Appendix B for
"Testing functional inequalities conditional on estimated functions"
Examples of estimators that satisfy Assumption 1

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Abstract

This appendix presents examples of estimators and the conditions under which they satisfy the restrictions in Assumption 1 in the paper. The examples we include are OLS, GMM, density-weighted average derivatives, and a semiparametric multiple-index estimator.

In the examples that follow, all expectations are taken with respect to a generic distribution $F \in \mathcal{F}$. At times, to simplify our exposition we omit denoting explicitly the dependence of these (and other) functionals on F.

B1 A convenient definition

Take a collection of column vectors $(v_\ell)_{\ell=1}^{d}$ where $v_\ell \in \mathbb{R}^d$ for each ℓ, and let

$$\underbrace{v \equiv (v_1', v_2', \ldots, v_d')'}_{d^2\times 1}.$$

For any such v we will define

$$\underbrace{H_d(v)}_{d\times d} \equiv \begin{pmatrix} v_1' \\ v_2' \\ \vdots \\ v_d' \end{pmatrix} \text{ and, when it exists, we will denote } M_d(v) \equiv H_d(v)^{-1}. \tag{B1.1}$$

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B2 OLS

Consider an iid sample $(Z_{1i}, Z_{2i})_{i=1}^n$ where $Z_i \equiv (Z_{1i}, Z_{2i}) \sim F$, with $Z_{1i} \in \mathbb{R}$ and $Z_{2i} \in \mathbb{R}^k$. Denote the ℓ^{th} element in Z_{2i} as $Z_{2i,\ell}$. Define

$$\underbrace{\overline{G}_\ell}_{k\times 1} \equiv \frac{1}{n}\sum_{i=1}^n Z_{2i}Z_{2i,\ell}, \quad \underbrace{\lambda_{\ell,F}}_{k\times 1} \equiv E_F[Z_2 Z_{2,\ell}],$$
$$\overline{G} \equiv \underbrace{\left(\overline{G}_1', \overline{G}_2', \ldots, \overline{G}_k'\right)'}_{k^2\times 1} \quad \text{and} \quad \lambda_F \equiv \underbrace{\left(\lambda_{1,F}', \lambda_{2,F}', \ldots, \lambda_{k,F}'\right)'}_{k^2\times 1}.$$

Assumption LS *$\exists\ \overline{M}_{z_2z_2}$ such that $\left\|(E_F[Z_2Z_2'])^{-1}\right\| \le \overline{M}_{z_2z_2}\ \forall\ F \in \mathcal{F}$. For some $q \ge 2$, there exist $\overline{\mu}_{z_2z_2}$ and $\overline{\mu}_{z_2\nu}$ such that, for each ℓ, m,*

$$E_F\left[\left|Z_{2,\ell}Z_{2,m} - E_F[Z_{2,\ell}Z_{2,m}]\right|^q\right] \le \overline{\mu}_{z_2z_2} \quad \textit{and} \quad E_F\left[\left|Z_{2,\ell}\nu\right|^q\right] \le \overline{\mu}_{z_2\nu} \quad \forall\ F \in \mathcal{F}.$$

There exists $\overline{M}_\lambda$ such that $\|M_k(\lambda_F)\| \le \overline{M}_\lambda$ for all $F \in \mathcal{F}$ and there exist $K_1 > 0$, $K_2 > 0$ and $\alpha > 0$ such that, for any $F \in \mathcal{F}$ and $v \in \mathbb{R}^{k^2}$,

$$\|v - \lambda_F\| \le K_1 \quad \Longrightarrow \quad \|M_k(v) - M_k(\lambda_F)\| \le K_2 \cdot \|v - \lambda_F\|^\alpha,$$

and there exists $K_3 < \infty$ such that

$$\sup_{v:\|v-\lambda_F\|\le K_1}\left\{\|M_k(v) - M_k(\lambda_F)\|\right\} \le K_3 \quad \forall\ F \in \mathcal{F}$$

Consider the OLS estimator

$$\widehat{\theta} = \left(\frac{1}{n}\sum_{i=1}^n Z_{2i}Z_{2i}'\right)^{-1} \frac{1}{n}\sum_{i=1}^n Z_{2i}Z_{1i},$$

and let

$$\theta_F^* \equiv (E_F[Z_2Z_2'])^{-1} \cdot E_F[Z_2Z_1],$$

and let us express $Z_1 = Z_2'\theta_F^* + \left(Z_1 - Z_2'\theta_F^*\right) \equiv Z_2'\theta_F^* + \nu$, where $\nu \equiv \left(Z_1 - Z_2'\theta_F^*\right)$. Note that $E_F[Z_2\nu] = 0$ by the definition of θ_F^*. In the usual linear regression model where we assume a structural relationship given by $Z_1 = Z_2'\beta_0 + \varepsilon$ with $E_F[Z_2\varepsilon] = 0\ \forall\ F \in \mathcal{F}$, we would have $\nu = \varepsilon$ and $\theta_F^* = \beta_0$ for all $F \in \mathcal{F}$.

Result OLS *Let $\theta_F^* \equiv \left(E_F[Z_2Z_2']\right)^{-1} \cdot E_F[Z_2Z_1]$, $\nu_i \equiv \left(Z_{1i} - Z_{2i}'\theta_F^*\right)$, and $\psi_F^\theta(Z_i) \equiv \left(E_F[Z_2Z_2']\right)^{-1} \cdot Z_{2i}\nu_i$.*

Note that $E_F[\psi_F^\theta(Z_i)] = 0$. Under Assumption LS, the OLS estimator satisfies

$$\widehat{\theta} = \theta_F^* + \frac{1}{n}\sum_{i=1}^{n}\psi_F^\theta(Z_i) + \varepsilon_n^\theta,$$

and the conditions in Assumption 1 of the paper are satisfied, with $\psi_F^\theta(Z_i) = \left(E_F[Z_2 Z_2']\right)^{-1} \cdot Z_{2i}\nu_i$, $r_n = n^{1/2}$, and for any τ and $\overline{\delta}$ such that $0 < \tau < \alpha/2$, and $0 < \overline{\delta} < (q-1)/2$.

Proof: Let $M_k(\cdot)$ be as defined in (B1.1) and note that $M_k(\lambda_F) = \left(E_F[Z_2 Z_2']\right)^{-1}$. We have

$$\begin{aligned}
\widehat{\theta} &= \theta_F^* + M_k(\lambda_F)\frac{1}{n}\sum_{i=1}^{n} Z_{2i}\nu_i + \left(M_k(\overline{G}) - M_k(\lambda_F)\right)\cdot\frac{1}{n}\sum_{i=1}^{n} Z_{2i}\nu_i \\
&\equiv \theta_F^* + \frac{1}{n}\sum_{i=1}^{n}\psi_F^\theta(Z_i) + \varepsilon_n^\theta, \quad \text{where} \\
\psi_F^\theta(Z_i) &\equiv M_k(\lambda_F)\cdot Z_{2i}\nu_i = (E_F[Z_2 Z_2'])^{-1}\cdot Z_{2i}\nu_i, \\
\varepsilon_n^\theta &\equiv \left(M_k(\overline{G}) - M_k(\lambda_F)\right)\cdot\frac{1}{n}\sum_{i=1}^{n} Z_{2i}\nu_i.
\end{aligned} \tag{B2.1}$$

Note that $E_F\left[\psi_F(Z_i)^\theta\right] = 0$. Take any $c > 0$. Using the conditions in Assumption LS,

$$\begin{aligned}
\mathbb{1}\left\{\|\varepsilon_n^\theta\| \geq c\right\} &\leq \mathbb{1}\left\{\left\|M_k(\overline{G}) - M_k(\lambda_F)\right\|\cdot\left\|\frac{1}{n}\sum_{i=1}^{n} Z_{2i}\nu_i\right\| \geq c\right\} \\
&= \underbrace{\mathbb{1}\left\{\left\|M_k(\overline{G}) - M_k(\lambda_F)\right\|\cdot\left\|\frac{1}{n}\sum_{i=1}^{n} Z_{2i}\nu_i\right\| \geq c\right\}\cdot\mathbb{1}\left\{\left\|M_k(\overline{G}) - M_k(\lambda_F)\right\| \leq K_3\right\}}_{\leq \mathbb{1}\left\{K_3\cdot\left\|\frac{1}{n}\sum_{i=1}^{n} Z_{2i}\nu_i\right\|\geq c\right\}} \\
&\quad + \underbrace{\mathbb{1}\left\{\left\|M_k(\overline{G}) - M_k(\lambda_F)\right\|\cdot\left\|\frac{1}{n}\sum_{i=1}^{n} Z_{2i}\nu_i\right\| \geq c\right\}\cdot\mathbb{1}\left\{\left\|M_k(\overline{G}) - M_k(\lambda_F)\right\| > K_3\right\}}_{\leq \mathbb{1}\left\{\|\overline{G}-\lambda_F\|\geq K_1\right\}} \\
&\leq \mathbb{1}\left\{\left\|\frac{1}{n}\sum_{i=1}^{n} Z_{2i}\nu_i\right\| \geq \frac{c}{K_3}\right\} + \mathbb{1}\left\{\left\|\overline{G} - \lambda_F\right\| \geq K_1\right\} \\
&\leq \mathbb{1}\left\{\left\|\frac{1}{n}\sum_{i=1}^{n} Z_{2i}\nu_i\right\| \geq \left(\frac{c}{K_3}\right)\wedge K_1\right\} + \mathbb{1}\left\{\left\|\overline{G} - \lambda_F\right\| \geq \left(\frac{c}{K_3}\right)\wedge K_1\right\}
\end{aligned} \tag{B2.2}$$

Take $b > 0$. Assumption LS and Chebyshev's inequality imply that, for all ℓ, m in $1, \ldots, k$,

$$\sup_{F\in\mathcal{F}} P_F\left(\left|\frac{1}{n}\sum_{i=1}^{n}\left(Z_{2i,\ell}Z_{2i,m} - E_F\left[Z_{2i,\ell}Z_{2i,m}\right]\right)\right| \geq b\right) \leq \frac{\overline{\mu}_{z_2z_2}}{\left(n^{1/2}\cdot b\right)^q}$$

Therefore,

$$\sup_{F\in\mathcal{F}} P_F\left(\left\|\overline{G} - \lambda_F\right\| \geq b\right) \leq \frac{\overline{M}_1}{\left(n^{1/2}\cdot b\right)^q} \tag{B2.3}$$

where $\overline{M}_1$ depends only on $\overline{\mu}_{z_2z_2}$ and k. Similarly, Assumption LS also implies that there exists a constant $\overline{M}_2$ which depends only on $\overline{\mu}_{z_2\nu}$ and k such that, for any $b > 0$

$$\sup_{F\in\mathcal{F}} P_F\left(\left\|\frac{1}{n}\sum_{i=1}^{n} Z_{2i}\nu_i\right\| \geq b\right) \leq \frac{\overline{M}_2}{\left(n^{1/2}\cdot b\right)^q} \tag{B2.4}$$

Combining (B2.3) and (B2.4) with (B2.2), we have that for any $c > 0$,

$$\sup_{F\in\mathcal{F}} P_F\left(\left\|\varepsilon_n^\theta\right\| \geq c\right) \leq \frac{\overline{M}_1 + \overline{M}_2}{\left(n^{1/2}\cdot\left(\left(\frac{c}{K_3}\right)\wedge K_1\right)\right)^q} = o\left(\frac{1}{n^{1/2+\delta}}\right) \quad \forall\ 0 < \delta < \frac{q-1}{2} \tag{B2.5}$$

Take $0 < \delta < \frac{q-1}{2}$ and consider a sequence $c_n > 0$ such that $n^{\frac{q-1-2\delta}{2q}} \cdot c_n \longrightarrow \infty$. Then, the result in (B2.5) would still hold for c_n. Thus,

$$\sup_{F\in\mathcal{F}} P_F\left(\left\|\varepsilon_n^\theta\right\| \geq c_n\right) = o\left(\frac{1}{n^{1/2+\delta}}\right) \quad \forall\ c_n : n^{\frac{q-1-2\delta}{2q}} \cdot c_n \longrightarrow \infty, \quad 0 < \delta < \frac{q-1}{2}$$

Take any $b > 0$. From our previous results,

$$\sup_{F\in\mathcal{F}} P_F\left(\left\|\frac{1}{n}\sum_{i=1}^{n}\psi_F^\theta(Z_i)\right\| \geq b\right) \leq \sup_{F\in\mathcal{F}} P_F\left(\left\|\frac{1}{n}\sum_{i=1}^{n} Z_{2i}\nu_i\right\| \geq \frac{b}{\overline{M}_\lambda}\right) \leq \frac{\overline{M}_2}{\left(n^{1/2}\cdot\left(\frac{b}{\overline{M}_\lambda}\right)\right)^q}$$

Thus, going back to the linear representation result in (B2.1), we have that for any $c > 0$,

$$\begin{aligned}\sup_{F\in\mathcal{F}} P_F\left(\left\|\widehat{\theta} - \theta_F^*\right\| \geq c\right) &\leq \sup_{F\in\mathcal{F}} P_F\left(\left\|\frac{1}{n}\sum_{i=1}^{n}\psi_F^\theta(Z_i)\right\| \geq \frac{c}{2}\right) + \sup_{F\in\mathcal{F}} P_F\left(\left\|\varepsilon_n^\theta\right\| \geq \frac{c}{2}\right)\\ &\leq \frac{\overline{M}_2}{\left(n^{1/2}\cdot\left(\frac{c}{2\overline{M}_\lambda}\right)\right)^q} + \frac{\overline{M}_1 + \overline{M}_2}{\left(n^{1/2}\cdot\left(\left(\frac{c}{2K_3}\right)\wedge K_1\right)\right)^q}\end{aligned}$$

And so,

$$\sup_{F\in\mathcal{F}} P_F\left(\left\|\widehat{\theta}-\theta_F^*\right\| \geq c\right) \longrightarrow 0 \quad \forall\, c>0.$$

Thus $\left\|\widehat{\theta}-\theta_F^*\right\| = o_p(1)$ uniformly over $\mathcal{F}$. Recall from its definition in (B2.1) that

$$\left\|\varepsilon_n^{\theta}\right\| \leq \left\|M_k(\overline{G})-M_k(\lambda_F)\right\| \cdot \left\|\frac{1}{n}\sum_{i=1}^{n} Z_{2i}\nu_i\right\|.$$

From (B2.4), we have

$$\left\|\frac{1}{n}\sum_{i=1}^{n} Z_{2i}\nu_i\right\| = O_p\left(n^{-1/2}\right), \quad \text{uniformly over } \mathcal{F}. \tag{B2.6}$$

Now let us analyze $\left\|M_k(\overline{G})-M_k(\lambda_F)\right\|$. Take any $b>0$. From the conditions in Assumption LS,

$$\begin{aligned}\mathbb{1}\left\{\left\|M_k(\overline{G})-M_k(\lambda_F)\right\| \geq b\right\} &\leq \max\left(\mathbb{1}\left\{K_2\cdot\left\|\overline{G}-\lambda_F\right\|^{\alpha} \geq b\right\},\, \mathbb{1}\left\{\left\|\overline{G}-\lambda_F\right\| \geq K_1\right\}\right)\\ &\leq \mathbb{1}\left\{\left\|\overline{G}-\lambda_F\right\| \geq \left(\frac{b}{K_2}\right)^{1/\alpha} \wedge K_1\right\}\end{aligned}$$

Take $\tau>0$. Then, from (B2.3),

$$\sup_{F\in\mathcal{F}} P_F\left(\left\|M_k(\overline{G})-M_k(\lambda_F)\right\| \geq n^{-\tau}\cdot b\right) \leq \frac{\overline{M}_1}{\left(n^{1/2}\cdot\left(\left(\frac{n^{-\tau}\cdot b}{K_2}\right)^{1/\alpha}\wedge K_1\right)\right)^q} \longrightarrow 0 \quad \forall\, \tau<\frac{\alpha}{2},$$

which means,

$$\left\|M_k(\overline{G})-M_k(\lambda_F)\right\| = o_p\left(n^{-\tau}\right) \quad \forall\, \tau<\frac{\alpha}{2}, \quad \text{uniformly over } \mathcal{F}.$$

Combining (B2.6) and the previous expression, we have that for any $0<\tau<\frac{\alpha}{2}$,

$$\left\|\varepsilon_n^{\theta}\right\| = o_p\left(\frac{1}{n^{1/2+\tau}}\right), \quad \text{uniformly over } \mathcal{F}$$

Together, (B2.1), (B2.5) and the previous expression show that the conditions in Assumption 1 of the paper are satisfied, with $\psi_F^{\theta}(Z_i) = \left(E_F[Z_2Z_2']\right)^{-1}\cdot Z_{2i}\nu_i$, $r_n = n^{1/2}$, $0<\tau<\alpha/2$, and $0<\overline{\delta}<(q-1)/2$. This proves Result OLS. ■

B3 GMM

Consider an iid sample $(Z_i)_{i=1}^n$ where $Z_i \sim F$. Let $\mathcal{S}_Z$ denote the support of Z and assume for simplicity that $\mathcal{S}_Z$ is the same for each $F \in \mathcal{F}$. Let us focus on an exactly-identified GMM model where $\theta \in \mathbb{R}^k$ and

$$\underbrace{g(Z_i,\theta)}_{k\times 1} \equiv \big(g_1(Z_i,\theta), g_2(Z_i,\theta), \ldots, g_k(Z_i,\theta)\big)'$$

is a collection of parametric moment functions satisfying $E_F[g(Z,\theta_F^*)] = 0$. We denote θ_F^* possibly as a functional of F for generality (to include, e.g, the OLS example described above). For simplicity we focus on an exactly-identified GMM model with as many moment restrictions as parameters which includes, e.g, MLE and NLS as special cases. Let Θ denote the parameter space and let

$$\overline{g}_\ell(\theta) \equiv \frac{1}{n}\sum_{i=1}^n g_\ell(Z_i,\theta), \quad \text{and} \quad \underbrace{\overline{g}(\theta)}_{k\times 1} \equiv \big(\overline{g}_1(\theta), \overline{g}_2(\theta), \ldots, \overline{g}_k(\theta)\big)'$$

denote the sample moments. Suppose the GMM estimator $\widehat{\theta} \in \Theta$ is characterized by the condition $\overline{g}(\widehat{\theta}) = 0$. Suppose the moment functions are differentiable with respect to θ and for the ℓ^{th} moment function g_ℓ denote

$$\underbrace{G_\ell(z,\theta)}_{k\times 1} \equiv \begin{pmatrix} \frac{\partial g_\ell(z,\theta)}{\partial \theta_1} & \frac{\partial g_\ell(z,\theta)}{\partial \theta_2} & \cdots & \frac{\partial g_\ell(z,\theta)}{\partial \theta_k} \end{pmatrix}',$$

$$\underbrace{\overline{G}_\ell(\theta)}_{k\times 1} \equiv \frac{1}{n}\sum_{i=1}^n G_\ell(Z_i,\theta) \quad \text{and} \quad \underbrace{\lambda_{\ell,F}(\theta)}_{k\times 1} = E_F\left[G_\ell(Z,\theta)\right]].$$

From here and the definition of $\widehat{\theta}$ we obtain the following mean value expression for the ℓ^{th} sample moment

$$0 = \overline{g}_\ell(\widehat{\theta}) = \overline{g}_\ell(\theta_F^*) + \overline{G}_\ell(\overline{\theta}_\ell)'(\widehat{\theta} - \theta_F^*) \quad \ell = 1, \ldots, k, \tag{B3.1}$$

where $\overline{\theta}_\ell$ belongs in the line segment connecting $\widehat{\theta}$ and θ_F^*. Take a collection $(\theta_\ell)_{\ell=1}^k$ where each $\theta_\ell \in \Theta$. This is a collection of k points in the parameter space Θ. For any such collection we will denote

$$\underset{\sim}{\theta} \equiv (\theta_1', \theta_2', \ldots, \theta_k')' \in \underbrace{\Theta \times \Theta \times \cdots \times \Theta}_{k \text{ products}} \equiv \Theta^k$$

In particular, we will denote $\underset{\sim}{\overline{\theta}} \equiv (\overline{\theta}_1', \overline{\theta}_2', \ldots, \overline{\theta}_k')'$, where $\overline{\theta}_\ell$ is as described in the mean-value approximation (B3.1), and $\underset{\sim}{\theta}_F^* \equiv \left(\theta_F^{*\prime}, \theta_F^{*\prime}, \ldots, \theta_F^{*\prime}\right)'$. For a given $\underset{\sim}{\theta} \equiv (\theta_1', \theta_2', \ldots, \theta_k')' \in \Theta^k$ let

$$\underbrace{\overline{G}(\underset{\sim}{\theta})}_{k^2\times 1} \equiv \left(\overline{G}_1(\theta_1)', \overline{G}_2(\theta_2)', \ldots, \overline{G}_k(\theta_k)'\right)', \quad \underbrace{\lambda_F(\underset{\sim}{\theta})}_{k^2\times 1} = (\lambda_{1,F}(\theta_1)', \lambda_{2,F}(\theta_2)', \ldots, \lambda_{k,F}(\theta_k)')'.$$

Consider the following restrictions.

