Using average derivatives to estimate incomplete information games with unobserved state variables

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Abstract

Average derivatives can be a powerful tool to identify and estimate incomplete information games with unobserved state variables (unobserved heterogeneity). We consider a static, binary choice game with P asymmetric players and unobserved state variables such that: (a) their support is not restricted to be finite, (b) unobserved state variables are allowed to be correlated with private types in unrestricted ways through a nonparametric joint distribution, (c) the competition effect is explicitly modeled as one of the unobserved state variables. Using average derivatives, we show identification results for players' payoff parameters, and we show that we can identify the direction of competition effects across players by using a "competition index", consisting of pairwise inner products of average derivatives across players. A simulation study shows good performance of well-known, existing average derivative estimators to identify payoff parameters and the direction of the competition effect for each player.

Keywords: Estimation of games, average derivatives, unobserved heterogeneity, semiparametric models. JEL classification: C1, C14, C57.

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1 Introduction

The econometric analysis of discrete, static games with *incomplete information* has been an active area of research for at least the last two decades. Such models have been used to study social interactions (e.g., Brock and Durlauf (2001a), Brock and Durlauf (2001b), Xu (2014), Lin, Tang, and Yu (2021)), as well as general interactions-based models and their applications. A partial list of the latter would include, Berry (1992), Seim (2006), Berry and Reiss (2007), Berry and Tamer (2007), Aradillas-López and Tamer (2008), Pesendorfer and Schmidt-Dengler (2008), Sweeting (2009), Bajari, Hong, Krainer, and Nekipelov (2010), Aradillas-López (2010), Tang (2010), Aradillas-López (2012), de Paula and Tang (2012), Xu and Wan (2014), Grieco (2014), Lewbel and Tang (2015), Aradillas-López and Gandhi (2016), Liu, Xu, and Vuong (2017), Xiao (2018), Aguirregabiria and Mira (2019), Marcoux (2019), Espín-Sánchez, Parra, and Wang (2023), Magnolfi and Roncoroni (2023). Existing work ranges from fully parametric to nonparametric models, and their objectives range from identification and estimation of finite-dimensional payoff parameters to nonparametric identification of payoff functions and/or conditional choice probabilities. While most papers focus on Bayesian Nash equilibrium (BNE) as the solution concept, alternative behavioral models considered include iterated dominance, rationalizability, and correlated equilibrium, and the results obtained range from point-identification to partial identification.

In the econometric analysis of incomplete information games, we refer to "state variables" as all those features, publicly observed by all players, that shift payoffs independently of the strategic or competition effect. The vast majority of proposed methods have two features in common: (i) the assumption that players' "types" (i.e., their privately observed signals) are independent of each other conditional on observable state variables, and (ii) the assumption that there are no unobserved state variables (i.e., ruling out unobserved heterogeneity). Combined, both assumptions imply that players' (unobserved) beliefs are conditioned on observable state variables. Notable exceptions exist, including Aradillas-López (2010), Grieco (2014), Xu and Wan (2014), Liu, Xu, and Vuong (2017), Aguirregabiria and Mira (2019), Marcoux (2019), Lin, Tang, and Yu (2021), and Magnolfi and Roncoroni (2023). Some of these papers allow for correlated private types (i.e., correlation across players' private information even after conditioning on all state variables),

while others allow for unobserved heterogeneity (i.e., unobserved state variables).

Existing work that allows for the presence of unobserved state variables has done so under very restrictive assumptions. Existing papers have either assumed that unobserved state variables have finite support (Aguirregabiria and Mira (2019)), that they are deterministic functions of observable covariates (Marcoux (2019), Lin, Tang, and Yu (2021)), or they have fully parameterized their distribution (Grieco (2014)). Importantly, no existing work, to my knowledge, explicitly models the competition effect itself as an unobserved state variable, mostly limiting the presence of unobserved state variables as an additive shock to payoffs. We contribute to the literature by presenting a discrete, incomplete information game that allows for the presence of unobserved state variables where: (a) their support is not restricted to be finite, (b) unobserved state variables are allowed to be correlated with private types in unrestricted ways through a nonparametric joint distribution, (c) the competition effect is explicitly treated as one of the unobserved state variables. While we do not allow for correlated private types conditional on all payoff shifters, we go beyond the usual assumption of private type independence, and we only require independence conditional on observed *and unobserved* state variables. Focusing on a binary choice game with a parametric expression for normal-form payoffs, where observed state variables shift payoffs through a linear index, we obtain a game-theoretic version of well-known identification results for average derivatives in linear index models (Powell, Stock, and Stoker (1989), Newey and Stoker (1993)). We show that average derivatives can allow us to identify, up to a proportionality constant, the payoff slope coefficients of observable state variables, as well as the direction of the competition effect. The use of average derivatives has significant computational advantages, as they do not require us to solve the game, and they allow the use of a multitude of existing nonparametric estimators. We proceed as follows. Section 2 describes our model, assumptions and main results. Section 3 presents a simulation study, and Section 4 concludes. Proofs are included in the Appendix.

2 A binary-choice game with P players

We consider a static, binary-choice game with $P \geq 2$ players, labeled $p = 1, \ldots, P$. The action space for p is $\mathcal{Y}_p \equiv \{0, 1\}$, and the joint action space is $\mathcal{Y} \equiv \{0, 1\}^P$. We

have a collection of *observable* features, given by $X \equiv \left((X_p')_{p=1}^P, X_c'\right)'$, where X_p denotes features of player p and X_c denotes features that describe the environment of the game (e.g., market characteristics). We will refer to X interchangeably as observable features, state variables, covariates, or payoff shifters. Throughout, let d_ξ denote the dimension, in Euclidean space, of a vector ξ, and let $Supp(\xi)$ denote the support of the distribution of a r.v. ξ. We will make the following assumptions about the distributional properties of X.

Assumption 1 ***(Some restrictions on the distribution of X)*** *No component of X is functionally determined by (i.e., a deterministic function of) other components of X. The joint distribution of X is absolutely continuous with respect to Lebesgue measure (a.c.l.m), with density function f_X, which is continuously differentiable. $Supp(X)$ is a nonempty, possibly unbounded, convex subset of $\mathbb{R}^{d_X}$.*

Our normal-form payoffs will depend on X through linear indices. Assumption 1 facilitates the use of density-weighted average derivative estimators, which have been extensively analyzed for linear index models, e.g., in Powell, Stock, and Stoker (1989), Newey and Stoker (1993), Cattaneo, Crump, and Jansson (2010), Cattaneo, Crump, and Jansson (2013), Cattaneo, Crump, and Jansson (2014). The restrictions in Assumption 1 can be relaxed in a variety of ways. For example, we can allow some components of X to be functionally determined by other components of X, and consequently allow for $Supp(X)$ to be non-convex by reparameterizing our payoff functions (see Ackerberg and Xu (2024)). Depending on the reparameterization, this may lead to a reinterpretation of our results. Convexity of $Supp(X)$ has been maintained in more general weighted average derivative methods, such as Newey and Stoker (1993). However, other types of average derivative estimators do not require convexity. This includes, for example, Banerjee (2007), who only requires compactness, and sieve-estimation procedures (e.g., see Chen, Pouzo, and Powell (2019), Chen, Chen, and Tamer (2023)), which do not appear to require convexity. Our focus here is on identification results through average derivatives, and we do not advocate for any particular estimator. Hence, subject to compatibility with our remaining restrictions, readers can replace or modify Assumption 1 to fit the requirements of their preferred average-derivative estimator.

Remark 1 ***(Discrete/categorical observable states)*** *If the original collection of observable state variables includes categorical or discrete elements, our analysis is implic-*

itly assumed to be performed conditional on these elements. For example, our analysis is performed conditional on the game being played in a "small market" or in a "large market", with the rest of the observable states, grouped in X, satisfying Assumption 1.

The game will also include unobservable payoff shifters which, for each player p, are captured by three scalar features, denoted as $(\Delta_p, \eta_p, \varepsilon_p)$. The distinction is that ε_p is only privately observed by p and is player p's "type", while $\Delta \equiv (\Delta_p)_{p=1}^P$ and $\eta \equiv (\eta_p)_{p=1}^P$ are publicly observed by all players. Δ_p will capture the competition effect for player p, modeled as random and unobserved. As usual, $-p$ will index all of player p's opponents, so if $\xi = (\xi_p)_{p=1}^P$, then $\xi_{-p} = (\xi_q)_{q \neq p}$.

2.1 Payoff functions

We consider a normal-form payoff function where observable state variables enter through linear indices. For each $y \in \mathcal{Y}$, player p's normal-form payoff function is,

$$u_p(y, X, \Delta_p, \eta_p, \varepsilon_p) = y_p \times \left(X_p' \beta_{p0} + X_c' \gamma_{p0} + \Delta_p \cdot \sum_{q \neq p} y_q + \eta_p - \varepsilon_p \right).$$

$(\beta_{p0}', \gamma_{p0}')_{p=1}^P$ are constant slope coefficients, while $(\Delta_p)_{p=1}^P$ are treated as unobserved state variables that measure the competition effect on each p. Our goal is to study identification and estimation of the slope coefficients, as well as identifiable features of the competition effects, through average derivatives. Express $X_p' \beta_{p0} + X_c' \gamma_{p0} \equiv X' \theta_{p0}$, where $\theta_{10} \equiv (\beta_{10}', 0', \gamma_{10}')'$, $\theta_{p0} \equiv (0', \beta_{p0}', 0', \gamma_{p0}')'$ for $2 \leq p \leq P-1$, and $\theta_{P0} \equiv (0', \beta_{P0}', \gamma_{P0}')'$. We will express payoffs compactly as, $u_p(y, X, \Delta_p, \eta_p, \varepsilon_p) = y_p \times \left(X' \theta_{p0} + \Delta_p \cdot \sum_{q \neq p} y_q + \eta_p - \varepsilon_p \right)$.

2.2 Players' information and unobserved state variables

The information available to players is described as follows.

Assumption 2 ***(Players' information)*** *The realization of $(X', \eta', \Delta')'$ is publicly observed by all players, who also know the true values of the parameters $\theta_0 \equiv \left((\theta_{p0}')_{p=1}^P \right)'$. The realization of ε_p is only privately observed by player p, and it denotes player p's "type". Players know the true distributional properties of $\varepsilon \equiv (\varepsilon_p)_{p=1}^P$, whose features will be described below.*

The unobserved state variables in our model are grouped as $U \equiv (\Delta', \eta')'$.

2.3 Behavior

Let $Y_p \in \{0,1\}$ denote p's choice. We will assume that players maximize their expected payoff given their beliefs. Then,

$$Y_p = \mathbb{1}\Big\{X'\theta_{p0} + \Delta_p \cdot \sum_{q \neq p} E_p[Y_q] + \eta_p - \varepsilon_p \geq 0\Big\} \tag{1}$$

where $E_p[\cdot]$ denotes p's subjective expectation (beliefs). Before discussing how beliefs are constructed, we will introduce an exclusion restriction.

Assumption 3 ***(An exclusion restriction)*** $\varepsilon_p|(\varepsilon_{-p}, X, U) \sim \varepsilon_p|U$ *for each* p.

The unobserved state variables U are allowed to be endogenous (correlated with players' private types ε_p) in unrestricted ways. We could allow for endogeneity of X by replacing the above restriction with $\varepsilon_p|(\varepsilon_{-p}, X, U) \sim \varepsilon_p|((X'\theta_{p0})_{p=1}^P, U)$. While a version of Proposition 1 will extend to this case, extending the competition-effect results in Proposition 2 would require additional restrictions. While we do not allow for unrestricted correlation across players' private types, we only require independence conditional on observed *and unobserved* state variables.

