*

$$E\big[\big\|p_n(X_1, \ldots, X_m)\big\|^2\big] = O(n),$$

then

$$\text{(i)} \quad U_n = \theta_n + o_p(1)$$

$$\text{(ii)} \quad U_n = \widehat{U}_n + o_p(n^{-1/2})$$