Assumption GMM

(i) There exists an integer $q \geq 2$ and a constant $\overline{\mu}_g < \infty$ such that, for each $\ell = 1, \ldots, k$ and each $F \in \mathcal{F}$, $E_F\left[g_\ell(Z, \theta_F^*)^q\right] \leq \overline{\mu}_g$. There exists a nonnegative function $\overline{V}(\cdot)$ such that, for each $\ell = 1, \ldots, k$ and $m = 1, \ldots, k$,

$$\left\|\frac{\partial g_\ell(z,\theta)}{\partial \theta_m} - \frac{\partial g_\ell(z,\theta')}{\partial \theta_m}\right\| \leq \overline{V}(z) \cdot \left\|\theta - \theta'\right\| \quad \forall\, z \in \mathcal{S}_Z \quad \text{and} \quad \theta, \theta' \in \Theta,$$

and there exists $\overline{\mu}_{\overline{V}}$ such that $E_F\left[\overline{V}(Z)^{4q}\right] \leq \overline{\mu}_{\overline{V}}$ $\forall\, F \in \mathcal{F}$, where q is the integer described above.

(ii) Let H_k and M_k be as defined in (B1.1). *$\exists\ \underline{d} > 0$, $\overline{M}_\lambda$, $K_3 > 0$ and $K_4 > 0$ and $\alpha_1 > 0$ such that, for every $F \in \mathcal{F}$,*

$$\inf_{\underset{\sim}{\theta} \in \Theta^k} \left|det\left(H_k(\lambda_F(\underset{\sim}{\theta}))\right)\right| \geq \underline{d} \quad \sup_{\underset{\sim}{\theta} \in \Theta^k} \left\|M_k(\lambda_F(\underset{\sim}{\theta}))\right\| \leq \overline{M}_\lambda$$

$$\forall\, \underset{\sim}{\theta} \in \Theta^k, \quad \|v - \lambda_F(\underset{\sim}{\theta})\| \leq K_3 \quad \Longrightarrow \quad \left\|M_k(\lambda_F(\underset{\sim}{\theta})) - M_k(v)\right\| \leq K_4 \cdot \|v - \lambda_F(\underset{\sim}{\theta})\|^{\alpha_1}.$$

And,

$$\sup_{\substack{v:\|v-\lambda_F(\underset{\sim}{\theta})\|\leq K_3 \\ \underset{\sim}{\theta}\in\Theta^k}} \left\{\left\|M_k(\lambda_F(\underset{\sim}{\theta})) - M_k(v)\right\|\right\} \leq K_5 < \infty$$

(iii) $\exists\ K_6 > 0$, $K_7 > 0$ and $\alpha_2 > 0$ such that, for every $F \in \mathcal{F}$,

$$\left\|\lambda_F(\underset{\sim}{\theta}) - \lambda_F(\underset{\sim}{\theta}_F^*)\right\| \leq K_7 \cdot \left\|\underset{\sim}{\theta} - \underset{\sim}{\theta}_F^*\right\|^{\alpha_2} \quad \forall\, \underset{\sim}{\theta} \in \Theta^k : \left\|\underset{\sim}{\theta} - \underset{\sim}{\theta}_F^*\right\| \leq K_6$$

Result GMM *Let θ_F^* be characterized by $E_F[g(Z,\theta_F^*)] = 0$ and define $\psi_F^\theta(Z_i) \equiv -\left(E_F\left[\frac{\partial g(Z,\theta_F^*)}{\partial\theta\partial\theta'}\right]\right)^{-1} \cdot g(Z_i, \theta_F^*)$. Under Assumption GMM, the estimator $\widehat{\theta}$ described by the sample moment conditions $\overline{g}(\widehat{\theta}) = 0$ satisfies*

$$\widehat{\theta} = \theta_F^* + \frac{1}{n}\sum_{i=1}^n \psi_F^\theta(Z_i) + \varepsilon_n^\theta,$$

and the conditions in Assumption 1 of the paper are satisfied, with $\psi_F^{\theta}(Z_i) = -\left(E_F\left[\frac{\partial g(Z,\theta_F^*)}{\partial\theta\partial\theta'}\right]\right)^{-1}\cdot g(Z_i,\theta_F^*)$, $r_n = n^{1/2}$, $0<\tau<\frac{\alpha_1}{2}\wedge\frac{\alpha_1\cdot\alpha_2}{2}$, and $0<\overline{\delta}<\frac{q-1}{2}$.

Proof: As defined above, let $\underset{\sim}{\theta}_F^* \equiv \left(\theta_F^{*\prime},\theta_F^{*\prime},\ldots,\theta_F^{*\prime}\right)'$ and note that $M_k(\lambda_F(\underset{\sim}{\theta}_F^*)) = \left(E_F\left[\frac{\partial g(Z,\theta_F^*)}{\partial\theta\partial\theta'}\right]\right)^{-1}$. Combining the mean-value expressions in (B3.1) for each of the $\ell=1,\ldots,k$ sample moments, we have

$$\overline{g}(\theta_F^*) + H_k\left(\overline{G}(\underset{\sim}{\overline{\theta}})\right)\left(\widehat{\theta}-\theta_F^*\right)=0.$$

From here,

$$\begin{aligned}
\widehat{\theta} =&\theta_F^* - M_k(\overline{G}(\underset{\sim}{\overline{\theta}}))\cdot\overline{g}(\theta_F^*)\\
\widehat{\theta} =&\theta_F^* - M_k(\lambda_F(\underset{\sim}{\theta}_F^*))\cdot\overline{g}(\theta_F^*) - \left[M_k(\lambda_F(\underset{\sim}{\overline{\theta}})) - M_k(\lambda_F(\underset{\sim}{\theta}_F^*))\right]\cdot\overline{g}(\theta_F^*)\\
&+\left[M_k(\lambda_F(\underset{\sim}{\overline{\theta}})) - M_k(\overline{G}(\underset{\sim}{\overline{\theta}}))\right]\cdot\overline{g}(\theta_F^*)\\
\equiv&\theta_F^* + \frac{1}{n}\sum_{i=1}^{n}\psi_F^{\theta}(Z_i) + \varepsilon_n^{\theta},\quad\text{where}
\end{aligned}\tag{B3.2}$$

$$\begin{aligned}
\psi_F^{\theta}(Z_i) \equiv& -M_k(\lambda_F(\underset{\sim}{\theta}_F^*))\cdot g(Z_i,\theta_F^*) = -\left(E_F\left[\frac{\partial g(Z,\theta_F^*)}{\partial\theta\partial\theta'}\right]\right)^{-1}\cdot g(Z_i,\theta_F^*),\\
\varepsilon_n^{\theta} \equiv&\left[M_k(\lambda_F(\underset{\sim}{\overline{\theta}})) - M_k(\lambda_F(\underset{\sim}{\theta}_F^*))\right]\cdot\overline{g}(\theta_F^*) + \left[M_k(\lambda_F(\underset{\sim}{\overline{\theta}})) - M_k(\overline{G}(\underset{\sim}{\overline{\theta}}))\right]\cdot\overline{g}(\theta_F^*)
\end{aligned}$$

Consider the class of functions

$$\mathscr{G}_{\ell,m} = \left\{f:\mathcal{S}_Z\to\mathbb{R}: f(z) = \frac{\partial g_\ell(z,\theta)}{\partial\theta_m}\ \text{ for some }\theta\in\Theta\right\}$$

By part (i) of Assumption GMM and Lemma 2.13 in Pakes and Pollard (1989), there exist positive constants A and V such that, for every the class $\mathscr{G}_{\ell,m}$ is Euclidean (A,V) for an envelope $\overline{W}(z)$ for which $\exists\ \overline{\mu}_{\overline{W}}<\infty$ such that $E_F\left[\overline{W}(Z)^{4q}\right]\le\overline{\mu}_{\overline{W}}$ (where q is the integer described in Assumption GMM). The conditions in Result S1 are satisfied for the integer q described in Assumption GMM and there exists a constant $\overline{D}<\infty$ such that, for each $\ell,m\in 1,\ldots,k$ and for any $b>0$,

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{\theta\in\Theta}\left|\frac{1}{n}\sum_{i=1}^{n}\left(\frac{\partial g_\ell(Z_i,\theta)}{\partial\theta_m} - E_F\left[\frac{\partial g_\ell(Z,\theta)}{\partial\theta_m}\right]\right)\right|\ge b\right)\le\frac{\overline{D}}{\left(n^{1/2}\cdot b\right)^q}\tag{B3.3}$$

Next, note that for any $b>0$, we have

$$\mathbb{1}\left\{\sup_{\underset{\sim}{\theta}\in\Theta^k}\left\|\overline{G}(\underset{\sim}{\theta})-\lambda_F(\underset{\sim}{\theta})\right\|\ge b\right\}\le\sum_{\ell=1}^{k}\sum_{m=1}^{k}\mathbb{1}\left\{\sup_{\theta\in\Theta}\left|\frac{1}{n}\sum_{i=1}^{n}\left(\frac{\partial g_\ell(Z_i,\theta)}{\partial\theta_m} - E_F\left[\frac{\partial g_\ell(Z,\theta)}{\partial\theta_m}\right]\right)\right|\ge m_k\cdot b\right\},$$

where m_k is a constant that depends only on k. Thus, from (B3.3) we have that there exists a constant $\overline{M}_1 < \infty$ such that, for all $b > 0$,

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{\underset{\sim}{\theta}\in\Theta^k} \left\|\overline{G}(\underset{\sim}{\theta}) - \lambda_F(\underset{\sim}{\theta})\right\| \geq b\right) \leq \frac{\overline{M}_1}{\left(n^{1/2}\cdot b\right)^q}, \tag{B3.4}$$

where q is the integer described in Assumption GMM. Using Chebyshev's inequality, part (i) of Assumption GMM also yields the following result for each $\ell = 1,\ldots,k$,

$$\sup_{F\in\mathcal{F}} P_F\left(\left|\frac{1}{n}\sum_{i=1}^n g_\ell(Z_i,\theta_F^*)\right| \geq b\right) \leq \frac{\overline{\mu}_g}{\left(n^{1/2}\cdot b\right)^q} \quad \forall\, b > 0. \tag{B3.5}$$

Next, note that

$$\mathbb{1}\left\{\left\|\frac{1}{n}\sum_{i=1}^n g(Z_i,\theta_F^*)\right\| \geq b\right\} \leq \sum_{\ell=1}^n \mathbb{1}\left\{\left|\frac{1}{n}\sum_{i=1}^n g_\ell(Z_i,\theta_F^*)\right| \geq c_k\cdot b\right\},$$

where c_k is a constant that depends only on k. By the conditions in Assumption GMM we have

$$\begin{aligned}\mathbb{1}\left\{\left\|\frac{1}{n}\sum_{i=1}^n \psi_F^\theta(Z_i)\right\| \geq b\right\} &\leq \mathbb{1}\left\{\left\|\frac{1}{n}\sum_{i=1}^n g(Z_i,\theta_F^*)\right\| \geq \frac{b}{\overline{M}_\lambda}\right\}\\ &\leq \sum_{\ell=1}^n \mathbb{1}\left\{\left|\frac{1}{n}\sum_{i=1}^n g_\ell(Z_i,\theta_F^*)\right| \geq c_k\cdot\left(\frac{b}{\overline{M}_\lambda}\right)\right\},\end{aligned}$$

From here and (B3.5) we have that there exist constants $\overline{M}_2 < \infty$ and $\overline{M}_3 < \infty$ such that, for all $b > 0$,

$$\sup_{F\in\mathcal{F}} P_F\left(\left\|\frac{1}{n}\sum_{i=1}^n g(Z_i,\theta_F^*)\right\| \geq b\right) \leq \frac{\overline{M}_2}{\left(n^{1/2}\cdot b\right)^q}, \quad \text{and} \quad \sup_{F\in\mathcal{F}} P_F\left(\left\|\frac{1}{n}\sum_{i=1}^n \psi_F^\theta(Z_i)\right\| \geq b\right) \leq \frac{\overline{M}_3}{\left(n^{1/2}\cdot b\right)^q}, \tag{B3.6}$$

where q is the integer described in Assumption GMM. Note that the above result implies that

$$\left\|\overline{g}(\theta_F^*)\right\| \equiv \left\|\frac{1}{n}\sum_{i=1}^n g(Z_i,\theta_F^*)\right\| = O_p\left(n^{-1/2}\right), \quad \text{and} \quad \left\|\frac{1}{n}\sum_{i=1}^n \psi_F^\theta(Z_i)\right\| = O_p\left(n^{-1/2}\right), \quad \text{uniformly over } \mathcal{F}. \tag{B3.7}$$

Now, take any $c > 0$ and note from Assumption GMM that

$$\left\|\varepsilon_n^\theta\right\| \leq 2\overline{M}_\lambda\cdot\left\|\overline{g}(\theta_F^*)\right\| + \left\|M_k(\lambda_F(\overline{\underset{\sim}{\theta}})) - M_k(\overline{G}(\overline{\underset{\sim}{\theta}}))\right\|\cdot\left\|\overline{g}(\theta_F^*)\right\|.$$

Therefore,

$$\begin{aligned}\mathbb{1}\left\{\|\varepsilon_n^{\theta}\| \geq c\right\} &\leq \max\left(\mathbb{1}\left\{2\overline{M}_{\lambda}\left\|\overline{g}(\theta_F^*)\right\| \geq \frac{c}{2}\right\}, \mathbb{1}\left\{\left\|M_k(\lambda_F(\overline{\underset{\sim}{\theta}})) - M_k(\overline{G}(\overline{\underset{\sim}{\theta}}))\right\| \cdot \left\|\overline{g}(\theta_F^*)\right\| \geq \frac{c}{2}\right\}\right) \\ &= \max\left(\mathbb{1}\left\{\left\|\overline{g}(\theta_F^*)\right\| \geq \frac{c}{4\overline{M}_{\lambda}}\right\}, \underbrace{\mathbb{1}\left\{\left\|M_k(\lambda_F(\overline{\underset{\sim}{\theta}})) - M_k(\overline{G}(\overline{\underset{\sim}{\theta}}))\right\| \cdot \left\|\overline{g}(\theta_F^*)\right\| \geq \frac{c}{2}\right\}}_{(III)}\right) \end{aligned} \tag{B3.8}$$

Let us examine the term (III) in (B3.8). From the conditions in Assumption GMM, we have

$$\begin{aligned}&\mathbb{1}\left\{\left\|M_k(\lambda_F(\overline{\underset{\sim}{\theta}})) - M_k(\overline{G}(\overline{\underset{\sim}{\theta}}))\right\| \cdot \left\|\overline{g}(\theta_F^*)\right\| \geq \frac{c}{2}\right\} = \\ &\underbrace{\mathbb{1}\left\{\left\|M_k(\lambda_F(\overline{\underset{\sim}{\theta}})) - M_k(\overline{G}(\overline{\underset{\sim}{\theta}}))\right\| \cdot \left\|\overline{g}(\theta_F^*)\right\| \geq \frac{c}{2}\right\} \cdot \mathbb{1}\left\{\left\|M_k(\lambda_F(\overline{\underset{\sim}{\theta}})) - M_k(\overline{G}(\overline{\underset{\sim}{\theta}}))\right\| \leq K_5\right\}}_{\leq \mathbb{1}\left\{\left\|\overline{g}(\theta_F^*)\right\| \geq \frac{c}{2K_5}\right\}} \\ &+\underbrace{\mathbb{1}\left\{\left\|M_k(\lambda_F(\overline{\underset{\sim}{\theta}})) - M_k(\overline{G}(\overline{\underset{\sim}{\theta}}))\right\| \cdot \left\|\overline{g}(\theta_F^*)\right\| \geq \frac{c}{2}\right\} \cdot \mathbb{1}\left\{\left\|M_k(\lambda_F(\overline{\underset{\sim}{\theta}})) - M_k(\overline{G}(\overline{\underset{\sim}{\theta}}))\right\| > K_5\right\}}_{\leq \mathbb{1}\left\{\|\overline{G}(\overline{\underset{\sim}{\theta}}) - \lambda_F(\overline{\underset{\sim}{\theta}})\| \geq K_3\right\}}\end{aligned}$$