Assumption 4 ***(Distributional properties of*** $\boldsymbol{\varepsilon_p|U}$***)*** *Conditional on almost every (a.e) realization of* U*, the distribution of* ε_p *is a.c.l.m, with unbounded support. The cdf of* $\varepsilon_p|U$ *is* $F_{\varepsilon_p|U}(\cdot|U)$*, with bounded density* $f_{\varepsilon_p|U}(\cdot|U)$*. The global bound of* $f_{\varepsilon_p|U}(\cdot|\cdot)$ *is denoted as* $\overline{f}_p$*. The distribution* $F_{\varepsilon_p|U}$ *will be left nonparametrically specified.*

2.4 Comparison of our setup with existing work

The previous assumptions describe the main features of our setup. We can compare it with existing work on static, incomplete information games that explicitly allow for unobserved state variables or at least discuss it as a potential extension. Grieco (2014) considers a two-player game and allows for unobserved state variables for each player, but the joint parametric distribution of state variables is fully parameterized. Xu and Wan (2014), Liu, Xu, and Vuong (2017), and Magnolfi and Roncoroni (2023) allow for correlated types, but they do not allow for unobserved state variables that can determine strategic effects; they only allow for publicly observed signals that contain information about other players' types. Marcoux

(2019) allows for an unobserved state variable for each player, but models it explicitly as the residual from a reduced-form equation that can be estimated in a first step. Aguirregabiria and Mira (2019) assume that the support of the unobservable state variables is finite, and they assume that the distribution of (independent) privately observed shocks is known. Lin, Tang, and Yu (2021) allow for a latent payoff shifter (the adjacency matrix) that is observed by all players but not by the researcher, but they assume that it is deterministic conditional on observable characteristics. Overall, to my knowledge, there is no existing method in the literature that allows for unobserved state variables where, simultaneously: (a) their support is not restricted to be finite, (b) the joint distribution of private types and unobserved state variables is nonparametric, (c) the competition effects themselves are unobserved state variables. While we do not allow for unrestricted correlation in players' private types, we only require independence conditional on observed and unobserved states.

2.4.1 Bayesian-Nash Equilibria (BNE)

Conditional on (X, U), the realization of ε_p contains no additional information for ε_{-p}. Thus, we will assume that players' beliefs are conditioned on (X, U), and that they correspond to a Bayesian-Nash Equilibrium (BNE). For a given $\pi \equiv (\pi_p)_{p=1}^P \in [0,1]^P$, denote $F_p(X, U, \pi_{-p}) \equiv F_{\varepsilon_p|U}\Big(X'\theta_{p0} + \Delta_p \cdot \sum_{q \neq p} \pi_q + \eta_p \Big| U\Big)$. Next, stack $F(X, U, \pi) \equiv (F_p(X, U, \pi_{-1}))_{p=1}^P$, and let $B(X, U, \pi) \equiv \pi - F(X, U, \pi)$. Conditional on (X, U), a BNE is any solution, in $\pi \in [0,1]^P$, to the system, $B(X, U, \pi) = 0$.

2.4.2 Existence of BNE

Existence of at least one BNE follows from Brouwer's Fixed Point Theorem: for any (X, U), $F(X, U, \cdot) : [0,1]^P \to [0,1]^P$ is a continuous self-map (since each F_p is a cdf), so it has a fixed point $\pi^* = F(X, U, \pi^*)$, which is equivalent to $B(X, U, \pi^*) = 0$.

2.4.3 Uniqueness and regular BNE

Let $J(X, U, \pi) \equiv \partial B(X, U, \pi)/\partial \pi$ denote the Jacobian of $B(X, U, \pi)$ w.r.t π. A solution π^* to the BNE system is *regular* if $J(X, U, \pi^*)$ is invertible. A regular solution is locally unique. To analyze uniqueness and regularity, the properties of the Jacobian

$J(X,U,\pi)$ are key. Denote $f_p(X,U,\pi_{-p}) \equiv f_{\varepsilon_p|U}\Big(X'\theta_{p0} + \Delta_p \cdot \sum_{q\neq p}\pi_q + \eta_p \Big| U\Big)$. Then,

$$J(X,U,\pi) = \begin{pmatrix} 1 & -\Delta_1 \cdot f_1(X,U,\pi_{-1}) & \cdots & -\Delta_1 \cdot f_1(X,U,\pi_{-1}) \\ -\Delta_2 \cdot f_2(X,U,\pi_{-2}) & 1 & \cdots & -\Delta_2 \cdot f_2(X,U,\pi_{-2}) \\ \vdots & \vdots & \ddots & \vdots \\ -\Delta_P \cdot f_P(X,U,\pi_{-P}) & -\Delta_P \cdot f_P(X,U,\pi_{-P}) & \cdots & 1 \end{pmatrix} \quad (2)$$

The following restriction will aid in the analysis of the principal minors of $J(X,U,\pi)$. It does not follow from our previous assumptions, since it involves the distribution of Δ.

Assumption 5 $1+\Delta_p \cdot f_p(X,U,\pi_{-p}) \neq 0$ *w.p.1, for each* $p = 1,\ldots,P$, *and any* $\pi \in [0,1]^P$.

Invoking Gale and Nikaido (1965, Theorem 4), a sufficient (not necessary) condition for uniqueness is for all the principal minors of the Jacobian $J(X,U,\pi)$ to be positive $\forall\ \pi \in [0,1]^P$, in which case $J(X,U,\pi)$ is a *P-matrix* over $\pi \in [0,1]^P$. Maintain Assumption 5 and, for each $\emptyset \neq \mathcal{S} \subseteq \{1,\ldots,P\}$ define,

$$s_{\mathcal{S}}(X,U,\pi) \equiv \sum_{p\in\mathcal{S}} \left(\frac{\Delta_p \cdot f_p(X,U,\pi_{-p})}{1+\Delta_p \cdot f_p(X,U,\pi_{-p})} \right), \text{ with } s(X,U,\pi) \equiv \sum_{p=1}^{P} \frac{\Delta_p \cdot f_p(X,U,\pi_{-p})}{1+\Delta_p \cdot f_p(X,U,\pi_{-p})}.$$

Take the principal submatrix of $J(X,U,\pi)$ with rows and columns indexed by $\mathcal{S}$. In the Appendix, we show that the corresponding principal minor is given by,

$$\mathcal{M}_{\mathcal{S}}(X,U,\pi) \equiv \prod_{p\in\mathcal{S}} \Big(1+\Delta_p \cdot f_p(X,U,\pi_{-p})\Big) \cdot (1 - s_{\mathcal{S}}(X,U,\pi)), \quad \emptyset \neq \mathcal{S} \subseteq \{1,\ldots,P\}. \quad (3)$$

Note that $\mathcal{M}_{\mathcal{S}}(X,U,\pi) \equiv 1$ whenever $|\mathcal{S}| = 1$, corresponding to the diagonal elements of $J(X,U,\pi)$. Thus, by the Gale-Nikaido P-matrix conditions, (X,U) will have a unique BNE if $\prod_{p\in\mathcal{S}} \Big(1+\Delta_p \cdot f_p(X,U,\pi_{-p})\Big) > 0$, and $\sum_{p\in\mathcal{S}} \left(\frac{\Delta_p \cdot f_p(X,U,\pi_{-p})}{1+\Delta_p \cdot f_p(X,U,\pi_{-p})} \right) < 1$ $\forall\ \pi \in [0,1]^P$, for all $|\mathcal{S}| \geq 2$. Next, let us analyze $J(X,U,\pi)^{-1}$. From (3), the determinant of $J(X,U,\pi)$ is $D(X,U,\pi) \equiv \prod_{p=1}^{P} \Big(1+\Delta_p \cdot f_p(X,U,\pi_{-p})\Big) \cdot (1 - s(X,U,\pi))$. By Assumption 5, $D(X,U,\pi) \neq 0\ \forall\ \pi \in [0,1]^P$ a.s. Hence, $J(X,U,\pi)$ is invertible, and every existing BNE is regular w.p.1. Denote the (p,q) element of the Jacobian inverse, $\Big(J(X,U,\pi)^{-1}\Big)_{pq}$, as $H_{pq}(X,U,\pi)$. Let $\delta_{pq} \equiv \mathbb{1}\{p = q\}$. In the Appendix, we

show that,

$$H_{pq}(X,U,\pi) = \frac{\delta_{pq}}{1+\Delta_p \cdot f_p(X,U,\pi_{-p})} + \frac{1}{1-s(X,U,\pi)} \cdot \frac{\Delta_p \cdot f_p(X,U,\pi_{-p})}{(1+\Delta_p \cdot f_p(X,U,\pi_{-p})) \cdot (1+\Delta_q \cdot f_q(X,U,\pi_{-q}))}. \tag{4}$$

For any regular BNE π^*, and any $(p,q) \in \{1,\ldots,P\}^2$, let

$$B_{pq}(X,U,\pi^*) \equiv H_{pq}(X,U,\pi^*) \cdot f_q(X,U,\pi^*_{-q}). \tag{5}$$

The Implicit Function Theorem (IFT) yields,

$$\frac{\partial \pi^*_p}{\partial X_q} = B_{pq}(X,U,\pi^*) \cdot \beta_{q0}, \quad \frac{\partial \pi^*_p}{\partial X_c} = \sum_{q=1}^{P} B_{pq}(X,U,\pi^*) \cdot \gamma_{q0}. \tag{6}$$

Let us go back to our behavioral assumptions, focusing on players' beliefs.

2.4.4 Players' beliefs

We assume that players' beliefs correspond to a regular BNE solution.

Assumption 6 ***(Regular BNE beliefs)***

(i) Let $\mathcal{B}(X,U)$ denote the total number of regular BNE given (X,U). There exists $\overline{\mathcal{E}} < \infty$ such that $\mathcal{B}(X,U) \leq \overline{\mathcal{E}}$ w.p.1.

(ii) Given (X,U), each player p selects a regular BNE solution as their beliefs. If there exist multiple regular BNE solutions, player p selects one through a selection mechanism, represented by a random device λ_p that indicates which solution is selected. The selection mechanisms λ_p and λ_q can differ for $p \neq q$.

(iii) Given (X,U), let $\mathcal{E}_p(X,U) \leq \overline{\mathcal{E}}$ denote the number of BNE that can be selected by λ_p with nonzero probability. λ_p is degenerate if $\mathcal{E}_p(X,U) \equiv 1$.

Given (X,U), let $\{\pi^{(1)}(X,U),\ldots,\pi^{\mathcal{E}_p(X,U)}(X,U)\}$ denote the collection of BNE that can be selected by λ_p with nonzero probability. Here, λ_p selects BNE $\pi^{(\ell)}(X,U)$ iff

$\lambda_p = \ell$. Then,

$$Y_p = \sum_{\ell=1}^{\mathcal{E}_p(X,U)} \mathbb{1}\Big\{X'\theta_{p0} + \Delta_p \cdot \sum_{q \neq p} \pi_q^{(\ell)}(X,U) + \eta_p - \varepsilon_p \geq 0\Big\} \cdot \mathbb{1}\Big\{\lambda_p = \ell\Big\} \tag{7}$$

If there is a unique BNE, we will denote it throughout as $\pi^*(X,U)$. In this case, (7) simplifies to $Y_p = \mathbb{1}\Big\{X'\theta_{p0} + \Delta_p \cdot \sum_{q \neq p} \pi_q^*(X,U) + \eta_p - \varepsilon_p \geq 0\Big\}$.

2.5 Distributional properties of the unobserved state variables

We proceed with the assumption that the unobserved state variables U have bounded, but not necessarily finite support.

Assumption 7 *$Supp(U)$ is bounded.*

Unlike existing work that requires unobserved state variables to have discrete support points (e.g., Liu, Xu, and Vuong (2017, Section 5.4), Aguirregabiria and Mira (2019)) when their distribution is nonparametric, we allow for $Supp(U)$ to be discrete or continuous, with or without point-masses, and we only require it to be bounded. Our next assumption involves the existence of an observable Z that controls for the dependence between the observable and unobservable state variables.

Assumption 8 ***(A control variable)*** *There exists an observable Z, possibly containing elements in common with X, such that,*

(i) *$U|(X,Z) \sim U|Z$, with cdf $F_{U|Z}(\cdot|Z)$, which is nonparametrically specified. It can be discrete or continuous, with or without point-masses.*

(ii) *$\lambda_p|(U,X,Z) \sim \lambda_p|(U,Z,\mathcal{B}(X,U))$.*

(iii) *$\varepsilon_p|(X,Z,U,\lambda_p) \sim \varepsilon_p|(Z,U,\lambda_p)$.*

(iv) *Conditional on a.e realization of (Z,U,λ_p), the distribution of ε_p is a.c.l.m, with unbounded support. Denote the cdf of $\varepsilon_p|(Z,U,\lambda_p)$ by $G_{\varepsilon_p|\Gamma_p}(\cdot|Z,U,\lambda_p)$, with density $g_{\varepsilon_p|\Gamma_p}(\cdot|Z,U,\lambda_p)$, which is bounded. $G_{\varepsilon_p|\Gamma_p}$ is nonparametrically specified.*

We allow for X and Z to have elements in common, but we rule out the case $Z \equiv X$. We will let $X^e \equiv \left((X_p^e)_{p=1}^P, X_c^e\right)$ denote the elements in X that are *excluded* from Z,

and we will let $X^I \equiv \left((X_p^I)_{p=1}^P, X_c^I\right)$ denote the elements in X that are *included* in Z. More precisely, no element in Z is functionally determined by any element in X^e, while some elements in Z are functionally determined by X^I. If the distribution of Z is a.c.l.m, then we will also assume that $Supp(X^e, Z)$ is convex.

2.5.1 An expression for $E[Y_p|X,Z,U]$

For any $\pi \in [0,1]^P$, denote,

$$\mu_p(X,Z,U,\pi_{-p},\lambda_p) \equiv G_{\varepsilon_p|\Gamma_p}\Big(X'\theta_{p0} + \Delta_p \cdot \sum_{q\neq p} \pi_q + \eta_p \,\Big|Z,U,\lambda_p\Big). \tag{8}$$

Denote $\sigma_p(\ell|U,Z) \equiv Pr(\lambda_p = \ell|U,Z)$ for $\ell = 1,\ldots,\mathcal{E}_p(X,U)$. From (7) and Assumption 8,

$$E[Y_p|X,Z,U] = \sum_{\ell=1}^{\mathcal{E}_p(X,U)} \mu_p\Big(X,Z,U,\pi_{-p}^{(\ell)}(X,U),\ell\Big)\cdot\sigma_p(\ell|U,Z). \tag{9}$$

2.6 Restrictions on $\mathcal{E}_p(X,U)$

Our results will rely on restrictions under which, at least over a subset of realizations $\mathcal{X} \subseteq Supp(X)$ of X, the number of existing BNE that can be selected by λ_p, which we have denoted as $\mathcal{E}_p(X,U)$, is either constant, or it depends on X only through X^I, so we can write it as $\mathcal{E}_p(X,U) \equiv \mathcal{E}_p(X^I,U)$. The simplest such case is when λ_p is degenerate.

Assumption 9 ***(Degenerate selection mechanism)*** *λ_p is degenerate, so $\mathcal{E}_p(X,U) \equiv 1$.*

Degeneracy is a behavioral assumption, but it will be trivially satisfied if the BNE system has a unique solution a.s. As we mentioned previously, if all principal minors of the Jacobian $J(X,U,\pi)$ are positive, the BNE for (X,U) will be unique. To present a sufficient condition for this P-matrix property, it is useful to simplify

the expression for the principal minors $\mathcal{M}_{\mathcal{S}}(X,U,\pi)$ given in (3). For $|\mathcal{S}| \geq 2$ let,

$$
\begin{aligned}
T_{\mathcal{S}}(X,U,\pi) \equiv & \sum_{\substack{p<q \\ (p,q)\in\mathcal{S}}} \Delta_p \cdot \Delta_q \cdot f_p(X,U,\pi_{-p}) \cdot f_q(X,U,\pi_{-q}) \\
& +2\cdot \sum_{\substack{p<q<r \\ (p,q,r)\in\mathcal{S}}} \Delta_p \cdot \Delta_q \cdot \Delta_r \cdot f_p(X,U,\pi_{-p}) \cdot f_q(X,U,\pi_{-q}) \cdot f_r(X,U,\pi_{-r}) \\
& +3\cdot \sum_{\substack{p<q<r<s \\ (p,q,r,s)\in\mathcal{S}}} \Delta_p \cdot \Delta_q \cdot \Delta_r \cdot \Delta_s \cdot f_p(X,U,\pi_{-p}) \cdot f_q(X,U,\pi_{-q}) \cdot f_r(X,U,\pi_{-r}) \cdot f_s(X,U,\pi_{-s}) \\
& + \cdots\cdots + \Big(|\mathcal{S}|-1\Big) \cdot \prod_{p\in\mathcal{S}} \Delta_p \cdot f_p(X,U,\pi_{-p}).
\end{aligned}
\tag{10}
$$

For any $|\mathcal{S}| \geq 2$, we show in the Appendix that simplifies to,

$$
\mathcal{M}_{\mathcal{S}}(X,U,\pi) = 1 - T_{\mathcal{S}}(X,U,\pi). \tag{11}
$$

Thus, $J(X,U,\pi)$ is a P-matrix iff $T_{\mathcal{S}}(X,U,\pi) < 1 \;\forall\; \pi \in [0,1]^P$, $\mathcal{S} \subseteq \{1,\ldots,P\} : |\mathcal{S}| \geq 2$. Hence, it can be useful to look for a bound for $T_{\mathcal{S}}(X,U,\pi)$ over any pre-specified subset $\mathcal{X} \subseteq Supp(X)$. For a given $\mathcal{X} \subseteq Supp(X)$, let $\overline{f}_p^{\mathcal{X}} \equiv \sup_{\substack{X\in\mathcal{X} \\ U\in Supp(U) \\ \pi_{-p}\in[0,1]^{P-1}}} f_p(X,U,\pi_{-p})$.

Note that $\overline{f}_p^{\mathcal{X}} \leq \overline{f}_p \;\forall\; \mathcal{X} \subseteq Supp(X)$, where $\overline{f}_p$ is the global bound for the density $f_{\varepsilon_p|U}$ described in Assumption 4. Let $(a \vee b) \equiv \max\{a,b\}$ and define,

$$
\begin{aligned}
\overline{T}_{\mathcal{X}}(\Delta) \equiv & \sum_{1\leq p<q\leq P} \Big((\Delta_p \cdot \Delta_q \cdot \overline{f}_p^{\mathcal{X}} \cdot \overline{f}_q^{\mathcal{X}}) \vee 0\Big) + 2 \cdot \sum_{1\leq p<q<r\leq P} \Big((\Delta_p \cdot \Delta_q \cdot \Delta_r \cdot \overline{f}_p^{\mathcal{X}} \cdot \overline{f}_q^{\mathcal{X}} \cdot \overline{f}_r^{\mathcal{X}}) \vee 0\Big) \\
& + 3 \cdot \sum_{1\leq p<q<r<s\leq P} \Big((\Delta_p \cdot \Delta_q \cdot \Delta_r \cdot \Delta_s \cdot \overline{f}_p^{\mathcal{X}} \cdot \overline{f}_q^{\mathcal{X}} \cdot \overline{f}_r^{\mathcal{X}} \cdot \overline{f}_s^{\mathcal{X}}) \vee 0\Big) + \cdots + (P-1) \cdot \Big(\Big(\prod_{p=1}^{P} \Delta_p \overline{f}_p^{\mathcal{X}}\Big) \vee 0\Big).
\end{aligned}
$$

We have $Pr(T_{\mathcal{S}}(X,U,\pi) < \overline{T}_{\mathcal{X}}(\Delta) \big| X \in \mathcal{X}) = 1 \;\forall \mathcal{S}$. Hence, if $Pr(\overline{T}_{\mathcal{X}}(\Delta) < 1 | X \in \mathcal{X}) = 1$, then whenever $X \in \mathcal{X}$, the underlying game will have a unique BNE w.p.1. We may assume that $Pr(\overline{T}_{\mathcal{X}}(\Delta) < 1) = 1$ for $\mathcal{X} = Supp(X)$, in which case the game will have a unique BNE for a.e X. Otherwise, if we have ex-ante information about the signs of some slope coefficients, we can use it to pre-specify a set $\mathcal{X} \subset Supp(X)$ for which

$\overline{f}_p^{\mathcal{X}} \ll \overline{f}_p$ is a plausible assumption, and the condition $Pr(\overline{T}_{\mathcal{X}}(\Delta) < 1 | X \in \mathcal{X}) = 1$ is more likely to hold.

Consider a set of the form $\overline{\mathcal{X}} \equiv \prod_{\ell=1}^{d_X} [\underline{x}_\ell, \overline{x}_\ell]$ (allowing for $\underline{x}_\ell = -\infty$ or $\overline{x}_\ell = +\infty$), and let $\mathcal{X} \equiv \overline{\mathcal{X}} \cap Supp(X) \neq \emptyset$. Suppose that, for at least one p, an economic theory predicts the sign of at least one nonzero slope coefficient in θ_{p0} for a regressor with large enough support (relative to the support of other regressors). Suppose the predicted sign is positive. We would then choose $\underline{x}_\ell$ large enough (and $\overline{x}_\ell = +\infty$) for the elements in X whose coefficient is predicted to be positive; choose $\overline{x}_\ell$ small enough (and $\underline{x}_\ell = -\infty$) for the elements in X whose coefficient is predicted to be negative; and set $[\underline{x}_\ell, \overline{x}_\ell]$ to a reasonably narrow interval for the remaining elements in X (whose slope coefficients have unknown sign). The goal would be to increase the likelihood that $X'\theta_{p0}$ is concentrated around large positive values over $\mathcal{X}$. If the regressor in question has a predicted negative slope coefficient, we can modify the bounds to increase the likelihood that $X'\theta_{p0}$ is concentrated around large negative values. In either case, our goal would be to design $\overline{\mathcal{X}}$ to increase the likelihood that $|X'\theta_{p0}|$ is bounded away from zero over $\mathcal{X}$, in an effort to have $\overline{f}_p^{\mathcal{X}} \ll \overline{f}_p$, so that the restriction $Pr(\overline{T}_{\mathcal{X}}(\Delta) < 1 | X \in \mathcal{X}) = 1$ is more likely to hold. We introduce this as a formal assumption.

Assumption 10 ***(A sufficient condition for BNE uniqueness over $\mathcal{X}$)*** *The set $\mathcal{X}$ is such that $Pr(\overline{T}_{\mathcal{X}}(\Delta) < 1 | X \in \mathcal{X}) = 1$.*

In the $P = 2$ case, Assumption 10 is immediately satisfied if $\Delta_1 \cdot \Delta_2 \leq 0$, and it reduces to the well-known condition $\Delta_1 \cdot \Delta_2 \cdot f_1(X, U, \pi_2) \cdot f_2(X, U, \pi_1) < 1 \ \forall \ \pi \in [0,1]^2$ whenever $\Delta_1 \cdot \Delta_2 > 0$ (see, e.g., Aradillas-López (2010), de Paula and Tang (2012)). Next, for each p, let $\overline{\Delta}_p \equiv \sup_{\Delta_p \in Supp(\Delta_p)} \left|\Delta_p\right|$. Each $\overline{\Delta}_p$ is finite since $Supp(\Delta)$ is bounded. The following condition is more restrictive than Assumption 10, but it is easy to verify for any collection of parametric densities $(f_{\varepsilon_p|U})_{p=1}^P$, and bounds $(\overline{\Delta}_p)_{p=1}^P$. As we will show below, it can also allow us to identify the directions of strategic interaction effects.

Assumption 10′ ***(A stronger, sufficient condition for BNE uniqueness over $\mathcal{X}$)*** *(I) $\overline{\Delta}_p \overline{f}_p^{\mathcal{X}} < 1$ for every $p = 1, \ldots, P$, and (II)* $\displaystyle\sum_{p=1}^{P} \frac{\overline{\Delta}_p \overline{f}_p^{\mathcal{X}}}{1 + \overline{\Delta}_p \overline{f}_p^{\mathcal{X}}} < 1.$

If Assumption 10′ holds, all principal minors are positive $\forall\ \pi \in [0,1]^P$ whenever $X \in \mathcal{X}$. This follows from (3), and we show it in the Appendix. If $P = 2$, condition (I) implies (II), so Assumption 10′ reduces to $\overline{\Delta}_p \cdot \overline{f}_p^{\mathcal{X}} < 1$ for $p = 1, 2$, which is stronger than the restriction $\Delta_1 \cdot \Delta_2 \cdot f_1(X, U, \pi_2) \cdot f_2(X, U, \pi_1) < 1\ \forall\ \pi \in [0,1]^2$ in Assumption 10. For the case $P \geq 3$, conditions (I) and (II) must be verified independently. If any of Assumptions 9, 10 or 10′ hold, λ_p will be degenerate whenever $X \in \mathcal{X}$. If λ_p is degenerate, let us write $\mu_p(X, Z, U, \pi_{-p}, \lambda_p) \equiv \mu_p(X, Z, U, \pi_{-p})$. Thus, under any of Assumptions 9, 10 or 10′, the expression in (9) simplifies to $E[Y_p|X, Z, U] = \mu_p(X, Z, U, \pi^*_{-p}(X, U))$ if $X \in \mathcal{X}$. If degeneracy or uniqueness do not hold, our main identification results for slope coefficients in Proposition 1 will still follow if the following restriction is satisfied.