Therefore,

$$\begin{aligned}\mathbb{1}\left\{\left\|M_k(\lambda_F(\overline{\underset{\sim}{\theta}})) - M_k(\overline{G}(\overline{\underset{\sim}{\theta}}))\right\| \cdot \left\|\overline{g}(\theta_F^*)\right\| \geq \frac{c}{2}\right\} &\leq \mathbb{1}\left\{\left\|\overline{g}(\theta_F^*)\right\| \geq \frac{c}{2K_5}\right\} + \mathbb{1}\left\{\left\|\overline{G}(\overline{\underset{\sim}{\theta}}) - \lambda_F(\overline{\underset{\sim}{\theta}})\right\| \geq K_3\right\} \\ &\leq \mathbb{1}\left\{\left\|\overline{g}(\theta_F^*)\right\| \geq \left(\frac{c}{2K_5}\right) \wedge K_3\right\} + \mathbb{1}\left\{\left\|\overline{G}(\overline{\underset{\sim}{\theta}}) - \lambda_F(\overline{\underset{\sim}{\theta}})\right\| \geq \left(\frac{c}{2K_5}\right) \wedge K_3\right\} \\ &\leq \mathbb{1}\left\{\left\|\overline{g}(\theta_F^*)\right\| \geq \left(\frac{c}{2K_5}\right) \wedge K_3\right\} + \mathbb{1}\left\{\sup_{\underset{\sim}{\theta} \in \Theta^k}\left\|\overline{G}(\underset{\sim}{\theta}) - \lambda_F(\underset{\sim}{\theta})\right\| \geq \left(\frac{c}{2K_5}\right) \wedge K_3\right\}\end{aligned} \tag{B3.9}$$

Combining (B3.8) and (B3.9), we have

$$\mathbb{1}\left\{\left\|\varepsilon_n^{\theta}\right\| \geq c\right\} \leq \mathbb{1}\left\{\left\|\overline{g}(\theta_F^*)\right\| \geq \left(K_3 \wedge \left(\frac{1}{2K_5} \wedge \frac{1}{4\overline{M}_{\lambda}}\right) \cdot c\right)\right\} + \mathbb{1}\left\{\sup_{\underset{\sim}{\theta} \in \Theta^k}\left\|\overline{G}(\underset{\sim}{\theta}) - \lambda_F(\underset{\sim}{\theta})\right\| \geq \left(K_3 \wedge \left(\frac{1}{2K_5} \wedge \frac{1}{4\overline{M}_{\lambda}}\right) \cdot c\right)\right\}$$

From here, combining (B3.4) and (B3.6), we have that for any $c > 0$,

$$\sup_{F \in \mathcal{F}} P_F\left(\left\|\varepsilon_n^{\theta}\right\| \geq c\right) \leq \frac{\overline{M}_1 + \overline{M}_2}{\left(n^{1/2} \cdot \left(K_3 \wedge \left(\frac{1}{2K_5} \wedge \frac{1}{4\overline{M}_{\lambda}}\right) \cdot c\right)\right)^q} = o\left(\frac{1}{n^{1/2+\delta}}\right) \quad \forall\ 0 < \delta < \frac{q-1}{2}$$

Take $0 < \delta < \frac{q-1}{2}$ and consider a sequence $c_n > 0$ such that $n^{\frac{q-1-2\delta}{2q}} \cdot c_n \longrightarrow \infty$. Then, the result in the previous expression would still hold for c_n. Thus,

$$\sup_{F\in\mathcal{F}} P_F\left(\left\|\varepsilon_n^{\theta}\right\| \geq c_n\right) = o\left(\frac{1}{n^{1/2+\delta}}\right) \quad \forall\, c_n : n^{\frac{q-1-2\delta}{2q}} \cdot c_n \longrightarrow \infty, \quad 0<\delta<\frac{q-1}{2} \tag{B3.10}$$

From our previous results we have

$$\begin{aligned}\sup_{F\in\mathcal{F}} P_F\left(\left\|\widehat{\theta}-\theta_F^*\right\| \geq c\right) &\leq \sup_{F\in\mathcal{F}} P_F\left(\left\|\frac{1}{n}\sum_{i=1}^{n}\psi_F^{\theta}(Z_i)\right\| \geq \frac{c}{2}\right) + \sup_{F\in\mathcal{F}} P_F\left(\left\|\varepsilon_n^{\theta}\right\| \geq \frac{c}{2}\right)\\ &\leq \frac{\overline{M}_3}{\left(n^{1/2}\cdot\frac{c}{2}\right)^q} + \frac{\overline{M}_1+\overline{M}_2}{\left(n^{1/2}\cdot\left(K_3 \wedge \left(\frac{1}{2K_5}\wedge\frac{1}{4\overline{M}_\lambda}\right)\cdot\frac{c}{2}\right)\right)^q} \longrightarrow 0 \quad \forall\, c>0\end{aligned} \tag{B3.11}$$

Thus, $\left\|\widehat{\theta}-\theta_F^*\right\| = o_p(1)$ uniformly over $\mathcal{F}$. Next, from the definition of ε_n^{θ} in (B3.2), we have

$$\left\|\varepsilon_n^{\theta}\right\| \leq \left\|M_k(\lambda_F(\overline{\underset{\sim}{\theta}})) - M_k(\lambda_F(\underset{\sim}{\theta}_F^*))\right\|\cdot\left\|\overline{g}(\theta_F^*)\right\| + \left\|M_k(\lambda_F(\overline{\underset{\sim}{\theta}})) - M_k(\overline{G}(\overline{\underset{\sim}{\theta}}))\right\|\cdot\left\|\overline{g}(\theta_F^*)\right\| \tag{B3.12}$$

As we stated in (B3.7), $\left\|\overline{g}(\theta_F^*)\right\| = O_p\left(n^{-1/2}\right)$ uniformly over $\mathcal{F}$. Let us examine the asymptotic properties of $\left\|M_k(\lambda_F(\overline{\underset{\sim}{\theta}})) - M_k(\lambda_F(\underset{\sim}{\theta}_F^*))\right\|$ and $\left\|M_k(\lambda_F(\overline{\underset{\sim}{\theta}})) - M_k(\overline{G}(\overline{\underset{\sim}{\theta}}))\right\|$ under Assumption GMM. Take any $b>0$, then from Assumption GMM,

$$\begin{aligned}\mathbb{1}\left\{\sup_{\underset{\sim}{\theta}\in\Theta^k}\left\|M_k(\overline{G}(\underset{\sim}{\theta})) - M_k(\lambda_F(\underset{\sim}{\theta}))\right\| \geq b\right\} &\leq \mathbb{1}\left\{\sup_{\underset{\sim}{\theta}\in\Theta^k}\left(K_4\cdot\left\|\overline{G}(\underset{\sim}{\theta})-\lambda_F(\underset{\sim}{\theta})\right\|^{\alpha_1}\right)\geq b\right\} + \mathbb{1}\left\{\sup_{\underset{\sim}{\theta}\in\Theta^k}\left\|\overline{G}(\underset{\sim}{\theta})-\lambda_F(\underset{\sim}{\theta})\right\|\geq K_3\right\}\\ &\leq \mathbb{1}\left\{\sup_{\underset{\sim}{\theta}\in\Theta^k}\left\|\overline{G}(\underset{\sim}{\theta})-\lambda_F(\underset{\sim}{\theta})\right\| \geq K_3 \wedge\left(\frac{b}{K_4}\right)^{1/\alpha_1}\right\}\end{aligned}$$

Therefore, from (B3.4),

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{\underset{\sim}{\theta}\in\Theta^k}\left\|M_k(\overline{G}(\underset{\sim}{\theta})) - M_k(\lambda_F(\underset{\sim}{\theta}))\right\| \geq n^{-\tau}\cdot b\right) \leq \frac{\overline{M}_1}{\left(n^{1/2}\cdot\left(K_3\wedge\left(\frac{n^{-\tau}\cdot b}{K_4}\right)^{1/\alpha_1}\right)\right)^q} \longrightarrow 0 \quad \forall\, \tau<\frac{\alpha_1}{2}$$

which means

$$\sup_{\underset{\sim}{\theta}\in\Theta^k}\left\|M_k(\overline{G}(\underset{\sim}{\theta})) - M_k(\lambda_F(\underset{\sim}{\theta}))\right\| = o_p\left(n^{-\tau}\right) \quad \forall\, \tau<\frac{\alpha_1}{2}, \quad \text{uniformly over } \mathcal{F}.$$

In particular,

$$\left\|M_k(\overline{G}(\overline{\underset{\sim}{\theta}})) - M_k(\lambda_F(\overline{\underset{\sim}{\theta}}))\right\| = o_p\left(n^{-\tau}\right) \quad \forall\;\, \tau<\frac{\alpha_1}{2}, \quad \text{uniformly over } \mathcal{F}. \tag{B3.13}$$

Next, recall from the definitions of $\overline{\underset{\sim}{\theta}}$ and $\underset{\sim}{\theta}_F^*$ that, for any $\delta>0$, we have $\mathbb{1}\left\{\left\|\overline{\underset{\sim}{\theta}}-\underset{\sim}{\theta}_F^*\right\|\geq\delta\right\} \leq$

$\mathbb{1}\left\{\left\|\widehat{\theta}-\theta_F^*\right\| \geq d_k \cdot \delta\right\}$, where d_k is a constant that depends only on k. From here and Assumption GMM we have that, for any $\eta > 0$,

$$
\begin{aligned}
\mathbb{1}\left\{\left\|\lambda_F(\overline{\underset{\sim}{\theta}}) - \lambda_F(\underset{\sim}{\theta}_F^*)\right\| \geq \eta\right\} &\leq \mathbb{1}\left\{K_7 \cdot \left\|\overline{\underset{\sim}{\theta}} - \underset{\sim}{\theta}_F^*\right\|^{\alpha_2} \geq \eta\right\} + \mathbb{1}\left\{\left\|\overline{\underset{\sim}{\theta}} - \underset{\sim}{\theta}_F^*\right\| \geq K_6\right\} \\
&\leq \mathbb{1}\left\{\left\|\overline{\underset{\sim}{\theta}} - \underset{\sim}{\theta}_F^*\right\| \geq K_6 \wedge \left(\frac{\eta}{K_7}\right)^{1/\alpha_2}\right\} \\
&\leq \mathbb{1}\left\{\left\|\widehat{\theta} - \theta_F^*\right\| \geq d_k \cdot \left(K_6 \wedge \left(\frac{\eta}{K_7}\right)^{1/\alpha_2}\right)\right\}
\end{aligned}
$$

Take $c > 0$. Using Assumption GMM and the result in the previous expression,

$$
\begin{aligned}
\mathbb{1}\left\{\left\|M_k(\lambda_F(\overline{\underset{\sim}{\theta}})) - M_k(\lambda_F(\underset{\sim}{\theta}_F^*))\right\| \geq c\right\} &\leq \mathbb{1}\left\{K_4 \cdot \left\|\lambda_F(\overline{\underset{\sim}{\theta}}) - \lambda_F(\underset{\sim}{\theta}_F^*)\right\|^{\alpha_1} \geq c\right\} + \mathbb{1}\left\{\left\|\lambda_F(\overline{\underset{\sim}{\theta}}) - \lambda_F(\underset{\sim}{\theta}_F^*)\right\| \geq K_3\right\} \\
&\leq \mathbb{1}\left\{\left\|\lambda_F(\overline{\underset{\sim}{\theta}}) - \lambda_F(\underset{\sim}{\theta}_F^*)\right\| \geq K_3 \wedge \left(\frac{c}{K_4}\right)^{1/\alpha_1}\right\} \\
&\leq \mathbb{1}\left\{\left\|\widehat{\theta} - \theta_F^*\right\| \geq d_k \cdot \left(K_6 \wedge \left(\frac{K_3 \wedge \left(\frac{c}{K_4}\right)^{1/\alpha_1}}{K_7}\right)^{1/\alpha_2}\right)\right\} \\
&\leq \mathbb{1}\left\{\left\|\widehat{\theta} - \theta_F^*\right\| \geq d_k \cdot \left(K_6 \wedge \left[\left(K_3 \wedge (c/K_4)^{1/\alpha_1}\right)\Big/K_7\right]^{1/\alpha_2}\right)\right\}
\end{aligned}
$$

From here, using (B3.11), for any $b > 0$ we have

$$
\begin{aligned}
&\sup_{F \in \mathcal{F}} P_F\left(\left\|M_k(\lambda_F(\overline{\underset{\sim}{\theta}})) - M_k(\lambda_F(\underset{\sim}{\theta}_F^*))\right\| \geq b\right) \\
&\qquad \leq \frac{\overline{M}_3}{\left(n^{1/2} \cdot \frac{1}{2} \cdot d_k \cdot \left(K_6 \wedge \left[\left(K_3 \wedge (b/K_4)^{1/\alpha_1}\right)\Big/K_7\right]^{1/\alpha_2}\right)\right)^q} \\
&\qquad + \frac{\overline{M}_1 + \overline{M}_2}{\left(n^{1/2} \cdot \left(K_3 \wedge \left(\frac{1}{2K_5} \wedge \frac{1}{4\overline{M}_\lambda}\right) \cdot \frac{1}{2} \cdot d_k \cdot \left(K_6 \wedge \left[\left(K_3 \wedge (b/K_4)^{1/\alpha_1}\right)\Big/K_7\right]^{1/\alpha_2}\right)\right)\right)^q}
\end{aligned}
$$

Therefore,

$$
\sup_{F \in \mathcal{F}} P_F\left(\left\|M_k(\lambda_F(\overline{\underset{\sim}{\theta}})) - M_k(\lambda_F(\underset{\sim}{\theta}_F^*))\right\| \geq n^{-\tau} \cdot b\right) \longrightarrow 0 \quad \forall\, \tau < \frac{\alpha_1 \cdot \alpha_2}{2}
$$

which means,

$$
\left\|M_k(\lambda_F(\overline{\underset{\sim}{\theta}})) - M_k(\lambda_F(\underset{\sim}{\theta}_F^*))\right\| = o_p\left(n^{-\tau}\right) \quad \forall\, \tau < \frac{\alpha_1 \cdot \alpha_2}{2}
$$

Thus, combining (B3.7), (B3.12), (B3.13) and the previous expression, we have that for any $0 < \tau < \frac{\alpha_1}{2} \wedge \frac{\alpha_1 \cdot \alpha_2}{2}$,

$$
\left\|\varepsilon_n^\theta\right\| = o_p\left(\frac{1}{n^{1/2+\tau}}\right) \quad \text{uniformly over } \mathcal{F}.
$$

Together, (B3.2), (B3.10) and the result in the previous expression show that the conditions in Assumption 1 of the paper are satisfied, with $\psi_F^{\theta}(Z_i) = -\left(E_F\left[\frac{\partial g(Z,\theta_F^*)}{\partial\theta\partial\theta'}\right]\right)^{-1} \cdot g(Z_i,\theta_F^*)$, $r_n = n^{1/2}$, $0 < \tau < \frac{\alpha_1}{2} \wedge \frac{\alpha_1 \cdot \alpha_2}{2}$, and $0 < \overline{\delta} < \frac{q-1}{2}$. This proves Result GMM. ■

B4 Density-weighted average derivatives

Consider an iid sample $(Z_{1i}, Z_{2i})_{i=1}^n$ where $Z_{1i} \in \mathbb{R}$, $Z_{2i} \in \mathbb{R}^d$ and $Z_i \equiv (Z_{1i}, Z_{2i}) \sim F \in \mathcal{F}$. As in our previous discussions, we will let $\mathcal{S}_\xi$ denote the support of the r.v ξ. We will group $Z \equiv (Z_1, Z_2)$ and we will assume that $\mathcal{S}_Z$ is the same for all $F \in \mathcal{F}$. Suppose that, for each $F \in \mathcal{F}$, we have $E_F[Z_1|Z_2] \equiv \mu_F(Z_2) = G_F(Z_2'\beta_0)$, where G_F is unknown but smooth as described in Powell, Stock, and Stoker (1989) (we will be precise about these smoothness conditions below). Let f_{z_2} denote the density of Z_2, assumed to be absolutely continuous with respect to Lebesgue measure, and denote $\delta_F \equiv E_F\left[f_{z_2}(Z_2)G_F'(Z_2'\beta_0)\right]$ and $\theta_F^* \equiv \delta_F \cdot \beta_0$. Using integration by parts, under the conditions described in Powell, Stock, and Stoker (1989), we have

$$\theta_F^* = -2 \cdot E_F\left[Z_1 \cdot \frac{\partial f_{z_2}(Z_2)}{\partial Z_2}\right].$$

Let $K : \mathbb{R}^d \to \mathbb{R}$ be a kernel function (whose conditions we will describe below) and let $\sigma > 0$ be a strictly positive scalar. For a pair of observations $i \neq j$ in the sample let

$$p(Z_i, Z_j; \sigma) \equiv \left(Z_{1i} - Z_{1j}\right) \cdot K^{(1)}\left(\frac{Z_{2j} - Z_{2i}}{\sigma}\right).$$

Let $\sigma_n \to 0$ be a bandwidth sequence. The estimator for θ_F^* proposed in Powell, Stock, and Stoker (1989) is of the form

$$\widehat{\theta} = \binom{n}{2}^{-1} \frac{1}{\sigma_n^{d+1}} \sum_{i<j} p(Z_i, Z_j; \sigma_n). \tag{B4.1}$$

Assumption DWAD

(i) There exists an integer $q \geq 2$ and a constant $\overline{\mu}_{z_1} < \infty$ such that $E_F\left[Z_1^{4q}\right] \leq \overline{\mu}_{z_1}$ for all $F \in \mathcal{F}$.

(ii) The kernel $K : \mathbb{R}^d \to \mathbb{R}$ is a multiplicative kernel of the form $K(\psi) = \prod_{\ell=1}^d \kappa(\psi_\ell)$ (with $\psi \equiv (\psi_1, \ldots, \psi_k)$). $\kappa(\cdot)$ is a bounded kernel function satisfying $|\kappa(v)| \leq \overline{\kappa} < \infty$ for all v. The kernel function $\kappa(\cdot)$ is also of bounded variation and it has support of the form $[-S, S]$. $\kappa(\cdot)$ is symmetric around zero and has the properties of a bias-reducing kernel of order L: $\int_{-S}^S v^j \kappa(v)dv = 0$ for $j = 1, \ldots, L-1$ and $\int_{-S}^S |v|^L \kappa(v)dv < \infty$. In addition, $\kappa(\cdot)$ is differentiable, with first derivative denoted as $\kappa'(\cdot)$. The function $\kappa^{'}(\cdot)$ is bounded, satisfying $|k'(v)| \leq \overline{\kappa}_1 < \infty$ for all v, and it is also

of bounded variation. Since $\kappa(\cdot)$ is symmetric around zero, $\kappa'(\cdot)$ is antisymmetric around zero, satisfying $\kappa'(v) = -\kappa'(-v)$ for all $v \in [-S,S]$. Thus, if we let $K^{(1)}$ denote the Jacobian of K, then $K^{(1)}(\psi) = -K^{(1)}(-\psi)$ for all $\psi \in \mathbb{R}^d$. We have $\left|K(\psi)\right| \leq \overline{K} < \infty$ and $\left\|K^{(1)}(\psi)\right\| \leq \overline{K}_1 < \infty$ for all $\psi \in \mathbb{R}^d$.