Assumption 11 ***(Constant number of regular BNE over $\mathcal{X}$)*** *The number of existing regular BNE, $\mathcal{B}(X, U)$, is the same for any (X, U) such that $X \in \mathcal{X}$. We will denote this number as $\overline{\mathcal{B}}_{\mathcal{X}}$.*

BNE uniqueness over $\mathcal{X}$ is a special case. Recall from Assumption 8 that $\lambda_p|(U, X, Z) \sim \lambda_p|(U, Z, \mathcal{B}(X, U))$. Thus, under Assumption 11, $\mathcal{E}_p(X, U)$ depends on X only through X^I. Hence, if $X \in \mathcal{X}$, we have $\mathcal{E}_p(X, U) \equiv \mathcal{E}_p(X^I, U)$, and equation (9) becomes $E[Y_p|X, Z, U] = \sum_{\ell=1}^{\mathcal{E}_p(X^I,U)} \mu_p\Big(X, Z, U, \pi^{(\ell)}_{-p}(X, U), \ell\Big) \cdot \sigma_p(\ell|U, Z)$.

2.7 Average derivatives for $E[Y_p|X, Z]$

Under our restrictions, if $X \in \mathcal{X}$,

$$
\begin{aligned}
E[Y_p|X, Z] &= \int \mu_p(X, Z, U, \pi^*_{-p}(X, U)) dF_{U|Z}(U|Z) \ \text{ if any of Assumptions 9, 10,} \\
&\quad \text{or 10′ hold} \\
E[Y_p|X, Z] &= \sum_{b=1}^{\overline{\mathcal{B}}_{\mathcal{X}}} \sum_{\ell=1}^{b} \int \Big(\mu_p(X, Z, U, \pi^{(\ell)}_{-p}(X, U), \ell) \cdot \sigma_p(\ell|U, Z) \cdot \mathbb{1}\{\mathcal{E}_p(X^I, U) = b\}\Big) dF_{U|Z}(U|Z) \\
&\quad \text{if Assumption 11 holds.}
\end{aligned}
\tag{12}
$$

We will focus on the average derivatives of X^e, the elements in X excluded from Z. Let $\beta^e_{p0} \subseteq \beta_{p0}$ and $\gamma^e_{p0} \subseteq \gamma_{p0}$ denote, respectively, the slope coefficients of X^e_p

and X_c^e for player p. If λ_p is degenerate, write $g_{\varepsilon_p|\Gamma_p}(\cdot|Z,U,\lambda_p) \equiv g_{\varepsilon_p|Z,U}(\cdot|Z,U)$. Let $B_{pq}(X,Z,\pi)$ be as defined in (5). For $X \in \mathcal{X}$, define $\Psi_{pp}(X,Z)$ as follows.

If Assumption 11 holds:

$$\Psi_{pp}(X,Z) \equiv$$
$$\int \Bigg(\sum_{b=1}^{\overline{\mathcal{B}}_{\mathcal{X}}} \sum_{\ell=1}^{b} \Big[1 + \Delta_p \cdot \sum_{q \neq p} B_{qp}(X,U,\pi^{(\ell)}(X,U)) \Big] \cdot g_{\varepsilon_p|\Gamma_p} \Big(X'\theta_{p0} + \Delta_p \cdot \sum_{q \neq p} \pi_q^{(\ell)}(X,U) + \eta_p \,\Big|\, Z,U,\ell \Big)$$
$$\cdot \sigma_p(\ell|U,Z) \mathbb{1}\{\mathcal{E}_p(X^I,U) = b\} \Bigg) dF_{U|Z}(U|Z).$$

If any of Assumptions 9, 10, or 10′ hold:

$$\Psi_{pp}(X,Z) \equiv$$
$$\int \Big[1 + \Delta_p \cdot \sum_{q \neq p} B_{qp}(X,U,\pi^*(X,U)) \Big] \cdot g_{\varepsilon_p|Z,U} \Big(X'\theta_{p0} + \Delta_p \cdot \sum_{q \neq p} \pi_q^*(X,U) + \eta_p \,\Big|\, Z,U \Big) dF_{U|Z}(U|Z).$$

And, for each $(p,q) \in \{1,\ldots,P\}^2$ define $\Psi_{pq}(X,Z)$ as follows.

If Assumption 11 holds:

$$\Psi_{pq}(X,Z) \equiv$$
$$\int \Bigg(\Delta_p \cdot \sum_{b=1}^{\overline{\mathcal{B}}_{\mathcal{X}}} \sum_{\ell=1}^{b} \Big[\sum_{r \neq p} B_{rq}(X,U,\pi^{(\ell)}(X,U)) \Big] \cdot g_{\varepsilon_p|\Gamma_p} \Big(X'\theta_{p0} + \Delta_p \cdot \sum_{q \neq p} \pi_q^{(\ell)}(X,U) + \eta_p \,\Big|\, Z,U,\ell \Big)$$
$$\cdot \sigma_p(\ell|U,Z) \cdot \mathbb{1}\{\mathcal{E}_p(X^I,U) = b\} \Bigg) dF_{U|Z}(U|Z).$$

If any of Assumptions 9, 10 or 10′ hold:

$$\Psi_{pq}(X,Z) \equiv$$
$$\int \Delta_p \cdot \Big[\sum_{r \neq p} B_{rq}(X,U,\pi^*(X,U)) \Big] \cdot g_{\varepsilon_p|Z,U} \Big(X'\theta_{p0} + \Delta_p \cdot \sum_{q \neq p} \pi_q^*(X,U) + \eta_p \,\Big|\, Z,U \Big) dF_{U|Z}(U|Z).$$

Proposition 1 *Suppose Assumptions 1-8 hold, and that any one of Assumptions 9, 10, 10', or 11 is also satisfied. Then, if $X \in \mathcal{X}$, for any $(p,q) \in \{1,\ldots,P\}^2$, we have,*

$$\begin{aligned} &\frac{\partial E[Y_p|X,Z]}{\partial X_p^e} = \beta_{p0}^e \cdot \Psi_{pp}(X,Z), \quad \frac{\partial E[Y_p|X,Z]}{\partial X_q^e} = \beta_{q0}^e \cdot \Psi_{pq}(X,Z), \\ &\frac{\partial E[Y_p|X,Z]}{\partial X_c^e} = \gamma_{p0}^e \cdot \Psi_{pp}(X,Z) + \sum_{q \neq p} \gamma_{q0}^e \cdot \Psi_{pq}(X,Z) \end{aligned} \tag{13}$$

Let $\phi_{\mathcal{X}}$ be a weight function for (X,Z) such that $\phi_{\mathcal{X}}(X,Z) > 0$ if $X \in \mathcal{X}$ and $\phi_{\mathcal{X}}(X,Z) \equiv 0$ otherwise. Assume $E[|\Psi_{pp}(X,Z)| \cdot \phi_{\mathcal{X}}(X,Z)] < \infty$ and $E[|\Psi_{pq}(X,Z)| \cdot \phi_{\mathcal{X}}(X,Z)] < \infty$ for each $(p,q) \in \{1,\ldots,P\}^2$. Let $\delta_{pp}^{\phi_{\mathcal{X}}} \equiv E[\Psi_{pp}(X,Z) \cdot \phi_{\mathcal{X}}(X,Z)]$ and $\delta_{pq}^{\phi_{\mathcal{X}}} \equiv E[\Psi_{pq}(X,Z) \cdot \phi_{\mathcal{X}}(X,Z)]$. Then,

$$\begin{aligned} &E\left[\frac{\partial E[Y_p|X,Z]}{\partial X_p^e} \cdot \phi_{\mathcal{X}}(X,Z)\right] = \beta_{p0}^e \cdot \delta_{pp}^{\phi_{\mathcal{X}}}, \\ &E\left[\frac{\partial E[Y_p|X,Z]}{\partial X_q^e} \cdot \phi_{\mathcal{X}}(X,Z)\right] = \beta_{q0}^e \cdot \delta_{pq}^{\phi_{\mathcal{X}}}, \\ &E\left[\frac{\partial E[Y_p|X,Z]}{\partial X_c^e} \cdot \phi_{\mathcal{X}}(X,Z)\right] = \gamma_{p0}^e \cdot \delta_{pp}^{\phi_{\mathcal{X}}} + \sum_{q \neq p} \gamma_{q0}^e \cdot \delta_{pq}^{\phi_{\mathcal{X}}} \end{aligned} \tag{14}$$

Proof: (13) follows (12) using the definition of μ_p in (8) and the expressions in (6). The result in (14) follows immediately from (13). ■

Our assumptions yield a game-theoretic version of well known identification properties of average derivatives in linear-index models. For observable payoff shifters excluded from Z, we have two results. First, the average derivatives of player-specific payoff shifters identify their payoff slope coefficients, up to a proportionality constant. Second, the average derivatives of common payoff shifters identify a linear combination, across players, of their payoff slope coefficients. If a subset $\widetilde{X}_c^e \subseteq X_c^e$ of common payoff shifters is assumed to have the same slope coefficients, $\widetilde{\gamma}_0^e$, for all players, then $\widetilde{\gamma}_0^e$ is identified up to a proportionality constant, since $E\left[\frac{\partial E[\sum_{p=1}^P Y_p|X,Z]}{\partial \widetilde{X}_c^e} \cdot \phi_{\mathcal{X}}(X,Z)\right] = \widetilde{\gamma}_0^e \cdot \sum_{p=1}^P \left(\delta_{pp}^{\phi_{\mathcal{X}}} + \sum_{q \neq p} \delta_{pq}^{\phi_{\mathcal{X}}}\right) \equiv \widetilde{\gamma}_0^e \cdot \overline{\delta}^{\phi_{\mathcal{X}}}$.

2.8 Identifying the direction of competition effects using average derivatives

Proposition 1 suggests that the direction of the strategic effect Δ_p can be identified by comparing $E\left[\frac{\partial E[Y_p|X,Z]}{\partial X_q^e}\cdot\phi_{\mathcal{X}}(X,Z)\right]$ and $E\left[\frac{\partial E[Y_q|X,Z]}{\partial X_q^e}\cdot\phi_{\mathcal{X}}(X,Z)\right]$ for $p\neq q$. However, identifying the sign of Δ_p from this comparison requires knowing the signs of the proportionality constants $\delta_{qq}^{\phi_{\mathcal{X}}}$ and $\delta_{pq}^{\phi_{\mathcal{X}}}$, which are unrestricted in Proposition 1. These signs can be identified for any p if we have knowledge about the predicted signs of at least one slope coefficient in β_{p0}^e, and β_{q0}^e for some $q\neq p$. More generally, however, if we specifically maintain Assumption 10′, and we assume that the sign of Δ_p is constant for player p. In this case, $\delta_{pp}^{\phi_{\mathcal{X}}}>0$, and the sign of $\delta_{pq}^{\phi_{\mathcal{X}}}$ will correspond to the sign of Δ_p. We present this result next.

Proposition 2 *Maintain Assumption 10′ in the context of Proposition 1. Then, we have $\delta_{pp}^{\phi_{\mathcal{X}}}>0\ \forall\ p$. Fix any p and assume that the sign of Δ_p is constant w.p.1 (i.e., either $\Delta_p\geq 0$ w.p.1, or $\Delta_p\leq 0$ w.p.1). Then, for this p, and any $q\neq p$, we have: (i) $\delta_{pq}^{\phi_{\mathcal{X}}}\geq 0$ if $\Delta_p\geq 0$ a.s. (ii) $\delta_{pq}^{\phi_{\mathcal{X}}}\leq 0$ if $\Delta_p\leq 0$ a.s. And (iii) $\delta_{pq}^{\phi_{\mathcal{X}}}=0$ iff $\Delta_p=0$ for $F_{U|Z}(\cdot|Z)$-almost every U, and almost every Z in the support of $Z\mid X\in\mathcal{X}$.*

Proof: In the Appendix.