(iii) Let L be the constant described above. Then, both $f_{z_2}(z_2)$ and $\mu_F(z_2)$ are L-times continuously differentiable with respect to z_2 for F–a.e $z_2 \in \mathcal{S}_Z$, with derivatives that are uniformly bounded over $\mathcal{S}_Z$ for all $F \in \mathcal{F}$.

(iv) Let L be as described above. The bandwidth sequence $\sigma_n > 0$ satisfies $\sigma_n \longrightarrow 0$ and is such that $n^{1/2-\Delta} \cdot \sigma_n^{d+1} \longrightarrow \infty$, $n^{1/2+\Delta} \cdot \sigma_n^L \longrightarrow 0$ and $n^{1+\Delta} \cdot \sigma_n^{d+1} \cdot \sigma_n^L \longrightarrow 0$ for some $0 < \Delta < 1/2$. In addition, the integer q described above and Δ are such that $q\Delta > \frac{1}{2}$.

Result DWAD *For a given $(z_1,z_2) \in \mathcal{S}_Z$, let $\varphi_F(z_1,z_2) \equiv (z_1 - \mu_F(z_2)) \cdot f_{z_2}(z_2)$, and let $\psi_F^{\theta}(Z_i) \equiv -2 \cdot \left(\frac{\varphi_F(Z_{1i},Z_{2i})}{\partial Z_2} - E_F\left[\frac{\varphi_F(Z_{1i},Z_{2i})}{\partial Z_2}\right]\right)$. Under Assumption DWAD, the estimator $\widehat{\theta}$ described in equation* (B4.1) *satisfies*

$$\widehat{\theta} = \theta_F^* + \frac{1}{n}\sum_{i=1}^{n} \psi_F^{\theta}(Z_i) + \varepsilon_n^{\theta},$$

and the conditions in Assumption 1 of the paper are satisfied, with $\psi_F^{\theta}(Z_i) = -2\cdot\left(\frac{\varphi_F(Z_{1i},Z_{2i})}{\partial Z_2} - E_F\left[\frac{\varphi_F(Z_{1i},Z_{2i})}{\partial Z_2}\right]\right)$, $r_n = n \cdot \sigma_n^{d+1}$, $0 < \tau < \Delta$, and $0 < \overline{\delta} < q\Delta - \frac{1}{2}$.

Proof: For a given $\sigma > 0$ let

$$\begin{aligned} r_{1,F}(Z_i;\sigma) &= E_F\left[p(Z_i,Z_j;\sigma)|Z_i\right] - E_F\left[p(Z_i,Z_j;\sigma)\right], \\ r_{2,F}(Z_i,Z_j;\sigma) &= \left(p(Z_i,Z_j;\sigma) - E_F\left[p(Z_i,Z_j;\sigma)\right]\right) - r_{1,F}(Z_i;\sigma) - r_{1,F}(Z_j;\sigma), \\ U_{2,n}(\sigma) &= \binom{n}{2}^{-1} \sum_{i<j} r_{2,F}(Z_i,Z_j;\sigma). \end{aligned}$$

$U_{2,n}(\sigma)$ is a degenerate U-statistic of order 2 and $\{U_{2,n}(\sigma) : \sigma > 0\}$ is a degenerate U-process of order 2. Going forward we will denote $U_{2,n}(\sigma_n) \equiv U_{2,n}$. A Hoeffding decomposition (see Serfling (1980, pages 177-178) or Sherman (1994, equations (6)-(7))) of the U-statistic in (B4.1) yields

$$\widehat{\theta} = \frac{1}{\sigma_n^{d+1}} \cdot E_F\left[p(Z_i,Z_j;\sigma_n)\right] + \frac{2}{n \cdot \sigma_n^{d+1}} \sum_{i=1}^{n} r_{1,F}(Z_i;\sigma_n) + \frac{1}{\sigma_n^{d+1}} \cdot U_{2,n}. \tag{B4.2}$$

For a given $(z_1,z_2) \in \mathcal{S}_Z$, let $\varphi_F(z_1,z_2) \equiv (z_1 - \mu_F(z_2)) \cdot f_{z_2}(z_2)$. Under the smoothness conditions and the higher-order properties of the kernel described in Assumption DWAD, an M^{th}–order approximation yields the following re-expression of (B4.2),

$$\begin{aligned}\widehat{\theta} =&\theta^*_F - \frac{2}{n}\sum_{i=1}^{n}\left(\frac{\varphi_F(Z_{1i},Z_{2i})}{\partial Z_2} - E_F\left[\frac{\varphi_F(Z_{1i},Z_{2i})}{\partial Z_2}\right]\right) + \frac{1}{\sigma_n^{d+1}}\cdot U_{2,n} + B_{n,F}\\ \equiv&\theta^*_F + \frac{1}{n}\sum_{i=1}^{n}\psi^{\theta}_F(Z_i) + \varepsilon^{\theta}_n, \quad \text{where}\\ \psi^{\theta}_F(Z_i) \equiv& -2\cdot\left(\frac{\varphi_F(Z_{1i},Z_{2i})}{\partial Z_2} - E_F\left[\frac{\varphi_F(Z_{1i},Z_{2i})}{\partial Z_2}\right]\right),\\ \varepsilon^{\theta}_n \equiv& \frac{1}{\sigma_n^{d+1}}\cdot U_{2,n} + B_{n,F},\end{aligned} \tag{B4.3}$$

where $B_{n,F}$ is a bias aggregate term which, by the smoothness conditions and the higher-order properties of the kernel described in Assumption DWAD, is such that there exists a constant $\overline{C}_H > 0$ such that

$$||B_{n,F}|| \leq \overline{C}_H \cdot \sigma_n^L \quad \forall\ F \in \mathcal{F}. \tag{B4.4}$$

Let us examine the properties of the degenerate U-process $\{U_{2,n}(\sigma) : \sigma > 0\}$ under Assumption DWAD. By Lemma 22 in Nolan and Pollard (1987), if $\lambda(\cdot)$ is a real-valued function of bounded variation on $\mathbb{R}$, the class of all functions of the form $z_2 \rightarrow \lambda(\gamma' z_2)$ with γ ranging over $\mathbb{R}^d$ is Euclidean for the constant envelope $\overline{\lambda} \equiv \sup_{b\in\mathbb{R}}|\lambda(b)|$. Combining this with the closure properties of Euclidean classes described in Lemma 2.14 in Pakes and Pollard (1989), the conditions in Assumption DWAD imply that the class of functions

$$\mathscr{H} = \left\{ f : \mathcal{S}^2_Z \longrightarrow \mathbb{R} : f(z_a, z_b) = (z_{1a} - z_{1b})\cdot K^{(1)}\left(\frac{z_{2b} - z_{2a}}{\sigma}\right) \text{ for some } \sigma > 0 \right\}$$

is Euclidean for envelope $G(z_a, z_b) = \overline{K}_1 \cdot (|z_{1a}| + |z_{1b}|)$. By Assumption DWAD, there exists $\overline{\mu}_G < \infty$ such that $E_F\left[G(Z_{1i}, Z_{1j})^{4q}\right] \leq \overline{\mu}_G$. From here, the conditions in Result S1 are satisfied for the integer q described in Assumption DWAD and we have that, for all $b > 0$, there exists $\overline{D}_H < \infty$ such that

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{\sigma>0}\left|U_{2,n}(\sigma)\right| \geq b\right) \leq \frac{\overline{D}_H}{(n\cdot b)^q}$$

From here it follows, in particular,

$$\sup_{F\in\mathcal{F}} P_F\left(\left|U_{2,n}\right| \geq b\right) \leq \frac{\overline{D}_H}{(n\cdot b)^q} \quad \forall\ b > 0 \tag{B4.5}$$

Note that (B4.5) implies

$$\left|U_{2,n}\right| = O_p\left(\frac{1}{n}\right) \quad \text{uniformly over } \mathcal{F}. \tag{B4.6}$$

Take any $c>0$. From (B4.4), we have

$$\mathbb{1}\left\{\left\|\varepsilon_n^{\theta}\right\|\geq c\right\}\leq \mathbb{1}\left\{\sup_{\sigma>0}\left\|U_{2,n}(\sigma)\right\|\geq \sigma_n^{d+1}\cdot\left(c-\overline{C}_H\cdot\sigma_n^L\right)\right\}$$

Let n_0 be the smallest integer such that $\overline{C}_H\cdot\sigma_n^L<c$. Then, from the previous expression and (B4.5),

$$\sup_{F\in\mathcal{F}}\ P_F\left(\left\|\varepsilon_n^{\theta}\right\|\geq c\right)\leq \frac{\overline{D}_H}{\left(n\cdot\sigma_n^{d+1}\cdot\left(c-\overline{C}_H\cdot\sigma_n^L\right)\right)^q}\quad \forall\ n\geq n_0 \tag{B4.7}$$

Note from Assumption DWAD(iv) that $\frac{n^{1/2+\overline{\delta}}}{(n\cdot\sigma_n^{d+1})}\to 0$ for any $0<\overline{\delta}<q\Delta-\frac{1}{2}$. Next, combining (B4.4) and (B4.6), we have

$$\left\|\varepsilon_n^{\theta}\right\|=O_p\left(\frac{1}{n\cdot\sigma_n^{d+1}}\right)+O\left(\sigma_n^L\right)\quad\text{uniformly over } \mathcal{F}.$$

Therefore, by the conditions in Assumption DWAD, for any $0<\tau<\Delta$,

$$\left\|\varepsilon_n^{\theta}\right\|=o_p\left(\frac{1}{n^{1/2+\tau}}\right),\quad\text{uniformly over } \mathcal{F}.$$

Together, (B4.3), (B4.7) and the previous expression show that the conditions in Assumption 1 are satisfied, with $\psi_F^{\theta}(Z_i)=-2\cdot\left(\frac{\varphi_F(Z_{1i},Z_{2i})}{\partial Z_2}-E_F\left[\frac{\varphi_F(Z_{1i},Z_{2i})}{\partial Z_2}\right]\right)$, $r_n=n\cdot\sigma_n^{d+1}$, $0<\tau<\Delta$, and $0<\overline{\delta}<q\Delta-\frac{1}{2}$. This proves Result DWAD. ■

B5 A semiparametric multiple-index estimator

Consider a collection of d single-valued indices $(m_\ell(W_\ell,\theta_\ell))_{\ell=1}^d$ where $\theta_\ell\in\mathbb{R}^{k_\ell}$. Each m_ℓ has a known parametric functional form (e.g, $m_\ell(W_\ell,\theta_\ell)=W_\ell'\theta_\ell$). Group $\cup_{\ell=1}^d W_\ell\equiv Z_2$ and let $\theta\equiv\left(\theta_1',\theta_2',\ldots,\theta_d'\right)'\in\mathbb{R}^k$ and denote

$$m(Z_2,\theta)\equiv(m_1(W_1,\theta_1),m_2(W_2,\theta_2),\ldots,m_d(W_d,\theta_d))'\in\mathbb{R}^d.$$

For simplicity let us focus on the case where Z_2 is a vector of jointly continuously distributed random variables. Let Z_1 be a scalar random variable and group $Z\equiv(Z_1,Z_2)\sim F\in\mathcal{F}$. Let Θ denote the parameter space for θ, assume Θ to be bounded and consider a model where there exists a $\theta^*\in\Theta$ such that

$$E_F[Z_1|Z_2]=E_F\left[Z_1\middle|m(Z_2,\theta^*)\right]\quad\forall\ F\in\mathcal{F}$$

For a given $\theta\in\Theta$ let $\mu_F(m(Z_2,\theta))\equiv E_F[Z_1|m(Z_2,\theta)]$. Our model therefore assumes $E_F[Z_1|Z_2]=\mu_F(m(Z_2,\theta^*))$. Let $\phi\in\mathbb{R}^k$ denote a vector of pre-specified instrument functions and consider an estimator based on the moment conditions

$$E_F\left[\phi(Z_2)\cdot\left(Z_1-\mu_F(m(Z_2,\theta^*))\right)\right]=0 \tag{B5.1}$$

Suppose we have a random sample $(Z_{1i},Z_{2i})_{i=1}^n$ where $Z_i\equiv(Z_{1i},Z_{2i})\sim F\in\mathcal{F}$. Let $\mathcal{S}_\xi$ denote the support of the r.v ξ and for simplicity assume throughout that $\mathcal{S}_Z$ is the same for all $F\in\mathcal{F}$. Suppose that the instrument functions are designed such that $\phi(z_2)=0\ \forall\ z_2\notin\mathcal{Z}_2$, where $\mathcal{Z}_2\subset\mathcal{S}_{Z_2}$ is a pre-specified set belonging in the interior of $\mathcal{S}_{Z_2}$ for all $F\in\mathcal{F}$. We refer to $\mathcal{Z}_2$ as our *inference range*. Thus, the instrument functions also serve as trimming functions to keep inference confined to the set $\mathcal{Z}_2$. Finally, suppose $\|\phi(z_2)\|\le\overline{\phi}\ \forall\ z_2$. Let

$$\mathcal{M}\equiv\left\{m\in\mathbb{R}^d\colon\ m=m(z_2,\theta)\ \text{ for some } (z_2,\theta)\in\mathcal{Z}_2\times\Theta\right\}.$$

$\mathcal{M}$ is the range of all possible values of the index $m(z_2,\theta)$ over our inference range and the parameter space. Let $\sigma_n\to0$ denote a bandwidth sequence and let K denote a kernel function. For a given $\theta\in\Theta$ and $z_2\in\mathcal{Z}_2$, let $f_m(m(Z_2,\theta))$ denote the density of $m(Z_2,\theta)$. Consider a kernel-based estimator of $\mu_F(m(z_2,\theta))$ of the form

$$\begin{aligned}
\widehat{\mu}(m(z_2,\theta))&=\frac{\widehat{R}(m(z_2,\theta))}{\widehat{f}_m(m(z_2,\theta))},\quad\text{where}\\
\widehat{R}(m(z_2,\theta))&=\frac{1}{n\cdot\sigma_n^d}\sum_{i=1}^n Z_{1i}K\left(\frac{m(Z_{2i},\theta)-m(z_2,\theta)}{\sigma_n}\right),\\
\widehat{f}_m(m(z_2,\theta))&=\frac{1}{n\cdot\sigma_n^d}\sum_{i=1}^n K\left(\frac{m(Z_{2i},\theta)-m(z_2,\theta)}{\sigma_n}\right).
\end{aligned}$$

Consider an estimator $\widehat{\theta}$ defined by the sample analog moment conditions to (B5.1),

$$\frac{1}{n}\sum_{i=1}^n\phi(Z_{2i})\cdot\left(Z_{1i}-\widehat{\mu}(m(Z_{2i},\widehat{\theta}))\right)=0. \tag{B5.2}$$

Assumption SMIM1 *For some $q\ge2$, we have $E_F[Z_1^{4q}]\le\overline{\mu}_{4q}<\infty$ for all $F\in\mathcal{F}$. Also, there exist constants $\underline{f}_m>0$, $\overline{f}_m<\infty$ and $\overline{\mu}<\infty$ such that $\overline{f}_m\ge f_m(m)\ge\underline{f}_m$ and $|\mu_F(m)|\le\overline{\mu}\ \ \forall\ m\in\mathcal{M}$ and all $F\in\mathcal{F}$. Also assume that both $f_m(m)$ and $\mu_F(m)$ are L-times continuously differentiable with respect to m for $F-$a.e $m\in\mathcal{M}$, with derivatives that are uniformly bounded over $\mathcal{M}$ for all $F\in\mathcal{F}$. The kernel $K:\mathbb{R}^d\to\mathbb{R}$ is a multiplicative kernel of the form $K(\psi)=\prod_{\ell=1}^d\kappa(\psi_\ell)$ (with $\psi\equiv(\psi_1,\ldots,\psi_d)$), where $\kappa(\cdot)$ is a function of bounded-variation, a bias-reducing kernel of order L with support of the form $[-S,\,S]$ (i.e, $\int_{-S}^S v^j\kappa(v)dv=0$ for $j=1,\ldots,L-1$ and $\int_{-S}^S|v|^L\kappa(v)dv<\infty$) and symmetric around zero. We have $\sup_{\psi\in\mathbb{R}^d}\left|K(\psi)\right|\le\overline{K}$. The bandwidth sequence $\sigma_n>0$ satisfies $\sigma_n\longrightarrow0$, with $n^{1/2+\Delta}\cdot\sigma_n^L\longrightarrow0$ and*

$n^{1/2-\Delta} \cdot \sigma_n^d \longrightarrow \infty$ *for some* $0 < \Delta < 1/2$. q *and* Δ *are such that* $q\Delta > \frac{1}{2}$.