2.9 Using Proposition 2 to identify the direction of strategic effects

Take any pair (p,q) for whom the conditions of Proposition 2 are satisfied. Let $X_{q,j}^e$ denote the j^{th} component of X_q^e, with slope coefficient $\beta_{q0,j}^e$. Define

$$\vartheta_{pq}^j\equiv E\left[\frac{\partial E[Y_p|X,Z]}{\partial X_{q,j}^e}\cdot\phi_{\mathcal{X}}(X,Z)\right]\times E\left[\frac{\partial E[Y_q|X,Z]}{\partial X_{q,j}^e}\cdot\phi_{\mathcal{X}}(X,Z)\right]=\delta_{pq}^{\phi_{\mathcal{X}}}\cdot\delta_{qq}^{\phi_{\mathcal{X}}}\cdot\left(\beta_{q0,j}^e\right)^2,$$
$$\vartheta_{pq}\equiv\sum_j\vartheta_{pq}^j=\delta_{pq}^{\phi_{\mathcal{X}}}\cdot\delta_{qq}^{\phi_{\mathcal{X}}}\cdot\left\|\beta_{q0}^e\right\|^2,$$

where $\|\cdot\|$ is the Euclidean norm. We refer to ϑ_{pq} as the *average derivative competition index*. It is the inner product of p and q's weighted average derivatives w.r.t X_q^e. By Proposition 2, if $\|\beta_{q0}^e\|\neq 0$, the sign of ϑ_{pq} identifies the sign of Δ_p, with $\vartheta_{pq}\geq 0$

if $\Delta_p \geq 0$, $\vartheta_{pq} \leq 0$ if $\Delta_p \leq 0$, and $\vartheta_{pq} = 0$ iff $\Delta_p = 0$ for $F_{U|Z}(\cdot|Z)$-almost every U, and almost every Z in the support of $Z \mid X \in \mathcal{X}$. If we assume that every Δ_p has constant sign w.p.1 $\forall\ p$, we can aggregate $\vartheta_p \equiv \sum_{q \neq p} \sum_{j=1}^{d_{X_q^e}} \vartheta_{pq}^j = \sum_{q \neq p} \delta_{pq}^{\phi_{\mathcal{X}}} \cdot \delta_{qq}^{\phi_{\mathcal{X}}} \cdot \left\| \beta_{q0}^e \right\|^2$. If $\|\beta_{q0}^e\| \neq 0$ for some $q \neq p$, the sign of ϑ_p identifies the sign of Δ_p by our previous arguments.

3 A simulation study

We apply our results to simulated data from a two-player game.

3.1 Payoff functions and payoff shifters

Payoff functions are,

$$u_p(y, X, \Delta_p, \eta_p) = y_p \times \Big(\beta_{p1_0} X_{p1} + \beta_{p2_0} X_{p2} + \gamma_{p1_0} X_{c1} + \gamma_{p2_0} X_{c2} + \Delta_p \cdot y_{-p} + \eta_p - \varepsilon_p \Big),$$

with two observable, player-specific payoff shifters, (X_{p1}, X_{p2}), and two observable, common payoff shifters, (X_{c1}, X_{c2}). The slope coefficients were fixed as follows. For player 1, we set $\beta_{11_0} = 1/2$, $\beta_{12_0} = -1/4$, $\gamma_{11_0} = 3/4$, $\gamma_{12_0} = 1/4$. For player 2, we set $\beta_{21_0} = 1/3$, $\beta_{22_0} = -1/3$, $\gamma_{21_0} = 3/4$, $\gamma_{22_0} = 1/4$. Note that $\gamma_{11_0} = \gamma_{21_0} = 3/4$, and $\gamma_{12_0} = \gamma_{22_0} = 1/4$. Thus, we will denote $\gamma_{11_0} = \gamma_{21_0} \equiv \gamma_{1_0}$ and $\gamma_{12_0} = \gamma_{22_0} \equiv \gamma_{2_0}$, where $\gamma_{1_0} = 3/4$ and $\gamma_{2_0} = 1/4$.

We generated the payoff shifters in X and η from multimodal distributions (a common feature in observational studies), using mixtures of distributions. Let $\xi \sim \sum_{i=1}^{k} w_i \cdot F_i$ indicate that ξ is distributed as a mixture of the distributions $(F_i)_{i=1}^k$, with mixture weights $(w_i)_{i=1}^k$. The components in X were generated as mutually independent mixtures of Normal distributions. For player 1, (X_{11}, X_{12}) were generated as, $X_{11} \sim (1/4) \cdot \mathcal{N}(-2, 2) + (1/4) \cdot \mathcal{N}(0, 1) + (1/2) \cdot \mathcal{N}(2, 2)$, and $X_{12} \sim (3/4) \cdot \mathcal{N}(-1, 3) + (1/4) \cdot \mathcal{N}(1, 3)$. For player 2, (X_{21}, X_{22}) were generated as $X_{21} \sim (3/5) \cdot \mathcal{N}(-2, 2) + (1/5) \cdot \mathcal{N}(0, 1) + (1/5) \cdot \mathcal{N}(2, 2)$, and $X_{22} \sim (1/3) \cdot \mathcal{N}(-1, 3) + (2/3) \cdot \mathcal{N}(1, 3)$. Note that X_{p1} is a mixture of $\mathcal{N}(-2, 2)$, $\mathcal{N}(0, 1)$ and $\mathcal{N}(2, 2)$, and X_{p2} is a mixture of $\mathcal{N}(-1, 3)$ and $\mathcal{N}(1, 3)$, with different mixture weights for each p. The common payoff shifters, (X_{c1}, X_{c2}) were generated as $X_{c1} \sim (1/6) \cdot \mathcal{N}(-1, 2) + (1/6) \cdot \mathcal{N}(0, 1) + (2/3) \cdot \mathcal{N}(2, 1/3)$, and $X_{c2} \sim (1/2) \cdot \mathcal{N}(-1, 1) + (1/2) \cdot \mathcal{N}(1, 2)$. From here, we generated

$Z = X_{c1} + X_{c2} + \nu_z$, where $\nu_z \sim \mathcal{N}(0,1)$, independent of X. Z will capture the dependence between the observable and unobservable state variables, as we describe next.

The unobserved state variables (η_1, η_2) were generated as follows. First, we produced $(\nu_{\eta_1}, \nu_{\eta_2})$ as a pair of independent mixtures of Uniform distributions, with $\nu_{\eta_1} \sim (1/3)\cdot U[-2,0] + (1/3)\cdot U[-1,1] + (1/3)\cdot U[0,2]$, and $\nu_{\eta_2} \sim (1/4)\cdot U[-2,0] + (1/4)\cdot U[-1,1] + (1/2)\cdot U[0,2]$. From here, we generated $\eta_1 = \nu_{\eta_1} + \Phi(Z)$ and $\eta_2 = \nu_{\eta_2} + \Phi(Z)$, where Φ is the $\mathcal{N}(0,1)$ cdf. As assumed in our main results, (η_1, η_2) have bounded support, with $\eta_p \in [-2,3]$. The competition effects were generated as $\Delta_1 = \nu_{\Delta_1} + \Phi(Z)$ and $\Delta_2 = \nu_{\Delta_2} + \Phi(Z)$, where $(\nu_{\Delta_1}, \nu_{\Delta_2})$ are mutually independent Uniform random variables, with $\nu_{\Delta_1} \sim U[0, 1.5]$ and $\nu_{\Delta_2} \sim U[0,1]$. Note that $\Delta_1 \in [0, 2.5]$ and $\Delta_2 \in [0,2]$. Z captures the dependence between (η, Δ) and X through $\Phi(Z) \in (0,1)$. Finally, the privately observed payoff shocks $(\varepsilon_1, \varepsilon_2)$ were generated as iid $\mathcal{N}(0,1)$ random variables, independent of all other covariates.

3.2 Designs analyzed: substitutes, complements and opposite-sign competition effects

The case $\Delta_p \geq 0$ in our basic design corresponds to a strategic-complements game. We will also present results for a strategic-substitutes design, where we multiply Δ_p by -1, leaving all other design features unchanged, and for designs where Δ_1 and Δ_2 have opposite signs. In each case, we simulated 1000 samples of sizes $n = 500, 1000, 2000$. For each simulation, we generated an iid sample $(Y_{1i}, Y_{2i}, X_i, Z_i)_{i=1}^n$ of realizations from the corresponding game.

3.3 BNE uniqueness

Since $\overline{f}_p = 1/\sqrt{2\pi}$, $\overline{\Delta}_1 = 2.5$ and $\overline{\Delta}_2 = 2$, Assumption 10′ is satisfied for all X. Thus, Propositions 1 and 2 hold for $\mathcal{X} \equiv Supp(X)$ in each one of our designs.

3.4 Estimation of weighted average derivatives

We use Z as the control variable described in Assumption 8. Note that Z is not functionally determined by (i.e., a deterministic function of) X. Thus, $X^e = X$. We use density weighting, $\phi(X,Z) \equiv f_{X,Z}(X,Z)$, and we set $\mathcal{X} = Supp(X)$. Since

$Supp(X,Z)$ is convex (in fact, unbounded), we used the kernel-based estimator described in Powell, Stock, and Stoker (1989) (PSS), perhaps the simplest, easiest to compute nonparametric estimator available. Using equation (3.1) in PSS, for $\xi = X_p, X_q, X_c$, we estimate $\widehat{E}\left[\frac{\partial E[Y_p|X,Z]}{\partial \xi} \cdot f_{X,Z}(X,Z)\right] = -\frac{2}{n}\sum_{i=1}^{n} \frac{\partial \widehat{f}^{-i}_{X,Z}(X_i,Z_i)}{\partial \xi} Y_{pi}$, where $\widehat{f}^{-i}_{X,Z}(x,z)$ is given by $\widehat{f}^{-i}_{X,Z}(x,z) = \frac{1}{n-1}\sum_{j\neq i} K\left(\frac{X_j - x}{h_n}, \frac{Z_j - z}{h_n}\right)$. The kernel K and bandwidth h_n satisfy the restrictions in Theorem 3.3 of PSS for $dim(X,Z) = 7$. For each $\xi_\ell \in (X,Z)$, we used $h_n = \widehat{\sigma}(\xi_\ell) \cdot n^{-1/10+1/1000}$. We used $K(\psi_1, \ldots, \psi_7) = \prod_{\ell=1}^{7} k(\psi_\ell)$, where k is symmetric around zero, with support $[-S, S]$, and bias-reducing of order 6. We used $k(\psi) = \mathbb{1}\{|\psi| \leq S\} \cdot (c_1(S^2 - \psi^2)^2 + c_2(S^2 - \psi^2)^4 + c_3(S^2 - \psi^2)^6)$. For its support we set $S = 10$, and c_1, c_2, c_3 were chosen so $k(\cdot)$ is bias-reducing of order 6.

3.5 Results

3.5.1 Payoff slope coefficients

Since all slope coefficients are nonzero, we focus on their ratios, which are point-identified by Proposition 1. For the slopes of player-specific payoff shifters we estimated, $\widehat{\left(\frac{\beta_{11_0}}{\beta_{12_0}}\right)} = \widehat{E}\left[\frac{\partial E[Y_1|X,Z]}{\partial X_{11}} \cdot f_{X,Z}(X,Z)\right] \Big/ \widehat{E}\left[\frac{\partial E[Y_1|X,Z]}{\partial X_{12}} \cdot f_{X,Z}(X,Z)\right]$ for player 1, and $\widehat{\left(\frac{\beta_{21_0}}{\beta_{22_0}}\right)} = \widehat{E}\left[\frac{\partial E[Y_2|X,Z]}{\partial X_{21}} \cdot f_{X,Z}(X,Z)\right] \Big/ \widehat{E}\left[\frac{\partial E[Y_2|X,Z]}{\partial X_{22}} \cdot f_{X,Z}(X,Z)\right]$ for player 2. Next, since $\gamma_{1_0} = \gamma_{2_0} \equiv \gamma_0$, for the slope coefficients of common shifters X_c, we estimated, $\widehat{\left(\frac{\gamma_{1_0}}{\gamma_{2_0}}\right)} = \widehat{E}\left[\frac{\partial E[Y_1+Y_2|X,Z]}{\partial X_{c1}} \cdot f_{X,Z}(X,Z)\right] \Big/ \widehat{E}\left[\frac{\partial E[Y_1+Y_2|X,Z]}{\partial X_{c2}} \cdot f_{X,Z}(X,Z)\right]$. Table 1 summarizes results for player-specific payoff shifters. Our results show that the slope ratios are estimated with good precision, even for a small sample size, across all strategic designs. Using MSE, the strategic-substitutes design was the most precise, and the strategic-complements case was relatively less precise. These are specific features of our DGPs and not a general result. Overall, median estimates were close to their true values across the board, and the 5%-95% quantile range predicted the true sign, and shrank steadily with n in all cases. This was also true for wider quantile ranges. Table 2 includes results for common payoff shifters. Once again, our estimates performed well across the board. In this case, the strategic-complements design yielded the relatively most precise results, and strategic-substitutes was relatively less precise; however, the results were reasonably accurate in all cases.

Table 1: Results for slope coefficients of player-specific payoff shifters.