Denote $R_F(m) \equiv \mu_F(m) \cdot f_m(m)$ and note from Assumption SMIM1 that $|R_F(m)| \leq \overline{\mu} \cdot \overline{f}_m \equiv \overline{R} \; \forall \; m \in \mathcal{M}$ and all $F \in \mathcal{F}$. Fix $m \in \mathcal{M}$. A second order aproximation yields

$$\widehat{\mu}(m) = \mu_F(m) + \frac{1}{f_m(m)} \cdot \left(\widehat{R}(m) - R_F(m)\right) - \frac{\mu_F(m)}{f_m(m)} \cdot \left(\widehat{f}_m(m) - f_m(m)\right)$$
$$- \frac{\left(\widehat{R}(m) - R_F(m)\right) \cdot \left(\widehat{f}_m(m) - f_m(m)\right)}{\widetilde{f}_m(m)^2} + \frac{\widetilde{R}(m) \cdot \left(\widehat{f}_m(m) - f_m(m)\right)^2}{\widetilde{f}_m(m)^3},$$

where $\widetilde{f}_m(m)$ is an intermediate point between $\widehat{f}_m(m)$ and $f_m(m)$, and $\widetilde{R}(m)$ is an intermediate point between $\widehat{R}(m)$ and $R_F(m)$. From here, we have that, for any given $(z_2, \theta) \in \mathcal{Z}_2 \times \Theta$,

$$\widehat{\mu}(m(z_2,\theta)) - \mu_F(m(z_2,\theta)) = \frac{1}{n \cdot \sigma_n^d} \sum_{i=1}^{n} \frac{(Z_{1i} - \mu_F(m(z_2,\theta)))}{f_m(m(z_2,\theta))} \cdot K\left(\frac{m(Z_{2i},\theta) - m(z_2,\theta)}{\sigma_n}\right) + \varepsilon_n^{\mu}(m(z_2,\theta)),$$

where

$$\varepsilon_n^{\mu}(m(z_2,\theta)) \equiv - \frac{\left(\widehat{R}(m(z_2,\theta)) - R_F(m(z_2,\theta))\right) \cdot \left(\widehat{f}_m(m(z_2,\theta)) - f_m(m(z_2,\theta))\right)}{\widetilde{f}_m(m(z_2,\theta))^2}$$
$$+ \frac{\widetilde{R}(m(z_2,\theta)) \cdot \left(\widehat{f}_m(m(z_2,\theta)) - f_m(m(z_2,\theta))\right)^2}{\widetilde{f}_m(m(z_2,\theta))^3} \tag{B5.3}$$

For the next few lines let us omit the dependence of $\widehat{f}_m$, $\widehat{R}$, $\widetilde{f}_m$, $\widetilde{R}$, ε_n^{μ}, f_m and R_F on $m(z_2,\theta)$ to simplify the exposition. Note first that

$$\left|\varepsilon_n^{\mu}\right| \leq \frac{|\widehat{R} - R_F| \cdot |\widehat{f}_m - f_m|}{|\widetilde{f}_m|^2} + \frac{\overline{R} \cdot |\widehat{f}_m - f_m|^2}{|\widetilde{f}_m|^3} + \frac{|\widetilde{R} - R_F| \cdot |\widehat{f}_m - f_m|^2}{|\widetilde{f}_m|^3}$$
$$\leq \frac{|\widehat{R} - R_F| \cdot |\widehat{f}_m - f_m|}{|\widetilde{f}_m|^2} + \frac{\overline{R} \cdot |\widehat{f}_m - f_m|^2}{|\widetilde{f}_m|^3} + \frac{|\widehat{R} - R_F| \cdot |\widehat{f}_m - f_m|^2}{|\widetilde{f}_m|^3}$$

where the last inequality follows because, by definition, $|\widetilde{R} - R_F| \leq |\widehat{R} - R_F|$. Next, note that, $\forall \; c > 0$, we have $\mathbb{1}\{|\xi_1|+|\xi_2|+|\xi_3|\geq c\}\leq \max\Big(\mathbb{1}\{|\xi_1|\geq \frac{c}{3}\}\,,\,\mathbb{1}\{|\xi_2|\geq \frac{c}{3}\}\,,\,\mathbb{1}\{|\xi_3|\geq \frac{c}{3}\}\Big)$. Therefore, for any $c>0$,

$$\mathbb{1}\left\{\left|\varepsilon_n^{\mu}\right|\geq c\right\}\leq \max\Bigg(\underbrace{\mathbb{1}\left\{\left|\widehat{R}-R_F\right|\cdot\left|\widehat{f}_m-f_m\right|\geq \frac{|\widetilde{f}_m|^2\cdot c}{3}\right\}}_{(IV)}\,,\,\underbrace{\mathbb{1}\left\{\left|\widehat{f}_m-f_m\right|^2\geq \frac{|\widetilde{f}_m|^3\cdot c}{3\overline{R}}\right\}}_{(V)}\,,\,\underbrace{\mathbb{1}\left\{\left|\widehat{R}-R_F\right|\cdot\left|\widehat{f}_m-f_m\right|^2\geq \frac{|\widetilde{f}_m|^3\cdot c}{3}\right\}}_{(VI)}\Bigg) \tag{B5.4}$$

Next, note that, $\forall\ c>0$, we have $\mathbb{1}\{|\xi_1|\cdot|\xi_2|\geq c\}\leq \max\Big(\mathbb{1}\{|\xi_1|\geq c^{1/2}\}\,,\,\mathbb{1}\{|\xi_2|\geq c^{1/2}\}\Big)$. Therefore,

$$\underbrace{\mathbb{1}\left\{\left|\widehat{R}-R_F\right|\cdot\left|\widehat{f}_m-f_m\right|\geq \frac{|\widetilde{f}_m|^2\cdot c}{3}\right\}}_{(IV)} \leq \max\left(\mathbb{1}\left\{\left|\widehat{R}-R_F\right|\geq\left(\frac{|\widetilde{f}_m|^2\cdot c}{3}\right)^{1/2}\right\}\,,\,\mathbb{1}\left\{\left|\widehat{f}_m-f_m\right|\geq\left(\frac{|\widetilde{f}_m|^2\cdot c}{3}\right)^{1/2}\right\}\right) \tag{B5.5A}$$

$$\underbrace{\mathbb{1}\left\{\left|\widehat{f}_m-f_m\right|^2\geq \frac{|\widetilde{f}_m|^3\cdot c}{3\overline{R}}\right\}}_{(V)}=\mathbb{1}\left\{\left|\widehat{f}_m-f_m\right|\geq\left(\frac{|\widetilde{f}_m|^3\cdot c}{3\overline{R}}\right)^{1/2}\right\} \tag{B5.5B}$$

$$\underbrace{\mathbb{1}\left\{\left|\widehat{R}-R_F\right|\cdot\left|\widehat{f}_m-f_m\right|^2\geq \frac{|\widetilde{f}_m|^3\cdot c}{3}\right\}}_{(VI)} \leq \max\left(\mathbb{1}\left\{\left|\widehat{R}-R_F\right|\geq\left(\frac{|\widetilde{f}_m|^3\cdot c}{3}\right)^{1/2}\right\}\,,\,\mathbb{1}\left\{\left|\widehat{f}_m-f_m\right|\geq\left(\frac{|\widetilde{f}_m|^3\cdot c}{3}\right)^{1/4}\right\}\right) \tag{B5.5C}$$

Note that $\min\left\{|\widetilde{f}_m|\,,\,|\widetilde{f}_m|^{3/2}\,,\,|\widetilde{f}_m|^{3/4}\right\}=\min\left\{|\widetilde{f}_m|^{3/2}\,,\,|\widetilde{f}_m|^{3/4}\right\}$, and $\min\left\{c^{1/4}\,,\,c^{1/2}\right\}\leq\min\left\{c^{1/4}\,,\,c\right\}$ for all $c>0$. Given this, let

$$\varphi^{\mu}(\widetilde{f}_m,c)\equiv\frac{1}{\sqrt{3}}\cdot\min\left\{\frac{1}{\overline{R}}\,,\,1\right\}\cdot\min\left\{|\widetilde{f}_m|^{3/2}\,,\,|\widetilde{f}_m|^{3/4}\right\}\cdot\min\left\{c^{1/4}\,,\,c\right\}.$$

Combining (B5.5A), (B5.5B) and (B5.5C) with (B5.4), we have

$$\mathbb{1}\left\{\left|\varepsilon_n^{\mu}\right| \geq c\right\} \leq \max\Bigg(\underbrace{\mathbb{1}\left\{\left|\widehat{R} - R_F\right| \geq \varphi^{\mu}(\widetilde{f}_m, c)\right\}}_{(VII)} \, , \, \underbrace{\mathbb{1}\left\{\left|\widehat{f}_m - f_m\right| \geq \varphi^{\mu}(\widetilde{f}_m, c)\right\}}_{(VIII)}\Bigg) \tag{B5.6}$$

Let us analyze term (VII) in (B5.6) first. Begin by expressing it as

$$\begin{aligned}\mathbb{1}\left\{\left|\widehat{R} - R_F\right| \geq \varphi^{\mu}(\widetilde{f}_m, c)\right\} = \;& \mathbb{1}\left\{\left|\widehat{R} - R_F\right| \geq \varphi^{\mu}(\widetilde{f}_m, c)\right\} \cdot \mathbb{1}\left\{\left|\widetilde{f}_m\right| \geq \left|f_m\right| - \frac{1}{2} \cdot \underline{f}_m\right\} \\ &+ \mathbb{1}\left\{\left|\widehat{R} - R_F\right| \geq \varphi^{\mu}(\widetilde{f}_m, c)\right\} \cdot \mathbb{1}\left\{\left|\widetilde{f}_m\right| < \left|f_m\right| - \frac{1}{2} \cdot \underline{f}_m\right\}\end{aligned} \tag{B5.7}$$

We begin with the first term on the right-hand side of (B5.7). Note first that $|f_m| - \frac{1}{2} \cdot \underline{f}_m \geq \frac{1}{2} \cdot \underline{f}_m$ and therefore $\mathbb{1}\left\{|\widetilde{f}_m| \geq |f_m| - \frac{1}{2} \cdot \underline{f}_m\right\} \leq \mathbb{1}\left\{|\widetilde{f}_m| \geq \frac{1}{2} \cdot \underline{f}_m\right\}$. Define

$$D^{\varepsilon^{\mu}} \equiv \frac{1}{\sqrt{3}} \cdot \min\left\{\frac{1}{\overline{\overline{R}}} \, , 1\right\} \cdot \min\left\{\left(\frac{1}{2} \cdot \underline{f}_m\right)^{3/2} \, , \left(\frac{1}{2} \cdot \underline{f}_m\right)^{3/4}\right\}$$

Then, the first term on the right-hand side of (B5.7) satisfies

$$\mathbb{1}\left\{\left|\widehat{R} - R_F\right| \geq \varphi^{\mu}(\widetilde{f}_m, c)\right\} \cdot \mathbb{1}\left\{\left|\widetilde{f}_m\right| \geq \left|f_m\right| - \frac{1}{2} \cdot \underline{f}_m\right\} \leq \mathbb{1}\left\{\left|\widehat{R} - R_F\right| \geq D^{\varepsilon^{\mu}} \cdot \min\left\{c^{1/4} \, , c\right\}\right\} \tag{B5.8A}$$

Next, we move on to the second term on the right-hand side of (B5.7). Recall that, by definition, we have $|\widehat{f}_m - f_m| \geq |\widetilde{f}_m - f_m|$. Therefore, $|\widehat{f}_m - f_m| \geq |\widetilde{f}_m - f_m| \geq |f_m| - |\widetilde{f}_m|$ and thus, $\mathbb{1}\{|\widetilde{f}_m| < |f_m| - \frac{1}{2} \cdot \underline{f}_m\} = \mathbb{1}\{|f_m| - |\widetilde{f}_m| > \frac{1}{2} \cdot \underline{f}_m\} \leq \mathbb{1}\{|\widetilde{f}_m - f_m| > \frac{1}{2} \cdot \underline{f}_m\} \leq \mathbb{1}\{|\widehat{f}_m - f_m| \geq \frac{1}{2} \cdot \underline{f}_m\}$. Therefore, the second term on the right-hand side of (B5.7) satisfies,

$$\begin{aligned}&\mathbb{1}\left\{\left|\widehat{R} - R_F\right| \geq \varphi^{\mu}(\widetilde{f}_m, c)\right\} \cdot \mathbb{1}\left\{\left|\widetilde{f}_m\right| < \left|f_m\right| - \frac{1}{2} \cdot \underline{f}_m\right\} \\ \leq &\mathbb{1}\left\{\left|\widehat{R} - R_F\right| \geq \varphi^{\mu}(\widetilde{f}_m, c)\right\} \cdot \mathbb{1}\left\{\left|\widehat{f}_m - f_m\right| \geq \frac{1}{2} \cdot \underline{f}_m\right\} \leq \mathbb{1}\left\{\left|\widehat{f}_m - f_m\right| \geq \frac{1}{2} \cdot \underline{f}_m\right\}\end{aligned} \tag{B5.8B}$$

Combining (B5.8A) and (B5.8B) with (B5.7), we have that the term (VII) in equation (B5.6) satisfies,

$$\begin{aligned}\underbrace{\mathbb{1}\left\{\left|\widehat{R} - R_F\right| \geq \varphi^{\mu}(\widetilde{f}_m, c)\right\}}_{(VII)} \leq \max\Bigg(&\mathbb{1}\left\{\left|\widehat{R} - R_F\right| \geq \min\left\{\frac{\underline{f}_m}{2} \, , D^{\varepsilon^{\mu}} c^{1/4} \, , D^{\varepsilon^{\mu}} c\right\}\right\} \\ &, \mathbb{1}\left\{\left|\widehat{f}_m - f_m\right| \geq \min\left\{\frac{\underline{f}_m}{2} \, , D^{\varepsilon^{\mu}} c^{1/4} \, , D^{\varepsilon^{\mu}} c\right\}\right\}\Bigg)\end{aligned} \tag{B5.9A}$$

Next, we analyze the term (VIII) in equation (B5.6). Similar to (B5.7), let us write it as

$$\mathbb{1}\left\{\left|\widehat{f}_m - f_m\right| \geq \varphi^{\mu}(\widetilde{f}_m, c)\right\} = \mathbb{1}\left\{\left|\widehat{f}_m - f_m\right| \geq \varphi^{\mu}(\widetilde{f}_m, c)\right\} \cdot \mathbb{1}\left\{\left|\widetilde{f}_m\right| \geq \left|f_m\right| - \frac{1}{2}\cdot \underline{f}_m\right\}$$
$$+ \mathbb{1}\left\{\left|\widehat{f}_m - f_m\right| \geq \varphi^{\mu}(\widetilde{f}_m, c)\right\} \cdot \mathbb{1}\left\{\left|\widetilde{f}_m\right| < \left|f_m\right| - \frac{1}{2}\cdot \underline{f}_m\right\}$$

Parallel steps to those leading to (B5.8A) and (B5.8B) now yield,

$$\underbrace{\mathbb{1}\left\{\left|\widehat{f}_m - f_m\right| \geq \varphi^{\mu}(\widetilde{f}_m, c)\right\}}_{(VIII)} \leq \mathbb{1}\left\{\left|\widehat{f}_m - f_m\right| \geq \min\left\{\frac{\underline{f}_m}{2}\ ,\ D^{\varepsilon^{\mu}} c^{1/4}\ ,\ D^{\varepsilon^{\mu}} c\right\}\right\} \tag{B5.9B}$$

Denote

$$\underline{\varphi}^{\mu}(c) \equiv \min\left\{\frac{\underline{f}_m}{2}\ ,\ D^{\varepsilon^{\mu}} c\ ,\ D^{\varepsilon^{\mu}} c^{1/4}\right\}. \tag{B5.10}$$

Combining (B5.9A) and (B5.9B) with (B5.6), we have that, for any $c > 0$,

$$\begin{aligned}
&\mathbb{1}\left\{\left|\varepsilon_n^{\mu}(m(z_2,\theta))\right| \geq c\right\}\\
&\leq \max\left(\mathbb{1}\left\{\left|\widehat{R}(m(z_2,\theta)) - R_F(m(z_2,\theta))\right| \geq \underline{\varphi}^{\mu}(c)\right\}\ ,\ \mathbb{1}\left\{\left|\widehat{f}_m(m(z_2,\theta)) - f_m(m(z_2,\theta))\right| \geq \underline{\varphi}^{\mu}(c)\right\}\right)\\
&\leq \mathbb{1}\left\{\left|\widehat{R}(m(z_2,\theta)) - R_F(m(z_2,\theta))\right| \geq \underline{\varphi}^{\mu}(c)\right\} + \mathbb{1}\left\{\left|\widehat{f}_m(m(z_2,\theta)) - f_m(m(z_2,\theta))\right| \geq \underline{\varphi}^{\mu}(c)\right\}\\
&\quad \forall\ (z_2,\theta) \in \mathcal{Z}_2 \times \Theta\ \forall\ F \in \mathcal{F}
\end{aligned} \tag{B5.11}$$

For $(z_2,\theta) \in \mathcal{Z}_2 \times \Theta$ and $\sigma > 0$, let

$$\begin{aligned}
&p^R(Z_i, z_2, \theta, \sigma) \equiv Z_{1i} \cdot K\left(\frac{m(Z_{2i},\theta) - m(z_2,\theta)}{\sigma}\right),\ p^{f_m}(Z_i, z_2, \theta, \sigma) \equiv K\left(\frac{m(Z_{2i},\theta) - m(z_2,\theta)}{\sigma}\right),\\
&\nu_n^R(z_2,\theta,\sigma) \equiv \frac{1}{n}\sum_{i=1}^{n}\left(p^R(Z_i, z_2, \theta, \sigma) - E_F\left[p^R(Z, z_2, \theta, \sigma)\right]\right), \quad \text{and}\\
&\nu_n^{f_m}(z_2,\theta,\sigma) \equiv \frac{1}{n}\sum_{i=1}^{n}\left(p^{f_m}(Z_i, z_2, \theta, \sigma) - E_F\left[p^{f_m}(Z, z_2, \theta, \sigma)\right]\right).
\end{aligned}$$

Assumption SMIM2 *Consider the following class of functions defined on* $\mathcal{S}_{Z_2}$

$$\mathscr{G}_1 = \left\{g : \mathcal{S}_{Z_2} \to \mathbb{R} : g(z_2) = K\left(\alpha \cdot m(z_2,\theta) + \beta \cdot m(v,\theta)\right)\ \textit{for some}\ \ v \in \mathcal{S}_{Z_2},\ \theta \in \Theta,\ \alpha,\beta \in \mathbb{R}\right\}$$

Then, $\mathscr{G}_1$ *is Euclidean for the constant envelope* $\overline{K}$.