Simulation results for our estimates of $\beta_{11_0}/\beta_{12_0}$ and $\beta_{21_0}/\beta_{22_0}$

Strategic-substitutes design: $\Delta_1 \in [-2.5, 0]$, $\Delta_2 \in [-2, 0]$								
	Player 1: $\beta_{11_0}/\beta_{12_0} = -2$				Player 2: $\beta_{21_0}/\beta_{22_0} = -1$			
n	5^{th} quantile	median	95^{th} quantile	MSE	5^{th} quantile	median	95^{th} quantile	MSE
500	-3.887	-2.092	-1.361	0.146	-1.860	-1.119	-0.716	0.045
1000	-3.118	-2.085	-1.523	0.083	-1.592	-1.133	-0.798	0.028
2000	-2.780	-2.095	-1.607	0.049	-1.617	-1.145	-0.876	0.026
Strategic-complements design: $\Delta_1 \in [0, 2.5]$, $\Delta_2 \in [0, 2]$								
	Player 1: $\beta_{11_0}/\beta_{12_0} = -2$				Player 2: $\beta_{21_0}/\beta_{22_0} = -1$			
n	5^{th} quantile	median	95^{th} quantile	MSE	5^{th} quantile	median	95^{th} quantile	MSE
500	-8.971	-2.026	-0.895	0.491	-2.559	-1.071	-0.446	0.103
1000	-5.059	-2.074	-1.176	0.240	-2.063	-1.063	-0.578	0.061
2000	-3.648	-2.071	-1.384	0.138	-1.865	-1.115	-0.747	0.036
Opposite strategic effects, design I: $\Delta_1 \in [0, 2.5]$, $\Delta_2 \in [-2, 0]$								
	Player 1: $\beta_{11_0}/\beta_{12_0} = -2$				Player 2: $\beta_{21_0}/\beta_{22_0} = -1$			
n	5^{th} quantile	median	95^{th} quantile	MSE	5^{th} quantile	median	95^{th} quantile	MSE
500	-7.705	-2.031	-0.933	0.436	-1.907	-1.134	-0.705	0.050
1000	-4.788	-2.091	-1.203	0.233	-1.669	-1.136	-0.811	0.031
2000	-3.486	-2.033	-1.401	0.114	-1.596	-1.156	-0.872	0.030
Opposite strategic effects, design II: $\Delta_1 \in [-2.5, 0]$, $\Delta_2 \in [0, 2]$								
	Player 1: $\beta_{11_0}/\beta_{12_0} = -2$				Player 2: $\beta_{21_0}/\beta_{22_0} = -1$			
n	5^{th} quantile	median	95^{th} quantile	MSE	5^{th} quantile	median	95^{th} quantile	MSE
500	-3.762	-2.061	-1.338	0.149	-3.102	-1.088	-0.453	0.109
1000	-3.160	-2.080	-1.490	0.079	-2.139	-1.086	-0.605	0.060
2000	-2.837	-2.083	-1.596	0.055	-1.824	-1.114	-0.734	0.038

Table 2: Results for slope coefficients of common payoff shifters.

Simulation results for our estimates of $\gamma_{1_0}/\gamma_{2_0} = 3$

Strategic-substitutes design: $\Delta_1 \in [-2.5, 0]$, $\Delta_2 \in [-2, 0]$				
n	5^{th} quantile	median	95^{th} quantile	MSE
500	1.761	3.299	10.429	0.697
1000	2.029	3.259	6.402	0.474
2000	2.192	3.040	4.953	0.226
Strategic-complements design: $\Delta_1 \in [0, 2.5]$, $\Delta_2 \in [0, 2]$				
n	5^{th} quantile	median	95^{th} quantile	MSE
500	1.972	3.197	6.298	0.431
1000	2.218	3.078	4.697	0.218
2000	2.255	2.881	3.854	0.123
Opposite strategic effects, design I: $\Delta_1 \in [0, 2.5]$, $\Delta_2 \in [-2, 0]$				
n	5^{th} quantile	median	95^{th} quantile	MSE
500	1.858	3.489	10.655	0.763
1000	2.142	3.389	7.009	0.399
2000	2.302	3.168	5.120	0.204
Opposite strategic effects, design II: $\Delta_1 \in [-2.5, 0]$, $\Delta_2 \in [0, 2]$				
n	5^{th} quantile	median	95^{th} quantile	MSE
500	1.839	3.461	9.095	0.796
1000	2.129	3.297	6.244	0.368
2000	2.236	3.132	4.809	0.211

3.5.2 Direction of competition effects

The average derivative competition index is estimated as described in Section 2.9, with $\widehat{\vartheta}_p = \sum_{j=1}^{2} \widehat{E}\left[\frac{\partial E[Y_p|X,Z]}{\partial X_{qj}} \cdot f_{X,Z}(X,Z)\right] \times \widehat{E}\left[\frac{\partial E[Y_q|X,Z]}{\partial X_{qj}} \cdot f_{X,Z}(X,Z)\right]$, $p \neq q$. For each design, we computed $Pr(\widehat{\vartheta}_p \lesseqgtr 0)$, the proportion of simulations for which $\widehat{\vartheta}_p \lesseqgtr 0$. Table 3 includes the results. Our competition index showed good predictive power for the true direction of the competition effect, with some variation across designs. For our particular DGPs, the competition index performed best in the strategic-substitutes case, where the true direction was correctly predicted with probability $> 97\%$ for both players even when $n = 500$, reaching almost 100% correct prediction for $n = 2000$. Across all cases and sample sizes, the lowest prediction probability was in the $\Delta_1 \geq 0$ and $\Delta_2 \leq 0$ design, where the true sign of Δ_2 was predicted with probability 70% for $n = 500$, which went up to about 85% for $n = 2000$ (and

approximately 90% in unreported results for $n = 3000$). The average probability of correctly predicting the sign of Δ_p across all our designs, and across $p = 1, 2$, was 85% for $n = 500$, 91% for $n = 1000$, and 95% for $n = 2000$.

Table 3: Results for the estimated direction of competition effects.

	Strategic-substitutes design $\Delta_1 \in [-2.5, 0]$, $\Delta_2 \in [-2, 0]$			Strategic-complements design $\Delta_1 \in [0, 2.5]$, $\Delta_2 \in [0, 2]$		
	$n = 500$	$n = 1000$	$n = 2000$	$n = 500$	$n = 1000$	$n = 2000$
$Pr(\widehat{\vartheta}_1 < 0)$	97.7%	99.8%	99.9%	18.5%	12.0%	4.8%
$Pr(\widehat{\vartheta}_1 > 0)$	2.3%	0.2%	0.1%	81.5%	88.0%	95.2%
$Pr(\widehat{\vartheta}_2 < 0)$	97.2%	99.8%	99.9%	22.2%	16.2%	10.0%
$Pr(\widehat{\vartheta}_2 > 0)$	2.8%	0.2%	0.1%	77.8%	83.8%	90.0%
	Opposite strategic effects, design I $\Delta_1 \in [0, 2.5]$, $\Delta_2 \in [-2, 0]$			Opposite strategic effects, design II $\Delta_1 \in [-2.5, 0]$, $\Delta_2 \in [0, 2]$		
	$n = 500$	$n = 1000$	$n = 2000$	$n = 500$	$n = 1000$	$n = 2000$
$Pr(\widehat{\vartheta}_1 < 0)$	13.0%	5.8%	1.3%	75.9%	86.5%	93.1%
$Pr(\widehat{\vartheta}_1 > 0)$	87.0%	94.2%	98.7%	24.1%	13.5%	6.9%
$Pr(\widehat{\vartheta}_2 < 0)$	70.4%	76.9%	84.7%	7.4%	1.5%	0.4%
$Pr(\widehat{\vartheta}_2 > 0)$	29.6%	23.1%	15.3%	92.6%	98.5%	99.6%

4 Concluding remarks

Allowing for unobserved state variables is essential for incomplete information games to be a sensible approximation to reality. Average derivatives are a largely unexplored tool in this literature. Their use allows us to incorporate unobserved state variables with a flexibility that, to my knowledge, has not been previously pointed out or explored. Their distribution can be continuous or discrete, and they can be allowed to be correlated with players' private types in unrestricted ways. Beyond modeling unobserved state variables simply as additive payoff shocks, we can also allow the competition effects themselves to be unobserved state variables.

Using a binary-choice game with P asymmetric players, we presented identification results for payoff parameters that extend well-known results in semiparametric index models to a game-theoretic setting, and we also showed how to identify the direction of competition effects across players through a *competition index*, consisting of pairwise inner products of average derivatives across players. Even though our analysis focused on binary-choice games, we trust that the arguments used here can show how to extend our results to alternative parameterization of payoffs, and to games with a richer action space by exploiting the properties of regular equilibria, or other solution concepts.

A Appendix: proofs

A.1 Obtaining expression (3) for the principal minors $\mathcal{M}_{\mathcal{S}}(X,U,\pi)$

For notational simplicity, let us omit the arguments (X,U,π) and write $f_p(X,U,\pi_{-p}) \equiv f_p$, $J(X,U,\pi) \equiv J$, etc. Denote $a_p \equiv \Delta_p f_p$, with $|\Delta_p| \leq \overline{\Delta}_p$, and $0 < f_p \leq \overline{f}_p^{\mathcal{X}}$ over $X \in \mathcal{X}$. Each Δ_p may be positive, negative, or zero, and $1 + a_p \neq 0$ w.p.1 $\forall\ p$ by Assumption 5. Let $\overline{a}_p \equiv \overline{\Delta}_p \overline{f}_p^{\mathcal{X}}$. Note that $|a_p| = |\Delta_p| f_p \leq \overline{\Delta}_p \overline{f}_p^{\mathcal{X}} = \overline{a}_p$ over $\mathcal{X}$. The Jacobian J is the $P \times P$ matrix given by, $J_{pp} = 1$ for every p, and $J_{pq} = -a_p$ for every $q \neq p$. For each $\emptyset \neq \mathcal{S} \subseteq \{1,\ldots,P\}$, let $J_{\mathcal{S}}$ denote the principal submatrix of J with rows and columns indexed by $\mathcal{S}$, and let $\det(J_{\mathcal{S}}) \equiv \mathcal{M}_{\mathcal{S}}$ denote the corresponding principal minor. Let $s_{\mathcal{S}} \equiv \sum_{p\in S} \frac{a_p}{1+a_p}$, with $s \equiv \sum_{p=1}^{P} \frac{a_p}{1+a_p}$. Take a set $\mathcal{S} = \{p_1,\ldots,p_k\}$ with $k \equiv |\mathcal{S}|$, and let $\iota_k = (1,\ldots,1)' \in \mathbb{R}^k$, $a \equiv (a_p)_{p\in\mathcal{S}}$. Let $Diag(1+a)$ denote the $k \times k$ diagonal matrix with diagonal vector $1 + a$. We can then write $J_{\mathcal{S}} = Diag(1+a) - a\iota_k'$. By Assumption 5, $Diag(1+a)$ is invertible w.p.1, and we can write $J_{\mathcal{S}} = Diag(1+a) \cdot (I_k - Diag(1+a)^{-1} a\iota_k')$, where I_k is the $k \times k$ identity matrix. Thus, $det(J_{\mathcal{S}}) = det(Diag(1+a)) \cdot det(I_k - Diag(1+a)^{-1} a\iota_k')$. Next, note (Horn and Johnson (2013, Section 0.8)) that the determinant of a block matrix $M \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is given by $det(M) = det(A) \cdot det(D - CA^{-1}B)$ if A is invertible, and $det(M) = det(A - BD^{-1}C) \cdot det(D)$ if D is invertible. Applying this to $M \equiv \begin{pmatrix} I_k & Diag(1+a)^{-1}a \\ \iota_k' & 1 \end{pmatrix}$ yields, $det(M) = det(I_k) \cdot det(1 - \iota_k' I_k^{-1} Diag(1+a)^{-1} a) = det(1 - \iota_k' Diag(1+a)^{-1} a) = 1 - \iota_k' Diag(1+a)^{-1} a$, and $det(M) = det(I_k - Diag(1+a)^{-1} a\iota_k') \cdot det(1) = det(I_k - Diag(1+$