For indices of the form $m(z_2,\theta) = z_2'\theta$, the condition in Assumption SMIM2 follows immediately from Lemma 22 in Nolan and Pollard (1987), who showed that if $\lambda(\cdot)$ is a real-valued function of bounded variation on $\mathbb{R}$, the class of all functions of the form $x \to \lambda(\gamma' x + \tau)$ with γ ranging over $\mathbb{R}^d$ and τ ranging over $\mathbb{R}$ is Euclidean for a constant envelope. By Assumption SMIM2, the class of functions

$$\left\{ g : \mathcal{S}_{Z_2} \to \mathbb{R} : g(z_2) = K\left(\frac{m(z_2,\theta) - m(v,\theta)}{\sigma}\right) \text{ for some } v \in \mathcal{S}_{Z_2}, \theta \in \Theta, \sigma > 0 \right\}$$

is Euclidean for the constant envelope $\overline{K}$ and the conditions in Result S1 are satisfied for any integer q (due to the constant nature of the envelope) and we have that, for all $b > 0$ and any integer q, there exists $\overline{M}_1 < \infty$ such that

$$\sup_{F \in \mathcal{F}} P_F\left(\sup_{\substack{(z_2,\theta) \in \mathcal{Z}_2 \times \Theta \\ \sigma > 0}} \left| \nu_n^{f_m}(z_2,\theta,\sigma) \right| > b \right) \leq \frac{\overline{M}_1}{\left(n^{1/2} \cdot b\right)^q} \tag{B5.12}$$

From here it follows that, for any $\varepsilon > 0$, there exists a finite $\Delta_\varepsilon > 0$ such that

$$\sup_{F \in \mathcal{F}} P_F\left(\sup_{\substack{(z_2,\theta) \in \mathcal{Z}_2 \times \Theta \\ \sigma > 0}} \left| n^{1/2} \nu_n^{f_m}(z_2,\theta,\sigma) \right| > \Delta_\varepsilon \right) \leq \varepsilon,$$

which means that

$$\sup_{(z_2,\theta) \in \mathcal{Z}_2 \times \Theta} \left| \nu_n^{f_m}(z_2,\theta,\sigma) \right| = O_p\left(\frac{1}{n^{1/2}}\right) \text{ uniformly over } \mathcal{F} \tag{B5.13}$$

By Pakes and Pollard (1989, Lemma 2.14), Assumption SMIM2 implies that the class of functions

$$\mathscr{G}_2 = \left\{ g : \mathcal{S}_{Z_1} \times \mathcal{S}_{Z_2} \to \mathbb{R} : g(z_1, z_2) = z_1 \cdot K\left(\frac{m(z_2,\theta) - m(v,\theta)}{\sigma}\right) \text{ for some } v \in \mathcal{S}_{Z_2}, \theta \in \Theta, \sigma > 0 \right\}$$

is also Euclidean for the envelope $G(z_1) = |z_1| \cdot \overline{K}$. By Assumption SMIM1, $E_F[G(Z_1)^{4q}] = \overline{K}^{4q} \cdot E_F[Z_1^{4q}] \leq \overline{K}^{4q} \cdot \overline{\mu}_{4q} < \infty$ for all $F \in \mathcal{F}$, from here, Result S1 implies the existence of $\overline{M}_2 < \infty$ such that, for the integer q described in Assumption SMIM1,

$$\sup_{F \in \mathcal{F}} P_F\left(\sup_{\substack{(z_2,\theta) \in \mathcal{Z}_2 \times \Theta \\ \sigma > 0}} \left| \nu_n^{R}(z_2,\theta,\sigma) \right| > b \right) \leq \frac{\overline{M}_2}{\left(n^{1/2} \cdot b\right)^q} \tag{B5.14}$$

which in turn also implies that

$$\sup_{(z_2,\theta)\in\mathcal{Z}_2\times\Theta}\left|\nu_n^R(z_2,\theta,\sigma)\right| = O_p\left(\frac{1}{n^{1/2}}\right) \text{ uniformly over } \mathcal{F}. \tag{B5.15}$$

We have

$$\widehat{R}(m(z_2,\theta)) = R_F(m(z_2,\theta)) + \frac{1}{\sigma_n^d}\cdot\nu_n^R(z_2,\theta,\sigma_n) + B_{n,F}^R(z_2,\theta),$$
$$\widehat{f_m}(m(z_2,\theta)) = f_m(m(z_2,\theta)) + \frac{1}{\sigma_n^d}\cdot\nu_n^{f_m}(z_2,\theta,\sigma_n) + B_{n,F}^{f_m}(z_2,\theta),$$

where

$$B_{n,F}^R(z_2,\theta) \equiv E_F\left[\frac{p^R(Z,z_2,\theta,\sigma_n)}{\sigma_n^d}\right] - R_F(z_2,\theta), \quad B_{n,F}^{f_m}(z_2,\theta) \equiv E_F\left[\frac{p^{f_m}(Z,z_2,\theta,\sigma_n)}{\sigma_n^d}\right] - f_m(z_2,\theta)$$

are the corresponding bias terms. By the smoothness conditions described above and the bias-reducing nature of the kernel K, there exists a constant $C_B^{\mu_a}$ such that

$$\sup_{(z_2,\theta)\in\mathcal{Z}_2\times\Theta}\left|B_{n,F}^R(z_2,\theta)\right| \le C_B^{\mu_a}\cdot\sigma_n^L, \quad \sup_{(z_2,\theta)\in\mathcal{Z}_2\times\Theta}\left|B_{n,F}^{f_m}(z_2,\theta)\right| \le C_B^{\mu_a}\cdot\sigma_n^L \quad \forall\, F\in\mathcal{F} \tag{B5.16}$$

Define

$$s_{1,n} \equiv C_B^{\mu_a}\cdot\sigma_n^L. \tag{B5.17}$$

Then, from (B5.13), (B5.15) and (B5.16)

$$\begin{aligned}
\sup_{(z_2,\theta)\in\mathcal{Z}_2\times\Theta}\left|\widehat{R}(m(z_2,\theta)) - R_F(m(z_2,\theta))\right| &\le \frac{1}{\sigma_n^d}\cdot\sup_{(z_2,\theta)\in\mathcal{Z}_2\times\Theta}\left|\nu_n^R(z_2,\theta,\sigma_n)\right| + s_{1,n} = O_p\left(\frac{1}{n^{1/2}\cdot\sigma_n^d}\right) + s_{1,n}\\
\sup_{(z_2,\theta)\in\mathcal{Z}_2\times\Theta}\left|\widehat{f_m}(m(z_2,\theta)) - f_m(m(z_2,\theta))\right| &\le \frac{1}{\sigma_n^d}\cdot\sup_{(z_2,\theta)\in\mathcal{Z}_2\times\Theta}\left|\nu_n^{f_m}(z_2,\theta,\sigma_n)\right| + s_{1,n} = O_p\left(\frac{1}{n^{1/2}\cdot\sigma_n^d}\right) + s_{1,n},
\end{aligned} \tag{B5.18}$$

uniformly over $\mathcal{F}$. Take any $b>0$ and let n_0 be the smallest integer such that $s_{1,n}<b$. Combining (B5.12), (B5.14) and (B5.18),

$$\begin{aligned}
\sup_{F\in\mathcal{F}} P_F\left(\sup_{(z_2,\theta)\in\mathcal{Z}_2\times\Theta}\left|\widehat{f_m}(m(z_2,\theta)) - f_m(m(z_2,\theta))\right| > b\right) &\le \frac{\overline{M}_1}{\left(n^{1/2}\cdot\sigma_n^d\cdot(b-s_{1,n})\right)^q} \quad \forall\, n\ge n_0,\\
\sup_{F\in\mathcal{F}} P_F\left(\sup_{(z_2,\theta)\in\mathcal{Z}_2\times\Theta}\left|\widehat{R}(m(z_2,\theta)) - R_F(m(z_2,\theta))\right| > b\right) &\le \frac{\overline{M}_2}{\left(n^{1/2}\cdot\sigma_n^d\cdot(b-s_{1,n})\right)^q} \quad \forall\, n\ge n_0.
\end{aligned} \tag{B5.19}$$

Going back to the definition of $\varepsilon_n^{\mu}(m(z_2,\theta))$ in (B5.3), recall that

$$\begin{aligned}\left|\varepsilon_n^{\mu}(m(z_2,\theta))\right| \leq & \frac{\left|\widehat{R}(m(z_2,\theta))-R_F(m(z_2,\theta))\right|\cdot\left|\widehat{f}_m(m(z_2,\theta))-f_m(m(z_2,\theta))\right|}{\left|\widetilde{f}_m(m(z_2,\theta))\right|^2} \\ &+\frac{\overline{R}\cdot\left|\widehat{f}_m(m(z_2,\theta))-f_m(m(z_2,\theta))\right|^2}{\left|\widetilde{f}_m(m(z_2,\theta))\right|^3} \\ &+\frac{\left|\widehat{R}(m(z_2,\theta))-R_F(m(z_2,\theta))\right|\cdot\left|\widehat{f}_m(m(z_2,\theta))-f_m(m(z_2,\theta))\right|^2}{\left|\widetilde{f}_m(m(z_2,\theta))\right|^3}\end{aligned} \tag{B5.20}$$

where $\widetilde{f}_m(m(z_2,\theta)$ is an intermediate point between $\widehat{f}_m(m(z_2,\theta)$ and $f_m(m(z_2,\theta)$ and $\overline{R}\equiv\overline{\mu}\cdot\overline{f}_m$. From (B5.18) and Assumption SMIM1, it immediately follows that $\sup_{(z_2,\theta)\in\mathcal{Z}_2\times\Theta}\left|\widetilde{R}(m(z_2,\theta))\right| = O_p(1)$ uniformly over $\mathcal{F}$. Assumption SMIM1 and the result in (B5.18) also imply that $\sup_{(z_2,\theta)\in\mathcal{Z}_2\times\Theta}\left|\frac{1}{\widehat{f}_m(m(z_2,\theta))}\right| = O_p(1)$ uniformly over $\mathcal{F}$. To see why, take any $\delta>0$ and note that

$$P_F\left(\sup_{(z_2,\theta)\in\mathcal{Z}_2\times\Theta}\left|\frac{1}{\widehat{f}_m(m(z_2,\theta))}\right| > \frac{1}{(1-\delta)\cdot\underline{f}_m}\right) \leq P_F\left(\sup_{(z_2,\theta)\in\mathcal{Z}_2\times\Theta}\left|\widehat{f}_m(m(z_2,\theta))-f_m(m(z_2,\theta))\right| > \delta\cdot\underline{f}_m\right)$$

Let n_0 be the smallest n such that $s_{1,n}<\delta\cdot\underline{f}_m$. Then,

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{(z_2,\theta)\in\mathcal{Z}_2\times\Theta}\left|\frac{1}{\widehat{f}_m(m(z_2,\theta))}\right| > \frac{1}{(1-\delta)\cdot\underline{f}_m}\right) \leq \frac{\overline{M}_1}{\left(n^{1/2}\cdot\sigma_n^d\cdot(\delta\cdot f_m - s_{1,n})\right)^q} \quad \forall\ n\geq n_0$$

Therefore, for any $\varepsilon>0$ there exists a small enough δ_ϵ and n_ε such that

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{(z_2,\theta)\in\mathcal{Z}_2\times\Theta}\left|\frac{1}{\widehat{f}_m(m(z_2,\theta))}\right| > \frac{1}{(1-\delta)\cdot\underline{f}_m}\right) \leq \varepsilon \quad \forall\ n\geq n_\varepsilon,$$

and so $\sup_{(z_2,\theta)\in\mathcal{Z}_2\times\Theta}\left|\frac{1}{\widehat{f}_m(m(z_2,\theta))}\right| = O_p(1)$ uniformly over $\mathcal{F}$. From here, (B5.18) and (B5.20) yield

$$\sup_{(z_2,\theta)\in\mathcal{Z}_2\times\Theta}\left|\varepsilon_n^{\mu}(m(z_2,\theta))\right| = O_p\left(\left(\frac{1}{n^{1/2}\cdot\sigma_n^d}+s_{1,n}\right)^2\right), \quad \text{uniformly over } \mathcal{F}. \tag{B5.21}$$

And going back to (B5.11) we have that, for any $c > 0$,

$$\mathbb{1}\left\{\left|\varepsilon_n^{\mu}(m(z_2,\theta))\right| \geq c\right\} \leq \mathbb{1}\left\{\left|\nu_n^{R}(z_2,\theta,\sigma_n)\right| \geq \sigma_n^{d}\cdot\left(\underline{\varphi}^{\mu}(c)-s_{1,n}\right)\right\} + \mathbb{1}\left\{\left|\nu_n^{f_m}(z_2,\theta,\sigma_n)\right| \geq \sigma_n^{d}\cdot\left(\underline{\varphi}^{\mu}(c)-s_{1,n}\right)\right\} \quad \forall\,(z_2,\theta)\in\mathcal{Z}_2\times\Theta,\ \forall\,F\in\mathcal{F}.$$

where $\underline{\varphi}^{\mu}(c)$ is defined in (B5.10). Thus, if we take any $c > 0$ and we let n_0 be the smallest integer such that $s_{1,n} < \underline{\varphi}^{\mu}(c)$, (B5.19) and the previous expression yield

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{(z_2,\theta)\in\mathcal{Z}_2\times\Theta}\left|\varepsilon_n^{\mu}(m(z_2,\theta))\right| \geq c\right) \leq \frac{\overline{M}_1+\overline{M}_2}{\left(n^{1/2}\cdot\sigma_n^{d}\cdot(\underline{\varphi}^{\mu}(c)-s_{1,n})\right)^{q}} \quad \forall\, n\geq n_0. \tag{B5.22}$$

For a given $\theta\in\Theta$ define

$$\nu_n^{\mu}(\theta) \equiv \frac{1}{n}\sum_{i=1}^{n}\phi(Z_{2i})\cdot(\widehat{\mu}(m(Z_{2i},\theta))-\mu_F(m(Z_{2i},\theta))). \tag{B5.23}$$

For a given $\sigma > 0$, let

$$\begin{aligned} m_F^{\mu}(Z_i,Z_j,\theta) &\equiv \frac{1}{2}\left(\phi(Z_{2j})\cdot\frac{\left(Z_{1i}-\mu_F(m(Z_{2j},\theta))\right)}{f_m(m(Z_{2j},\theta))}+\phi(Z_{2i})\cdot\frac{\left(Z_{1j}-\mu_F(m(Z_{2i},\theta))\right)}{f_m(m(Z_{2i},\theta))}\right),\\ p_F^{\mu}(Z_i,Z_j;\theta,\sigma) &\equiv m_F^{\mu}(Z_i,Z_j,\theta)\cdot K\left(\frac{m(Z_{2i},\theta)-m(Z_{2j},\theta)}{\sigma}\right),\\ U_{2,n}^{\mu}(\theta,\sigma) &= \binom{2}{n}^{-1}\sum_{i<j}p_F^{\mu}(Z_i,Z_j;\theta,\sigma), \end{aligned} \tag{B5.24}$$

and denote $U_{2,n}^{\mu}(\theta,\sigma_n)\equiv U_{2,n}^{\mu}(\theta)$. Let

$$Q_F(z,\theta)\equiv\phi(z_2)\cdot\frac{(z_1-\mu_F(m(z_2,\theta)))}{f_m(m(z_2,\theta)}.$$

From (B5.3), we have

$$\nu_n^{\mu}(\theta) = \frac{1}{\sigma_n^{d}}\cdot U_{2,n}^{\mu}(\theta)+\left(\frac{K(0)}{n\cdot\sigma_n^{d}}\right)\cdot\frac{1}{n}\sum_{i=1}^{n}Q_F(Z_i,\theta)+\frac{1}{n}\sum_{i=1}^{n}\phi(Z_{2i})\cdot\varepsilon_n^{\mu}(m(Z_{2i},\theta)) \tag{B5.25}$$

Recall that we have assumed that there exist constants $\underline{f}_m > 0$, $\overline{f}_m < \infty$ and $\overline{\mu} < \infty$ such that $\overline{f}_m \geq f_m(m) \geq \underline{f}_m$ and $|\mu_F(m)| \leq \overline{\mu}$ $\forall\, m\in\mathcal{M}$ and all $F\in\mathcal{F}$. We have also assumed that both

$f_m(m)$ and $\mu_F(m)$ are L-times continuously differentiable with respect to m for F−a.e $m \in \mathcal{M}$, with derivatives that are uniformly bounded over $\mathcal{M}$ for all $F \in \mathcal{F}$. Let

$$\eta_F(m(Z_2,\theta)) \equiv E_F[\phi(Z_2) \mid m(Z_2,\theta)],$$

and assume that, like the other functionals analyzed before, $\eta_F(m)$ is also L-times continuously differentiable with respect to m for F−a.e $m \in \mathcal{M}$ with derivatives that are uniformly bounded over $\mathcal{M}$ for all $F \in \mathcal{F}$. For a given $\theta \in \Theta$ and $\sigma > 0$, let

$$\begin{aligned}
r^{\mu}_{1,F}(Z_i;\theta,\sigma) &= E_F\left[p^{\mu}_F(Z_i,Z_j;\theta,\sigma)\middle|Z_i\right] - E_F\left[p^{\mu}_F(Z_i,Z_j;\theta,\sigma)\right],\\
r^{\mu}_{2,F}(Z_i,Z_j;\theta,\sigma) &= \left(p^{\mu}_F(Z_i,Z_j;\theta,\sigma) - E_F\left[p^{\mu}_F(Z_i,Z_j;\theta,\sigma)\right]\right) - r^{\mu}_{1,F}(Z_i;\theta,\sigma) - r^{\mu}_{1,F}(Z_j;\theta,\sigma),\\
V^{\mu}_{2,n}(\theta,\sigma) &\equiv \binom{2}{n}^{-1}\sum_{i<j} r^{\mu}_{2,F}(Z_i,Z_j;\theta,\sigma)
\end{aligned}$$

$V^{\mu}_{2,n}(\theta,\sigma)$ is a degenerate U-statistic of order 2 and $\left\{V^{\mu}_{2,n}(\theta,\sigma) : \theta \in \Theta,\, \sigma > 0\right\}$ is a degenerate U-process of order 2, and therefore compatible with the conditions for Result S1 under the assumptions we will describe below.

Let us denote $V^{\mu}_{2,n}(\theta,\sigma_n) \equiv V^{\mu}_{2,n}(\theta)$. A Hoeffding decomposition (see Serfling (1980, pages 177-178) or Sherman (1994, equations (6)-(7))) of the U-statistic $U^{\mu}_{2,n}(\theta)$ in (B5.24), combined with the higher-order kernel properties and the smoothness conditions described above yield the following result,

$$\frac{1}{\sigma_n^d}\cdot U^{\mu}_{2,n}(\theta) = \frac{1}{n}\sum_{i=1}^{n}\eta_F(m(Z_{2i},\theta))\cdot\Big(Z_{1i} - \mu_F(m(Z_{2i},\theta))\Big) + \frac{1}{\sigma_n^d}\cdot V^{\mu}_{2,n}(\theta) + B^{\mu}_{n,F}(\theta) \tag{B5.26}$$

where $B_{n,\mu}(\theta)$ is a bias term which, by our smoothness assumptions, is such that there exists a constant $C_B^{\mu_b}$ such that

$$\sup_{\theta\in\Theta}\left\|B^{\mu}_{n,F}(\theta)\right\| \le C_B^{\mu_b}\cdot\sigma_n^L \quad \forall\, F \in \mathcal{F} \tag{B5.27}$$

Plugging (B5.26) into (B5.25), we have

$$\begin{aligned}
\nu^{\mu}_n(\theta) &= \frac{1}{n}\sum_{i=1}^{n}\eta_F(m(Z_{2i},\theta))\cdot\Big(Z_{1i} - \mu_F(m(Z_{2i},\theta))\Big) + \varepsilon^{\nu^{\mu}}_n(\theta), \quad \text{where}\\
\varepsilon^{\nu^{\mu}}_n(\theta) &\equiv \frac{1}{\sigma_n^d}\cdot V^{\mu}_{2,n}(\theta) + \left(\frac{K(0)}{n\cdot\sigma_n^d}\right)\cdot\frac{1}{n}\sum_{i=1}^{n}\Big(Q_F(Z_i,\theta) - E_F\left[Q_F(Z,\theta)\right]\Big)\\
&\quad + \frac{1}{n}\sum_{i=1}^{n}\phi(Z_{2i})\cdot\varepsilon^{\mu}_n(m(Z_{2i},\theta)) + \left(\frac{K(0)}{n\cdot\sigma_n^d}\right)\cdot E_F\left[Q_F(Z,\theta)\right] + B^{\mu}_{n,F}(\theta)
\end{aligned} \tag{B5.28}$$

From our assumptions, it follows that there exists a constant $C_Q < \infty$ such that

$$\sup_{\theta\in\Theta} \left| E_F\left[Q_F(Z,\theta)\right] \right| \le C_Q \quad \forall\ F \in \mathcal{F}$$

Define

$$s_{2,n} \equiv \frac{|K(0)| \cdot C_Q}{n\cdot \sigma_n^d} + C_B^{\mu_b} \cdot \sigma_n^L. \tag{B5.29}$$

Thus, from (B5.27) and the previous expression,

$$\begin{aligned} \left\|\varepsilon_n^{\nu^\mu}(\theta)\right\| \le & \frac{1}{\sigma_n^d}\cdot \left\|V_{2,n}^\mu(\theta)\right\| + \left|\frac{K(0)}{n\cdot\sigma_n^d}\right| \cdot \left\| \frac{1}{n}\sum_{i=1}^n \Big(Q_F(Z_i,\theta) - E_F\left[Q_F(Z,\theta)\right]\Big)\right\| \\ & + \left\| \frac{1}{n}\sum_{i=1}^n \phi(Z_{2i})\cdot \varepsilon_n^\mu(m(Z_{2i},\theta))\right\| + s_{2,n} \end{aligned} \tag{B5.30}$$

Assumption SMIM3 *The index $m(z_2,\theta)$ is smooth with respect to θ and, for every $F\in\mathcal{F}$, the following Jacobians are well-defined for F−a.e $z_2 \in \mathcal{S}_{Z_2}$ and for all $\theta\in\Theta$,*

$$\underbrace{\nabla_\theta \mu_F(m(z_2;\theta))}_{1\times k} \equiv \begin{pmatrix} \frac{\partial \mu_F(m(z_2;\theta)'}{\partial\theta_1} & \frac{\partial \mu_F(m(z_2;\theta)'}{\partial\theta_2} & \cdots & \frac{\partial \mu_F(m(z_2;\theta)'}{\partial\theta_d}\end{pmatrix},$$

$$\underbrace{\nabla_\theta f_m(m(z_2;\theta))}_{1\times k} \equiv \begin{pmatrix} \frac{\partial f_m(m(z_2;\theta)'}{\partial\theta_1} & \frac{\partial f_m(m(z_2;\theta)'}{\partial\theta_2} & \cdots & \frac{\partial f_m(m(z_2;\theta)'}{\partial\theta_d}\end{pmatrix}$$

There exists a nonnegative function $\overline{H}_1(\cdot)$ such that, for each $F\in\mathcal{F}$,

$$\sup_{\theta\in\Theta} \left\|\nabla_\theta \mu_F(m(z_2;\theta))\right\| \le \overline{H}_1(z_2) \quad \forall\ z_2 \in \mathcal{Z}_2,$$

$$\sup_{\theta\in\Theta} \|\nabla_\theta f_m(m(z_2;\theta))\| \le \overline{H}_1(z_2) \quad \forall\ z_2 \in \mathcal{Z}_2,$$

and there exists $\overline{\mu}_{\overline{H}_1} < \infty$ such that $E_F[\overline{H}_1(Z_2)^{4q}] \le \overline{\mu}_{\overline{H}_1}$ $\forall\ F\in\mathcal{F}$, where q is the integer described in Assumption SMIM1.

If we let m_F^μ be as defined in (B5.24), and

$$\overline{G}_1(Z_i,Z_j) \equiv \left(\frac{(|Z_{1i}| + \overline{f}_m + \overline{\mu})}{2\underline{f}_m^2}\cdot \|\phi(Z_{2j})\|\cdot \overline{H}_1(Z_{2j}) + \frac{(|Z_{1j}| + \overline{f}_m + \overline{\mu})}{2\underline{f}_m^2}\cdot \|\phi(Z_{2i})\|\cdot \overline{H}_1(Z_{2i})\right)$$

where $\underline{f}_m$, $\overline{f}_m$ and $\overline{\mu}$ are as defined in Assumption SMIM1, then, for each $F\in\mathcal{F}$,

$$\left\|m_F^\mu(Z_i,Z_j,\theta) - m_F^\mu(Z_i,Z_j,\theta')\right\| \le \overline{G}_1(Z_i,Z_j)\cdot \left\|\theta-\theta'\right\| \quad \forall\ \theta,\theta' \in \Theta$$

Let q be the integer described in Assumption SMIM1. By Assumption SMIM3, there exists $\overline{\mu}_{\overline{G}_1} < \infty$ such that,

$$E_F\left[\overline{G}_1(Z_i, Z_j)^{4q}\right] \leq \overline{\mu}_{\overline{G}_1} \quad \forall\, F \in \mathcal{F}$$

For the ℓ^{th} component ($\phi_\ell(Z_2)$) of $\phi(Z_2)$ and for each F let

$$\mathscr{G}^{\ell}_{3,F} = \Bigg\{ g : \mathcal{S}_Z^2 \to \mathbb{R} : g(z_a, z_b) = \frac{1}{2}\left(\phi_\ell(z_{2b}) \cdot \frac{(z_{1a} - \mu_F(m(z_{2b}, \theta)))}{f_m(m(z_{2b}, \theta))} + \phi_\ell(z_{2a}) \cdot \frac{(z_{1b} - \mu_F(m(z_{2a}, \theta)))}{f_m(m(z_{2a}, \theta))}\right) \times K\left(\frac{m(z_{2a}, \theta) - m(z_{2b}, \theta)}{\sigma}\right) \quad \text{for some } \theta \in \Theta,\, \sigma > 0 \Bigg\}$$

By Assumptions SMIM2 and SMIM3, Lemmas 2.13 and 2.14 in Pakes and Pollard (1989), there exist positive constants A_3 and V_3 such that, for every $F \in \mathcal{F}$, the class $\mathscr{G}^{\ell}_{3,F}$ is Euclidean (A_3, V_3) for the envelope

$$G_3(z_a, z_b) = \frac{1}{2\underline{f}_m}\left(\left\|\phi(z_{2b})\right\| \cdot \left|(z_{1a} - \mu_F(m(z_{2b}, \theta_0)))\right| + \left\|\phi(z_{2a})\right\| \cdot \left|(z_{1b} - \mu_F(m(z_{2a}, \theta_0)))\right|\right) + M_0 \cdot \overline{G}_1(z_a, z_b)$$

where θ_0 is an arbitrary point of Θ and $M_0^\ell \equiv 2\sqrt{k}\sup_\Theta \|\theta - \theta_0\|$. Let q be the integer described in Assumption SMIM1. By the conditions described in AssumptionsSMIM1 and SMIM3, there exists $\overline{\mu}_{G_3} < \infty$ such that $E_F\left[G_3(Z_i, Z_j)^{4q}\right] \leq \overline{\mu}_{G_3}$ for all $F \in \mathcal{F}$. Next, let

$$\mathscr{G}^{\ell}_{4,F} = \left\{ f : \mathcal{S}_Z \to \mathbb{R} : f(z) = E_F\left[g(z, Z)\right] \text{ for some } g \in \mathscr{G}^{\ell}_{3,F} \right\}$$

By Lemma 20 in Nolan and Pollard (1987) (or Lemma 5 in Sherman (1994)), Assumptions SMIM2 and SMIM3 imply that there exist positive constants A_4 and V_4 such that $\mathscr{G}^{\ell}_{4,F}$ is Euclidean (A_4, V_4) for the envelope

$$G_4(z) = \sqrt{E_F\left[G_3(z, Z)^2\right]}$$

Let q be any positive integer. By Jensen's inequality, $G_4(z)^{4q} = \left(E_F\left[G_3(z, Z)^2\right]\right)^{2q} \leq E_F\left[G_3(z, Z)^{4q}\right]$. Therefore, $E_F\left[G_4(Z_i)^{4q}\right] \leq E_F\left[G_3(Z_i, Z_j)^{4q}\right] \leq \overline{\mu}_{G_3}$. The conditions in Result S1 are satisfied for the integer q described in Assumption SMIM2 and there exists a constant $\overline{M}_3 < \infty$ such that, for all $b > 0$,

$$\sup_{F \in \mathcal{F}} P_F\left(\sup_{\theta \in \Theta}\left\|V^{\mu}_{2,n}(\theta)\right\| > b\right) \leq \sup_{F \in \mathcal{F}} P_F\left(\sup_{\substack{\theta \in \Theta \\ h > 0}}\left\|V^{\mu}_{2,n}(\theta)\right\| > b\right) \leq \frac{\overline{M}_3}{(n \cdot b)^q} \tag{B5.31}$$

which in turn also implies that

$$\sup_{\theta \in \Theta}\left\|V^{\mu}_{2,n}(\theta)\right\| = O_p\left(\frac{1}{n}\right) \text{ uniformly over } \mathcal{F}. \tag{B5.32}$$

For ϕ_ℓ, the ℓ^{th} component of ϕ, let

$$\mathscr{G}^{\ell}_{5,F} = \left\{ g : \mathcal{S}_Z \to \mathbb{R} : g(z) = \phi_\ell(z_2) \cdot \frac{(z_1 - \mu_F(m(z_2,\theta)))}{f_m(m(z_2,\theta))} \quad \text{for some } \theta \in \Theta \right\}$$

Let

$$\overline{G}_2(z) \equiv \frac{(|z_1| + \overline{f}_m + \overline{\mu})}{2\underline{f}_m^2} \cdot \|\phi(z_2)\| \cdot \overline{H}_1(z_2),$$

where $\overline{H}_1(\cdot)$ is as described in Assumption SMIM3. By the conditions described there, for any $F \in \mathcal{F}$, we have

$$\left| \phi_\ell(z_2) \cdot \frac{(z_1 - \mu_F(m(z_2,\theta)))}{f_m(m(z_2,\theta))} - \phi_\ell(z_2) \cdot \frac{(z_1 - \mu_F(m(z_2,\theta')))}{f_m(m(z_2,\theta'))} \right| \leq \overline{G}_2(z) \cdot \left\| \theta - \theta' \right\| \quad \forall\, \theta, \theta' \in \Theta.$$

By Assumptions SMIM2 and SMIM3, Lemmas 2.13 and 2.14 in Pakes and Pollard (1989), there exist positive constants A_5 and V_5 such that, for every $F \in \mathcal{F}$, the class $\mathscr{G}^{\ell}_{5,F}$ is Euclidean (A_5, V_5) for the envelope

$$G_5(z) = \frac{1}{\underline{f}_m} \cdot \left\| \phi(z_2) \right\| \cdot \left| (z_1 - \mu_F(m(z_2,\theta_0))) \right| + M_0 \cdot \overline{G}_2(z)$$

where θ_0 is an arbitrary point of Θ and $M_0 \equiv 2\sqrt{k} \sup_\Theta \|\theta - \theta_0\|$. By the conditions in Assumption SMIM3, there exists $\overline{\mu}_{G_5} < \infty$ such that $E_F\left[G_5(Z)^{4q}\right] \leq \overline{\mu}_{G_5}$ for all $F \in \mathcal{F}$. Thus, conditions in Result S1 are satisfied for the integer q described in Assumption SMIM2 and there exists a constant $\overline{M}_4 < \infty$ such that, for all $b > 0$,

$$\sup_{F \in \mathcal{F}} P_F\left(\sup_{\theta \in \Theta} \left\| \frac{1}{n} \sum_{i=1}^{n} \Big(Q_F(Z_i,\theta) - E_F\left[Q_F(Z,\theta)\right] \Big) \right\| > b \right) \leq \frac{\overline{M}_4}{\left(n^{1/2} \cdot b\right)^q} \tag{B5.33}$$

which in turn also implies that

$$\sup_{\theta \in \Theta} \left\| \frac{1}{n} \sum_{i=1}^{n} \Big(Q_F(Z_i,\theta) - E_F\left[Q_F(Z,\theta)\right] \Big) \right\| = O_p\left(\frac{1}{n^{1/2}} \right) \text{ uniformly over } \mathcal{F}. \tag{B5.34}$$

Now, going back to (B5.30), we have

$$\begin{aligned} \sup_{\theta \in \Theta} \left\| \varepsilon_n^{\nu^\mu}(\theta) \right\| \leq & \frac{1}{\sigma_n^d} \cdot \sup_{\theta \in \Theta} \left\| V_{2,n}^{\mu}(\theta) \right\| + \left| \frac{K(0)}{n \cdot \sigma_n^d} \right| \cdot \sup_{\theta \in \Theta} \left\| \frac{1}{n} \sum_{i=1}^{n} \Big(Q_F(Z_i,\theta) - E_F\left[Q_F(Z,\theta)\right] \Big) \right\| \\ & + \overline{\phi} \cdot \sup_{(x,\theta) \in \mathcal{X} \times \Theta} \left| \varepsilon_n^{\mu}(m(x,\theta)) \right| + s_{2,n} \end{aligned}$$

And so, from (B5.21), (B5.32) and (B5.34), we have that, uniformly over $\mathcal{F}$,

$$\sup_{\theta\in\Theta} \left\|\varepsilon_n^{\nu^\mu}(\theta)\right\| = O_p\left(\frac{1}{n\cdot\sigma_n^d}\right) + O_p\left(\frac{1}{n^{3/2}\cdot\sigma_n^d}\right) + O_p\left(\left(\frac{1}{n^{1/2}\cdot\sigma_n^d} + s_{1,n}\right)^2\right) + s_{2,n} \tag{B5.35}$$

Take any $b>0$ and let n_0 be the smallest integer such that

$$s_{2,n} < b \quad \text{and} \quad s_{1,n} < \underbrace{\min\left\{\frac{\underline{f}_m}{2},\, D^{\varepsilon^\mu}\cdot\left(\frac{b-s_{2,n}}{3\overline{\phi}}\right),\, D^{\varepsilon^\mu}\cdot\left(\frac{b-s_{2,n}}{3\overline{\phi}}\right)^{1/4}\right\}}_{=\underline{\varphi}^\mu\left(\frac{b-s_{2,n}}{3\overline{\phi}}\right)\ \text{(see (B5.10))}}$$

Then, from (B5.22), (B5.31) and (B5.33),

$$\begin{aligned}\sup_{F\in\mathcal{F}} P_F\left(\sup_{\theta\in\Theta}\left\|\varepsilon_n^{\nu^\mu}(\theta)\right\| > b\right) \leq & \frac{\overline{M}_3}{\left(n\cdot\sigma_n^d\cdot\left(\frac{b-s_{2,n}}{3}\right)\right)^q} + \frac{\overline{M}_4}{\left(n^{3/2}\cdot\sigma_n^d\cdot\left(\frac{b-s_{2,n}}{3|K(0)|}\right)\right)^q} \\ & + \frac{\overline{M}_1+\overline{M}_2}{\left(n^{1/2}\cdot\sigma_n^d\cdot\left(\underline{\varphi}^\mu\left(\frac{b-s_{2,n}}{3\overline{\phi}}\right) - s_{1,n}\right)\right)^q} \qquad \forall\ n\geq n_0\end{aligned} \tag{B5.36}$$

Equipped with the result in (B5.36), let us go back to the analysis of the estimator $\widehat{\theta}$ described by (B5.2). Recall that we defined

$$\underbrace{\eta_F(m(Z_2,\theta))}_{k\times 1} \equiv E_F\left[\phi(Z_2)|m(Z_2,\theta)\right] = \begin{pmatrix} E_F\left[\phi_1(Z_2)|m(Z_2,\theta)\right] \\ E_F\left[\phi_2(Z_2)|m(Z_2,\theta)\right] \\ \vdots \\ E_F\left[\phi_k(Z_2)|m(Z_2,\theta)\right]\end{pmatrix} \equiv \begin{pmatrix}\eta_{1,F}(m(Z_2,\theta)) \\ \eta_{2,F}(m(Z_2,\theta)) \\ \vdots \\ \eta_{k,F}(m(Z_2,\theta))\end{pmatrix}$$

We add the following smoothness conditions to those described in Assumption SMIM3.

Assumption SMIM4 *For every $F\in\mathcal{F}$, the following Jacobians are well-defined for $F-$a.e $z_2\in\mathcal{S}_{Z_2}$ and everywhere on Θ,*

$$\underbrace{\nabla_\theta\eta_{\ell,F}(m(z_2,\theta))}_{1\times k} \equiv \left(\frac{\partial\eta_{\ell,F}(m(z_2,\theta)'}{\partial\theta_1} \quad \frac{\partial\eta_{\ell,F}(m(z_2,\theta)'}{\partial\theta_2} \quad \cdots \quad \frac{\partial\eta_{\ell,F}(m(z_2,\theta)'}{\partial\theta_d}\right), \quad \ell=1,\ldots,k \tag{B5.37}$$

Let

$$\underbrace{\nabla_\theta \eta_F(m(z_2,\theta)}_{k\times k} \equiv \begin{pmatrix} \nabla_\theta \eta_{1,F}(m(z_2,\theta)) \\ \nabla_\theta \eta_{2,F}(m(z_2,\theta)) \\ \vdots \\ \nabla_\theta \eta_{k,F}(m(z_2,\theta)) \end{pmatrix}$$

and express $\phi(z_2) \equiv (\phi_1(z_2), \phi_2(z_2), \ldots, \phi_k(z_2))' \in \mathbb{R}^k$. *For* $\ell = 1, \ldots, k$, *define*

$$\underbrace{T_{\ell,F}(Z,\theta)}_{1\times k} \equiv \Big(\phi_\ell(Z_2) - \eta_{\ell,F}(m(Z_2,\theta))\Big) \cdot \nabla_\theta \mu_F(m(Z_2,\theta)) + \nabla_\theta \eta_{\ell,F}(m(Z_2,\theta)) \cdot \Big(Z_1 - \mu_F(m(Z_2,\theta))\Big),$$

$$\underbrace{T_F(Z,\theta)}_{k^2\times 1} \equiv \Big(T_{1,F}(Z,\theta) \quad T_{2,F}(Z,\theta) \quad \cdots \quad T_{k,F}(Z,\theta)\Big)',$$

$$\underbrace{\lambda_{\ell,F}(\theta)}_{1\times k} \equiv E_F\left[T_{\ell,F}(Z,\theta)\right],$$

$$\underbrace{\lambda_F(\theta)}_{k^2\times 1} \equiv E\left[T_F(Z,\theta)\right] = \Big(\lambda_{1,F}(\theta) \quad \lambda_{2,F}(\theta) \quad \cdots \quad \lambda_{k,F}(\theta)\Big)'$$