$a)^{-1}a\imath_k')$. Hence, $det(I_k - Diag(1+a)^{-1}a\imath_k') = 1 - \imath_k' Diag(1+a)^{-1}a$. Thus, $det(J_{\mathcal{S}}) = det(Diag(1+a)) \cdot (1 - \imath_k' Diag(1+a)^{-1}a) = \left(\prod_{p\in\mathcal{S}}(1+a_p)\right) \cdot (1 - \imath_k' Diag(1+a)^{-1}a)$. Since $\imath_k' Diag(1+a)^{-1}a = \sum_{p\in\mathcal{S}} \frac{a_p}{1+a_p}$, we then obtain, $\mathcal{M}_{\mathcal{S}} \equiv det(J_{\mathcal{S}}) = \left(\prod_{p\in\mathcal{S}}(1+a_p)\right) \cdot (1 - s_{\mathcal{S}})$, where $s_{\mathcal{S}} \equiv \sum_{p\in S} \frac{a_p}{1+a_p}$. This proves the expression in (3). ■

A.2 Proof of equation (4)

As we defined in (4), for each $(p,q) \in \{1,\dots,P\}^2$, let $H_{pq} \equiv \frac{\delta_{pq}}{1+a_p} + \frac{1}{1-s} \cdot \frac{a_p}{(1+a_p)(1+a_q)}$. This is well-defined by Assumption 5, and H_{pq} is the (p,q) element of J^{-1} for every (p,q). We will show this by proving directly that $JH = I_P$, where H is the $P \times P$ matrix with (p,q) entry given by H_{pq}. Recall that the Jacobian J is a $P \times P$ matrix with diagonal entries equal to 1 and, in row p, every off-diagonal entry equal to $-a_p$. Thus, we can write its (p,r) entry as $J_{pr} = \delta_{pr} - a_p\left(1 - \delta_{pr}\right)$, where $\delta_{pr} \equiv \mathbb{1}\{p = r\}$. As previously, let $s \equiv \sum_{p=1}^{P} \frac{a_p}{1+a_p}$. Fix $(p,q) \in \{1,\dots,P\}^2$. Then, $(JH)_{pq} = \sum_{r=1}^{P} J_{pr}H_{rq} = \sum_{r=1}^{P}\Big[\delta_{pr} - a_p(1-\delta_{pr})\Big]H_{rq}$. Distributing the sum and writing $\sum_{r\neq p} H_{rq} = \sum_{r=1}^{P} H_{rq} - H_{pq}$, this simplifies to,

$$(JH)_{pq} = H_{pq} - a_p\left(\sum_{r=1}^{P} H_{rq} - H_{pq}\right) = (1+a_p)H_{pq} - a_p\sum_{r=1}^{P} H_{rq}. \tag{15}$$

Using the definition of H_{rq}, $\sum_{r=1}^{P} H_{rq} = \sum_{r=1}^{P}\left[\frac{\delta_{rq}}{1+a_r} + \frac{1}{1-s} \cdot \frac{a_r}{(1+a_r)(1+a_q)}\right]$. Next, note that $\sum_{r=1}^{P} \frac{\delta_{rq}}{1+a_r} = \frac{1}{1+a_q}$, and $\sum_{r=1}^{P} \frac{1}{1-s} \cdot \frac{a_r}{(1+a_r)(1+a_q)} = \frac{1}{(1-s)(1+a_q)} \sum_{r=1}^{P} \frac{a_r}{1+a_r} = \frac{s}{(1-s)(1+a_q)}$. Thus, $\sum_{r=1}^{P} H_{rq} = \frac{1}{1+a_q} + \frac{s}{(1-s)(1+a_q)} = \frac{(1-s)+s}{(1-s)(1+a_q)} = \frac{1}{(1-s)(1+a_q)}$. Substituting into (15), we obtain

$$(JH)_{pq} = (1+a_p)H_{pq} - a_p \cdot \frac{1}{(1-s)(1+a_q)}. \tag{16}$$

Next, from the definition of H_{pq}, we have $(1+a_p)H_{pq} = (1+a_p)\left[\frac{\delta_{pq}}{1+a_p} + \frac{1}{1-s} \cdot \frac{a_p}{(1+a_p)(1+a_q)}\right] = \delta_{pq} + \frac{a_p}{(1-s)(1+a_q)}$. Plugging back into (16), we obtain

$$(JH)_{pq} = \left[\delta_{pq} + \frac{a_p}{(1-s)(1+a_q)}\right] - \frac{a_p}{(1-s)(1+a_q)} = \delta_{pq}.$$

Since this is true for all $(p,q) \in \{1,\dots,P\}^2$, it follows that $JH = I_P$. ■

A.3 Proof of the equality in equation (11)

As before, let $a_p \equiv \Delta_p f_p$. For each $|\mathcal{S}| \geq 2$, define

$$T_{\mathcal{S}} \equiv \sum_{\substack{(p,q)\in\mathcal{S}\\ p<q}} a_p a_q + 2 \sum_{\substack{(p,q,r)\in\mathcal{S}\\ p<q<r}} a_p a_q a_r + 3 \sum_{\substack{(p,q,r,t)\in\mathcal{S}\\ p<q<r<t}} a_p a_q a_r a_t + \cdots + (|\mathcal{S}|-1)\prod_{p\in\mathcal{S}} a_p.$$

Denote $n \equiv |\mathcal{S}|$, and let $e_k(\mathcal{S}) \equiv \sum_{\substack{U\subseteq\mathcal{S}\\ |U|=k}} \prod_{u\in U} a_u$, with $e_0(\mathcal{S}) \equiv 1$. By definition, we have $T_{\mathcal{S}} = \sum_{k=2}^{n}(k-1)\, e_k(\mathcal{S})$. Expanding the product yields $\prod_{p\in\mathcal{S}}(1+a_p) = \sum_{U\subseteq\mathcal{S}}\prod_{u\in U} a_u = \sum_{k=0}^{n} e_k(\mathcal{S})$. Next, let us expand $s_{\mathcal{S}}\cdot\prod_{p\in\mathcal{S}}(1+a_p)$. Recall that, $s_{\mathcal{S}} = \sum_{p\in\mathcal{S}} \frac{a_p}{1+a_p}$, so $s_{\mathcal{S}}\cdot \prod_{p\in\mathcal{S}}(1+a_p) = \sum_{p\in\mathcal{S}} \frac{a_p}{1+a_p}\prod_{q\in\mathcal{S}}(1+a_q) = \sum_{p\in\mathcal{S}} a_p \prod_{q\in\mathcal{S}\setminus\{p\}}(1+a_q) = \sum_{p\in\mathcal{S}}\sum_{j=0}^{n-1} a_p e_j(\mathcal{S}\setminus\{p\})$. We claim that,

$$\sum_{p\in\mathcal{S}} a_p\, e_{k-1}(\mathcal{S}\setminus\{p\}) = k\, e_k(\mathcal{S}) \;\text{ for every } k = 1,\ldots,n. \tag{17}$$

To see this, expand the left-hand side using the definition of $e_{k-1}(\mathcal{S}\setminus\{p\})$:

$$\sum_{p\in\mathcal{S}} a_p\, e_{k-1}(\mathcal{S}\setminus\{p\}) = \sum_{p\in\mathcal{S}} \sum_{\substack{V\subseteq\mathcal{S}\setminus\{p\}\\ |V|=k-1}} a_p \prod_{v\in V} a_v = \sum_{p\in\mathcal{S}} \sum_{\substack{V\subseteq\mathcal{S}\setminus\{p\}\\ |V|=k-1}} \prod_{u\in V\cup\{p\}} a_u. \tag{18}$$

Each summand on the right-hand side of (18) is indexed by a pair (p, V) with $p \in \mathcal{S}$, $V \subseteq \mathcal{S}\setminus\{p\}$, $|V| = k-1$; equivalently, it is indexed by a k-subset $U = V\cup\{p\} \subseteq \mathcal{S}$ together with a choice of a distinct element $p \in U$. For a *fixed* k-subset $U \subseteq \mathcal{S}$, the term $\prod_{u\in U} a_u$ appears once for *each* distinct element $p \in U$, i.e., exactly $|U| = k$ times. Hence

$$\sum_{p\in\mathcal{S}} \sum_{\substack{V\subseteq\mathcal{S}\setminus\{p\}\\ |V|=k-1}} \prod_{u\in V\cup\{p\}} a_u = \sum_{\substack{U\subseteq\mathcal{S}\\ |U|=k}} k\cdot\prod_{u\in U} a_u = k \sum_{\substack{U\subseteq\mathcal{S}\\ |U|=k}} \prod_{u\in U} a_u = k\, e_k(\mathcal{S}), \tag{19}$$

which, combined with (18), proves (17). From here, we have

$$s_{\mathcal{S}}\cdot\prod_{p\in\mathcal{S}}(1+a_p) = \sum_{j=0}^{n-1}\sum_{p\in\mathcal{S}} a_p\, e_j(\mathcal{S}\setminus\{p\}) = \sum_{j=0}^{n-1}(j+1)\, e_{j+1}(\mathcal{S}) = \sum_{k=1}^{n} k\, e_k(\mathcal{S}), \tag{20}$$

where the last line re-labels $k \equiv j+1$. Combining our results,

$$\begin{aligned}\mathcal{M}_{\mathcal{S}} &= \prod_{p\in\mathcal{S}}(1+a_p)\cdot(1-s_{\mathcal{S}}) = \prod_{p\in\mathcal{S}}(1+a_p) - s_{\mathcal{S}}\cdot\prod_{p\in\mathcal{S}}(1+a_p) \\ &= \sum_{k=0}^{n} e_k(\mathcal{S}) \;-\; \sum_{k=1}^{n} k\, e_k(\mathcal{S}) = e_0(\mathcal{S}) + \sum_{k=1}^{n}(1-k)\, e_k(\mathcal{S}) \\ &= 1 - \sum_{k=1}^{n}(k-1)\, e_k(\mathcal{S}) = 1 - \sum_{k=2}^{n}(k-1)\, e_k(\mathcal{S}) = 1 - T_{\mathcal{S}}\end{aligned}$$

This proves the statement in equation (11). ■

A.4 Proof of sufficiency of Assumption 10′ for strict positivity of all principal minors of $J(X,U,\pi)$

Recall that, for any nonempty $\mathcal{S} \subseteq \{1,\ldots,P\}$, the corresponding principal minor is $\mathcal{M}_{\mathcal{S}} \equiv \det(J_{\mathcal{S}}) = \left(\prod_{p\in\mathcal{S}}(1+a_p)\right)\left(1-s_{\mathcal{S}}\right)$, where $s_{\mathcal{S}} \equiv \sum_{p\in\mathcal{S}} \frac{a_p}{1+a_p}$. We will show that both factors are strictly positive under Assumption 10′. First, note that, by condition (I) of Assumption 10′, $\overline{\Delta}_p \overline{f}_p^{\mathcal{X}} < 1$ for every $p = 1,\ldots,P$. Since $|a_p| \le \overline{\Delta}_p \overline{f}_p^{\mathcal{X}}$, it follows that $a_p \ge -\overline{\Delta}_p \overline{f}_p^{\mathcal{X}} > -1$ and hence, $1 + a_p > 1 - \overline{\Delta}_p \overline{f}_p^{\mathcal{X}} > 0$ for every $p \in \mathcal{S}$. Thus, Hence $\prod_{p\in S}(1+a_p) > 0$. Next, we will show that $1 - s_{\mathcal{S}} > 0$. To this end, note that the function $x \mapsto x/(1+x)$ is strictly increasing on $(-1,\infty)$, since its derivative is $1/(1+x)^2 > 0$. Since $a_p \in (-1,\infty)$ for every $p \in S$, and $a_p \le \overline{\Delta}_p \overline{f}_p^{\mathcal{X}}$, we then have $\frac{a_p}{1+a_p} \le \frac{\overline{\Delta}_p \overline{f}_p^{\mathcal{X}}}{1+\overline{\Delta}_p \overline{f}_p^{\mathcal{X}}}$ for every $p \in \mathcal{S}$. Every term on the right-hand side is nonnegative, so adding over p can only increase the total. Thus, $s_S = \sum_{p\in S} \frac{a_p}{1+a_p} \le \sum_{p\in S} \frac{\overline{\Delta}_p \overline{f}_p^{\mathcal{X}}}{1+\overline{\Delta}_p \overline{f}_p^{\mathcal{X}}} \le \sum_{p=1}^{P} \frac{\overline{\Delta}_p \overline{f}_p^{\mathcal{X}}}{1+\overline{\Delta}_p \overline{f}_p^{\mathcal{X}}} < 1$, where the last inequality is condition (II) of Assumption 10′. Hence $1 - s_{\mathcal{S}} > 0$. Combining our results, $\mathcal{M}_{\mathcal{S}} = \left(\prod_{p\in\mathcal{S}}(1+a_p)\right)\left(1-s_S\right) > 0$. Since $\emptyset \ne \mathcal{S} \subseteq \{1,\ldots,P\}$ was arbitrary, this holds for every principal minor of J. ■

A.5 Proof of Proposition 2

Recall that, under Assumption 10′, whenever $X \in \mathcal{X}$, we have

$$\Psi_{pp}(X,Z) = \int \left[1 + \Delta_p \cdot \sum_{q \neq p} B_{qp}(X,U,\pi^*(X,U))\right] \cdot g_{\varepsilon_p|Z,U}\left(X'\theta_{p0} + \Delta_p \cdot \sum_{q \neq p} \pi^*_q(X,U) + \eta_p \,\middle|\, Z,U\right) dF_{U|Z}(U|Z),$$

$$\Psi_{pq}(X,Z) = \int \Delta_p \cdot \left[\sum_{r \neq p} B_{rq}(X,U,\pi^*(X,U))\right] \cdot g_{\varepsilon_p|Z,U}\left(X'\theta_{p0} + \Delta_p \cdot \sum_{q \neq p} \pi^*_q(X,U) + \eta_p \,\middle|\, Z,U\right) dF_{U|Z}(U|Z)$$

Denote,

$$\varphi_{pp}(X,U) \equiv 1 + \Delta_p \cdot \sum_{q \neq p} B_{qp}(X,U,\pi^*(X,U)), \quad \varphi_{pq}(X,U) \equiv \sum_{r \neq p} B_{rq}(X,U,\pi^*(X,U)). \tag{21}$$

Then, we can express

$$\Psi_{pp}(X,Z) = \int \varphi_{pp}(X,U) \cdot g_{\varepsilon_p|Z,U}\left(X'\theta_{p0} + \Delta_p \cdot \sum_{q \neq p} \pi^*_q(X,U) + \eta_p \,\middle|\, Z,U\right) dF_{U|Z}(U|Z),$$

$$\Psi_{pq}(X,Z) = \int \Delta_p \cdot \varphi_{pq}(X,U) \cdot g_{\varepsilon_p|Z,U}\left(X'\theta_{p0} + \Delta_p \cdot \sum_{q \neq p} \pi^*_q(X,U) + \eta_p \,\middle|\, Z,U\right) dF_{U|Z}(U|Z)$$

The proof will proceed by showing that, under Assumption 10′, $\varphi_{pq}(X,U) > 0$ and $\varphi_{pp}(X,U) > 0$ a.s when $X \in \mathcal{X}$. Since $g_{\varepsilon_p|Z,U}$ is a density function with unbounded support for a.e (Z,U), it satisfies $g_{\varepsilon_p|Z,U}(\cdot|Z,U) > 0$ a.e. (Z,U). As before, we will omit the arguments $(X,Z,U,\pi^*(X,U))$, so we will write $f_p(X,U,\pi^*_{-p}(X,U)) \equiv f_p$, $B_{pq}(X,U,\pi^*(X,U)) \equiv B_{pq}$, $\varphi_{pq}(X,U) \equiv \varphi_{pq}$, etc. As before, for each p we will let $a_p \equiv \Delta_p f_p$. From Assumption 5, we have $1 + a_p \neq 0$ a.s for all p. Recall that $|\Delta_p| \leq \overline{\Delta}_p$ w.p.1, and $0 < f_p \leq \overline{f}_p^{\mathcal{X}}$ over $\mathcal{X}$. Recall that $\delta_{pq} \equiv \mathbb{1}\{p = q\}$, and $s \equiv \sum_{p=1}^{P} \frac{a_p}{1+a_p}$. In (4) and (5), we defined $H_{pq} = \frac{1}{1+a_p}\left(\delta_{pq} + \frac{1}{1-s} \cdot \frac{a_p}{1+a_q}\right)$, and $B_{pq} = H_{pq} f_q$. From (21), we have $\varphi_{pq} = \sum_{r \neq p} B_{rq}$, and $\varphi_{pp} = 1 + \Delta_p \sum_{q \neq p} B_{qp}$. Throughout, write $\overline{a}_p \equiv \overline{\Delta}_p \overline{f}_p^{\mathcal{X}}$, and note that $|a_p| = |\Delta_p| f_p \leq \overline{\Delta}_p \overline{f}_p^{\mathcal{X}} = \overline{a}_p$. We begin by showing that we can express $\varphi_{pq} = \frac{f_q}{(1+a_p)(1+a_q)(1-s)}$, and $\varphi_{pp} = \frac{1-s_{-p}}{(1+a_p)(1-s)}$, where $s_{-p} \equiv \sum_{r \neq p} \frac{a_r}{1+a_r} = s - \frac{a_p}{1+a_p}$. By definition,

$\varphi_{pq} = \sum_{r\neq p} B_{rq} = \sum_{r\neq p} \frac{f_q}{1+a_r}\left(\delta_{rq} + \frac{1}{1-s}\cdot\frac{a_r}{1+a_q}\right)$. Since $p \neq q$, we have $\sum_{r\neq p} \frac{f_q}{1+a_r}\cdot\delta_{rq} = \frac{f_q}{1+a_q}$. Next, note that, since $\sum_{r\neq p}\frac{a_r}{1+a_r} = s - \frac{a_p}{1+a_p}$, we have $\frac{1}{1-s}\cdot\sum_{r\neq p}\frac{f_q}{1+a_r}\cdot\frac{a_r}{1+a_q} = \frac{1}{1-s}\cdot\frac{f_q}{1+a_q}\cdot\left(s - \frac{a_p}{1+a_p}\right)$. Thus,

$$\begin{aligned}\varphi_{pq} &= \frac{f_q}{1+a_q}\left[1 + \frac{1}{1-s}\left(s - \frac{a_p}{1+a_p}\right)\right] = \frac{f_q}{1+a_q}\cdot\frac{(1-s)+s-\frac{a_p}{1+a_p}}{1-s} = \frac{f_q}{1+a_q}\cdot\frac{1-\frac{a_p}{1+a_p}}{1-s}\\ &= \frac{f_q}{(1+a_p)(1+a_q)(1-s)}.\end{aligned} \tag{22}$$

For φ_{pp} note that $\sum_{q\neq p} B_{qp} = \sum_{q\neq p}\frac{f_p}{1+a_q}\cdot\frac{1}{1-s}\cdot\frac{a_q}{1+a_p} = \frac{f_p}{(1+a_p)(1-s)}\sum_{q\neq p}\frac{a_q}{1+a_q} \equiv \frac{f_p\, s_{-p}}{(1+a_p)(1-s)}$. Hence, $\varphi_{pp} = 1 + \Delta_p\cdot\frac{f_p\, s_{-p}}{(1+a_p)(1-s)} = 1 + \frac{a_p\, s_{-p}}{(1+a_p)(1-s)}$. Using $s = \frac{a_p}{1+a_p} + s_{-p}$, we obtain, $\varphi_{pp} = \frac{(1+a_p)(1-s)+a_p\, s_{-p}}{(1+a_p)(1-s)}$. Simplifying the numerator, $(1+a_p)(1-s)+a_p s_{-p} = 1 - s_{-p}[(1+a_p) - a_p] = 1 - s_{-p}$. Thus,

$$\varphi_{pp} = \frac{1-s_{-p}}{(1+a_p)(1-s)}. \tag{23}$$

Next, we will show that, under Assumption 10′, if $X \in \mathcal{X}$, then we have $\varphi_{pq} > 0$ a.s for every $p \neq q$. From (22), since $f_q > 0$, it suffices to show that the denominator $(1+a_p)(1+a_q)(1-s)$ is strictly positive. We show this in two steps. First, by condition (I) of Assumption 10′, $\overline{\Delta}_r\overline{f}_r^{\mathcal{X}} < 1$ for every r. Since $|a_r| \leq \overline{\Delta}_r\overline{f}_r^{\mathcal{X}}$, in particular $a_r \geq -\overline{\Delta}_r\overline{f}_r^{\mathcal{X}} > -1$, so $1 + a_r > 1 - \overline{\Delta}_r\overline{f}_r^{\mathcal{X}} > 0$ for every $r = 1,\ldots,P$. In particular, $1 + a_p > 0$ and $1 + a_q > 0$, so $(1 + a_p)(1 + a_q) > 0$. Next, we show that $1 - s > 0$. To show this, note first that the function $x \mapsto x/(1 + x)$ is strictly increasing on $(-1,\infty)$, since its derivative is $1/(1 + x)^2 > 0$. As shown above, $a_r \in (-1,\infty)$ for every r, and $a_r \leq \overline{\Delta}_r\overline{f}_r^{\mathcal{X}}$, so $\frac{a_r}{1+a_r} \leq \frac{\overline{\Delta}_r\overline{f}_r^{\mathcal{X}}}{1+\overline{\Delta}_r\overline{f}_r^{\mathcal{X}}}$ for every $r = 1,\ldots,P$. Thus, $s = \sum_{r=1}^P \frac{a_r}{1+a_r} \leq \sum_{r=1}^P \frac{\overline{\Delta}_r\overline{f}_r^{\mathcal{X}}}{1+\overline{\Delta}_r\overline{f}_r^{\mathcal{X}}} < 1$, where the last inequality is condition (II) of Assumption 10′. Hence $1 - s > 0$. Combining our results, we have $(1+a_p)(1+a_q)(1-s) > 0$, and since $f_q > 0$, it follows that $\varphi_{pq} = \frac{f_q}{(1+a_p)(1+a_q)(1-s)} > 0$. Next, we show that $\varphi_{pp} > 0$ under Assumption 10′. First, we have already shown that the denominator in (23), $(1 + a_p)(1 - s)$, is strictly positive, so we only need to show that the numerator $1 - s_{-p}$ is strictly positive, i.e., $s_{-p} < 1$. But this follows immediately since $s_{-p} = \sum_{r\neq p}\frac{a_r}{1+a_r} \leq \sum_{r\neq p}\frac{\overline{\Delta}_r\overline{f}_r^{\mathcal{X}}}{1+\overline{\Delta}_r\overline{f}_r^{\mathcal{X}}} \leq \sum_{r=1}^P \frac{\overline{\Delta}_r\overline{f}_r^{\mathcal{X}}}{1+\overline{\Delta}_r\overline{f}_r^{\mathcal{X}}} < 1$, where the last inequality is condition (II). Thus $1 - s_{-p} > 0$. Hence,

$\varphi_{pp} = \frac{1-s_{-p}}{(1+a_p)(1-s)} > 0$ a.s. Thus, under Assumption 10′, if $X \in \mathcal{X}$, then w.p.1,

$$\Psi_{pp}(X,Z) = \int \underbrace{\varphi_{pp}(X,U) \cdot g_{\varepsilon_p|Z,U}\left(X'\theta_{p0} + \Delta_p \cdot \sum_{q \neq p} \pi_q^*(X,U) + \eta_p \,\middle|\, Z, U\right)}_{>0} dF_{U|Z}(U|Z) > 0,$$

$$\Psi_{pq}(X,Z) = \int \underbrace{\Delta_p \cdot \varphi_{pq}(X,U) \cdot g_{\varepsilon_p|Z,U}\left(X'\theta_{p0} + \Delta_p \cdot \sum_{q \neq p} \pi_q^*(X,U) + \eta_p \,\middle|\, Z, U\right)}_{>0} dF_{U|Z}(U|Z).$$

Thus, under Assumption 10′, if $X \in \mathcal{X}$, then w.p.1: $\Psi_{pp}(X,Z) > 0$ for each p and, if the sign of Δ_p is constant for p, then for any $q \neq p$ we have $\Psi_{pq}(X,Z) \geq 0$ if $\Delta_p \geq 0$ w.p.1, $\Psi_{pq}(X,Z) \leq 0$ if $\Delta_p \leq 0$ w.p.1, and $\Psi_{pq}(X,Z) = 0$ iff $\Delta_p = 0$ for $F_{U|Z}(\cdot|Z)$-almost every U, and almost every Z in the support of $Z \mid X \in \mathcal{X}$. The statement in Proposition 2 follows from here since $\delta_{pp}^{\phi_{\mathcal{X}}} \equiv E[\Psi_{pp}(X,Z) \cdot \phi_{\mathcal{X}}(X,Z)]$ and $\delta_{pq}^{\phi_{\mathcal{X}}} \equiv E[\Psi_{pq}(X,Z) \cdot \phi_{\mathcal{X}}(X,Z)]$, where $\phi_{\mathcal{X}}(X,Z) > 0$ if $X \in \mathcal{X}$ and $\phi_{\mathcal{X}}(X,Z) \equiv 0$ otherwise. ■

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