(i) There exists a nonnegative function $\overline{H}_2(\cdot)$ *such that, for each* $F \in \mathcal{F}$,

$$\sup_{\theta\in\Theta} \left\|\nabla_\theta \eta_F(m(z_2,\theta)\right\| \leq \overline{H}_2(z_2) \quad \forall\, z_2 \in \mathcal{Z}_2$$

and there exists $\overline{\mu}_{\overline{H}_6} < \infty$ *such that* $E_F\left[\overline{H}_2(Z_2)^{4q}\right] \leq \overline{\mu}_{\overline{H}_6}$ *for all* $F \in \mathcal{F}$, *where* q *is the integer described in Assumption SMIM1. Note that this condition, combined with Assumptions SMIM1 and SMIM3 imply that there exists a nonnegative function* $\overline{G}_6(\cdot)$ *such that, for all* $F \in \mathcal{F}$,

$$\left\|T_F(z,\theta) - T_F(z,\theta')\right\| \leq \overline{G}_6(z) \cdot \left\|\theta - \theta'\right\| \quad \forall\, z \in \mathcal{S}_Z \quad \text{and} \quad \theta,\theta' \in \Theta,$$

and there exists $\overline{\mu}_{\overline{G}_6} < \infty$ *such that* $E_F\left[\overline{G}_6(Z)^{4q}\right] \leq \overline{\mu}_{\overline{G}_6}$ $\forall\, F \in \mathcal{F}$, *where* q *is the integer described in Assumption SMIM1.*

(ii) Let H_k *and* M_k *be as defined in* (B1.1). *Assume that* $\exists\, \underline{d} > 0$, $\overline{M}_\lambda$, $K_5 > 0$, $K_6 > 0$ *and* $\alpha_1 > 0$ *such that, for every* $F \in \mathcal{F}$,

$$\begin{aligned} &\inf_{\theta\in\Theta}\left|det(H_k(\lambda_F(\theta)))\right| \geq \underline{d} \quad \sup_{\theta\in\Theta}\left\|M_k(\lambda_F(\theta))\right\| \leq \overline{M}_\lambda \\ &\left\|M_k(\lambda_F(\theta)) - M_k(v)\right\| \leq K_6 \cdot \|\lambda_F(\theta) - v\|^{\alpha_1} \quad \forall\, v,\theta : \|v - \lambda_F(\theta)\| \leq K_5,\ \theta \in \Theta. \end{aligned} \tag{B5.38}$$

And,

$$\sup_{\substack{v:\|v-\lambda_F(\theta)\|\leq K_5\\ \theta\in\Theta}}\left\{\left\|M_k(\lambda_F(\theta))-M_k(v)\right\|\right\}\leq K_7<\infty$$

(iii) $\exists\ K_8>0$, $K_9>0$ *and* $\alpha_2>0$ *such that, for every* $F\in\mathcal{F}$,

$$\left\|\lambda_F(\theta)-\lambda_F(\theta^*)\right\|\leq K_9\cdot\|\theta-\theta^*\|^{\alpha_2}\quad\forall\ \theta:\|\theta-\theta^*\|\leq K_8$$

Result SMIM *Define*

$$\zeta_F(Z_i)\equiv\Big(\phi(Z_{2i})-\eta_F\Big(m(Z_{2i},\theta^*)\Big)\Big)\cdot\Big(Z_{1i}-\mu_F\Big(m(Z_{2i},\theta^*)\Big)\Big),$$
$$\psi_F^{\theta}(Z_i)\equiv M_k\left(\lambda_F(\theta^*)\right)\cdot\zeta_F(Z_i).$$

Note that $E_F[\zeta_F(Z)]=E_F[\psi_F^{\theta}(Z)]=0$. *Under Assumptions SMIM1-SMIM4, the estimator defined by* (B5.2) *satisfies,*

$$\widehat{\theta}=\theta^*+\frac{1}{n}\sum_{i=1}^{n}\psi_F^{\theta}(Z_i)+\varepsilon_n^{\theta},$$

and the conditions in Assumption 1 of the paper are satisfied, with $\psi_F^{\theta}(Z_i)=M_k\left(\lambda_F(\theta^*)\right)\cdot\zeta_F(Z_i)$, $r_n=n^{1/2}\cdot\sigma_n^d$, *and for any* τ *and* $\overline{\delta}$ *such that* $0<\tau<\min\left\{\left(\frac{\alpha_1}{2}\right),\ (\alpha_1\cdot\alpha_2\cdot\Delta),\ \Delta\right\}$ *and* $0<\overline{\delta}<q\Delta-\frac{1}{2}$.

Proof: Let us go back to (B5.2), which defines the estimator $\widehat{\theta}$ by the sample-analog moment condition $\frac{1}{n}\sum_{i=1}^{n}\phi(Z_{2i})\cdot\Big(Z_{1i}-\widehat{\mu}(m(Z_{2i},\widehat{\theta}))\Big)=0$. From here we obtain,

$$\begin{aligned}0=&\frac{1}{n}\sum_{i=1}^{n}\phi(Z_{2i})\cdot\Big(Z_{1i}-\mu_F\Big(m(Z_{2i},\theta^*)\Big)\Big)+\frac{1}{n}\sum_{i=1}^{n}\phi(Z_{2i})\cdot\Big(\mu_F\Big(m(Z_{2i},\theta^*)\Big)-\mu_F\Big(m(Z_{2i},\widehat{\theta})\Big)\Big)\\&+\underbrace{\frac{1}{n}\sum_{i=1}^{n}\phi(Z_{2i})\cdot\Big(\mu_F\Big(m(Z_{2i},\widehat{\theta})\Big)-\widehat{\mu}\Big(m(Z_{2i},\widehat{\theta})\Big)\Big)}_{-\nu_n^{\mu}(\widehat{\theta})\text{ (see (B5.23))}}\end{aligned}\tag{B5.39}$$

From our result in (B5.28), we have

$$\nu_n^{\mu}(\widehat{\theta})=\frac{1}{n}\sum_{i=1}^{n}\eta_F(m(Z_{2i},\widehat{\theta}))\cdot\Big(Z_{1i}-\mu_F\Big(m(Z_{2i},\widehat{\theta})\Big)\Big)+\varepsilon_n^{\nu^{\mu}}(\widehat{\theta}).$$

Thus, (B5.39) becomes

$$
\begin{aligned}
0 = & \frac{1}{n}\sum_{i=1}^{n}\phi(Z_{2i})\cdot\Big(Z_{1i}-\mu_F\Big(m(Z_{2i},\theta^*)\Big)\Big)+\frac{1}{n}\sum_{i=1}^{n}\phi(Z_{2i})\cdot\mu_F\Big(m(Z_{2i},\theta^*)\Big)\\
& -\frac{1}{n}\sum_{i=1}^{n}\Big[\phi(Z_{2i})\cdot\mu_F\Big(m(Z_{2i},\widehat{\theta})\Big)+\eta_F\Big(m(Z_{2i},\widehat{\theta})\Big)\cdot\Big(Z_{1i}-\mu_F\Big(m(Z_{2i},\widehat{\theta})\Big)\Big)\Big]\\
& -\varepsilon_n^{\nu^\mu}(\widehat{\theta})
\end{aligned}
$$

And from here, using the Jacobians defined in (B5.37) and the Mean Value Theorem, the previous expression becomes,

$$
\begin{aligned}
0 = & \frac{1}{n}\sum_{i=1}^{n}\Big(\phi(Z_{2i})-\eta_F\Big(m(Z_{2i},\theta^*)\Big)\Big)\cdot\Big(Z_{1i}-\mu_F\Big(m(Z_{2i},\theta^*)\Big)\Big)\\
& -\Big\{\frac{1}{n}\sum_{i=1}^{n}\Big[\Big(\phi(Z_{2i})-\eta_F\Big(m(Z_{2i},\overline{\theta})\Big)\Big)\cdot\nabla_\theta\mu_F\Big(m(Z_{2i},\overline{\theta})\Big)+\nabla_\theta\eta_F\Big(m(Z_{2i},\overline{\theta})\Big)\cdot\Big(Z_{1i}-\mu_F\Big(m(Z_{2i},\overline{\theta})\Big)\Big)\Big]\Big\}\\
& \quad\times\Big(\widehat{\theta}-\theta^*\Big)\\
& -\varepsilon_n^{\nu^\mu}(\widehat{\theta})
\end{aligned}
\tag{B5.40}
$$

where $\overline{\theta}$ belongs in the line segment connecting $\widehat{\theta}$ and θ^* (thus $\overline{\theta}\in\Theta$). Let $T_{\ell,F}$ and T_F be as defined in Assumption SMIM4 and let

$$
\begin{aligned}
\underbrace{\overline{T}_\ell(\theta)}_{1\times k} & \equiv \frac{1}{n}\sum_{i=1}^{n}T_{\ell,F}(Z_i,\theta),\\
\underbrace{\overline{T}(\theta)}_{k^2\times 1} & \equiv \frac{1}{n}\sum_{i=1}^{n}T_F(Z_i,\theta)=\Big(\overline{T}_1(\theta)\quad\overline{T}_2(\theta)\quad\cdots\quad\overline{T}_k(\theta)\Big)'
\end{aligned}
$$

Using our definition of M_k in (B1.1), the expression in (B5.40) yields,

$$
\begin{aligned}
\widehat{\theta} = & \theta^*+M_k\Big(\overline{T}(\overline{\theta})\Big)\cdot\frac{1}{n}\sum_{i=1}^{n}\Big(\phi(Z_{2i})-\eta_F\Big(m(Z_{2i},\theta^*)\Big)\Big)\cdot\Big(Z_{1i}-\mu_F\Big(m(Z_{2i},\theta^*)\Big)\Big)\\
& -M_k\Big(\overline{T}(\overline{\theta})\Big)\cdot\varepsilon_n^{\nu^\mu}(\widehat{\theta})
\end{aligned}
\tag{B5.41}
$$

Define

$$
\begin{aligned}
\zeta_F(Z_i) & \equiv\Big(\phi(Z_{2i})-\eta_F\Big(m(Z_{2i},\theta^*)\Big)\Big)\cdot\Big(Z_{1i}-\mu_F\Big(m(Z_{2i},\theta^*)\Big)\Big),\\
\psi_F^\theta(Z_i) & \equiv M_k\left(\lambda_F(\theta^*)\right)\cdot\zeta_F(Z_i).
\end{aligned}
\tag{B5.42}
$$

Note that $E_F[\zeta_F(Z)] = E_F[\psi_F^\theta(Z)] = 0$. We can re-express (B5.41) as,

$$
\begin{aligned}
\widehat{\theta} &= \theta^* + \frac{1}{n}\sum_{i=1}^{n} \psi_F^\theta(Z_i) + \varepsilon_n^\theta, \quad \text{where} \\
\varepsilon_n^\theta &\equiv \left(M_k\left(\overline{T}(\overline{\theta})\right) - M_k\left(\lambda_F(\overline{\theta})\right)\right) \cdot \frac{1}{n}\sum_{i=1}^{n} \zeta_F(Z_i) \\
&\quad + \left(M_k\left(\lambda_F(\overline{\theta})\right) - M_k\left(\lambda_F(\theta^*)\right)\right) \cdot \frac{1}{n}\sum_{i=1}^{n} \zeta_F(Z_i) \\
&\quad - \left(M_k\left(\overline{T}(\overline{\theta})\right) - M_k\left(\lambda_F(\overline{\theta})\right)\right) \cdot \varepsilon_n^{\nu^\mu}(\widehat{\theta}) \\
&\quad - M_k\left(\lambda_F(\overline{\theta})\right) \cdot \varepsilon_n^{\nu^\mu}(\widehat{\theta})
\end{aligned}
\tag{B5.43}
$$

Recall from the conditions in (B5.38) that $\sup_{\theta\in\Theta}\left\|M_k(\lambda_F(\theta))\right\| \leq \overline{M}_\lambda$. Therefore,

$$
\begin{aligned}
\left\|\varepsilon_n^\theta\right\| \leq{}& \left\|M_k\left(\overline{T}(\overline{\theta})\right) - M_k\left(\lambda_F(\overline{\theta})\right)\right\| \cdot \left\|\frac{1}{n}\sum_{i=1}^{n} \zeta_F(Z_i)\right\| + 2\overline{M}_\lambda \cdot \left\|\frac{1}{n}\sum_{i=1}^{n} \zeta_F(Z_i)\right\| \\
&+ \left\|M_k\left(\overline{T}(\overline{\theta})\right) - M_k\left(\lambda_F(\overline{\theta})\right)\right\| \cdot \left\|\varepsilon_n^{\nu^\mu}(\widehat{\theta})\right\| + \overline{M}_\lambda \cdot \left\|\varepsilon_n^{\nu^\mu}(\widehat{\theta})\right\|
\end{aligned}
$$

Therefore, for any $c > 0$,

$$
\begin{aligned}
\mathbb{1}\left\{\left\|\varepsilon_n^\theta\right\| \geq c\right\} \leq{}& \mathbb{1}\left\{\left\|M_k\left(\overline{T}(\overline{\theta})\right) - M_k\left(\lambda_F(\overline{\theta})\right)\right\| \cdot \left\|\frac{1}{n}\sum_{i=1}^{n} \zeta_F(Z_i)\right\| \geq \frac{c}{4}\right\} \\
&+ \mathbb{1}\left\{2\overline{M}_\lambda \cdot \left\|\frac{1}{n}\sum_{i=1}^{n} \zeta_F(Z_i)\right\| \geq \frac{c}{4}\right\} \\
&+ \mathbb{1}\left\{\left\|M_k\left(\overline{T}(\overline{\theta})\right) - M_k\left(\lambda_F(\overline{\theta})\right)\right\| \cdot \left\|\varepsilon_n^{\nu^\mu}(\widehat{\theta})\right\| \geq \frac{c}{4}\right\} \\
&+ \mathbb{1}\left\{\overline{M}_\lambda \cdot \left\|\varepsilon_n^{\nu^\mu}(\widehat{\theta})\right\| \geq \frac{c}{4}\right\}
\end{aligned}
\tag{B5.44}
$$

Let K_5 and K_7 be as described in (B5.38) and note that by the conditions described there,

$$\begin{aligned}
&\mathbb{1}\left\{\left\|M_k\left(\overline{T}(\overline{\theta})\right)-M_k\left(\lambda_F(\overline{\theta})\right)\right\|\cdot\left\|\frac{1}{n}\sum_{i=1}^n\zeta_F(Z_i)\right\|\geq\frac{c}{4}\right\}\\
&=\underbrace{\mathbb{1}\left\{\left\|M_k\left(\overline{T}(\overline{\theta})\right)-M_k\left(\lambda_F(\overline{\theta})\right)\right\|\cdot\left\|\frac{1}{n}\sum_{i=1}^n\zeta_F(Z_i)\right\|\geq\frac{c}{4}\right\}\times\mathbb{1}\left\{\left\|M_k\left(\overline{T}(\overline{\theta})\right)-M_k\left(\lambda_F(\overline{\theta})\right)\right\|\leq K_7\right\}}_{\leq\mathbb{1}\left\{\left\|\frac{1}{n}\sum_{i=1}^n\zeta_F(Z_i)\right\|\geq\frac{c}{4K_7}\right\}}\\
&+\underbrace{\mathbb{1}\left\{\left\|M_k\left(\overline{T}(\overline{\theta})\right)-M_k\left(\lambda_F(\overline{\theta})\right)\right\|\cdot\left\|\frac{1}{n}\sum_{i=1}^n\zeta_F(Z_i)\right\|\geq\frac{c}{4}\right\}\times\mathbb{1}\left\{\left\|M_k\left(\overline{T}(\overline{\theta})\right)-M_k\left(\lambda_F(\overline{\theta})\right)\right\|>K_7\right\}}_{\leq\mathbb{1}\left\{\left\|\overline{T}(\overline{\theta})-\lambda_F(\overline{\theta})\right\|\geq K_5\right\}}\\
&\leq\mathbb{1}\left\{\left\|\frac{1}{n}\sum_{i=1}^n\zeta_F(Z_i)\right\|\geq\frac{c}{4K_7}\right\}+\mathbb{1}\left\{\left\|\overline{T}(\overline{\theta})-\lambda_F(\overline{\theta})\right\|\geq K_5\right\}.
\end{aligned}\tag{B5.45A}$$

Similarly,

$$\begin{aligned}
&\mathbb{1}\left\{\left\|M_k\left(\overline{T}(\overline{\theta})\right)-M_k\left(\lambda_F(\overline{\theta})\right)\right\|\cdot\left\|\varepsilon_n^{\nu^\mu}(\widehat{\theta})\right\|\geq\frac{c}{4}\right\}\\
&=\underbrace{\mathbb{1}\left\{\left\|M_k\left(\overline{T}(\overline{\theta})\right)-M_k\left(\lambda_F(\overline{\theta})\right)\right\|\cdot\left\|\varepsilon_n^{\nu^\mu}(\widehat{\theta})\right\|\geq\frac{c}{4}\right\}\times\mathbb{1}\left\{\left\|M_k\left(\overline{T}(\overline{\theta})\right)-M_k\left(\lambda_F(\overline{\theta})\right)\right\|\leq K_7\right\}}_{\leq\mathbb{1}\left\{\left\|\varepsilon_n^{\nu^\mu}(\widehat{\theta})\right\|\geq\frac{c}{4K_7}\right\}}\\
&+\underbrace{\mathbb{1}\left\{\left\|M_k\left(\overline{T}(\overline{\theta})\right)-M_k\left(\lambda_F(\overline{\theta})\right)\right\|\cdot\left\|\varepsilon_n^{\nu^\mu}(\widehat{\theta})\right\|\geq\frac{c}{4}\right\}\times\mathbb{1}\left\{\left\|M_k\left(\overline{T}(\overline{\theta})\right)-M_k\left(\lambda_F(\overline{\theta})\right)\right\|>K_7\right\}}_{\leq\mathbb{1}\left\{\left\|\overline{T}(\overline{\theta})-\lambda_F(\overline{\theta})\right\|\geq K_5\right\}}\\
&\leq\mathbb{1}\left\{\left\|\varepsilon_n^{\nu^\mu}(\widehat{\theta})\right\|\geq\frac{c}{4K_7}\right\}+\mathbb{1}\left\{\left\|\overline{T}(\overline{\theta})-\lambda_F(\overline{\theta})\right\|\geq K_5\right\}.
\end{aligned}\tag{B5.45B}$$

Combining (B5.45A) and (B5.45B) with (B5.44), for any $c>0$ we have

$$\begin{aligned}
\mathbb{1}\left\{\left\|\varepsilon_n^{\theta}\right\|\geq c\right\}\leq&\ \mathbb{1}\left\{\left\|\frac{1}{n}\sum_{i=1}^n\zeta_F(Z_i)\right\|\geq\left(K_5\wedge\left(\frac{1}{8\overline{M}_\lambda}\wedge\frac{1}{4K_7}\right)\cdot c\right)\right\}\\
&+\mathbb{1}\left\{\sup_{\theta\in\Theta}\left\|\frac{1}{n}\sum_{i=1}^n\Big(T_F(Z_i,\theta)-E_F\left[T_F(Z,\theta)\right]\Big)\right\|\geq\left(K_5\wedge\left(\frac{1}{8\overline{M}_\lambda}\wedge\frac{1}{4K_7}\right)\cdot c\right)\right\}\\
&+\mathbb{1}\left\{\sup_{\theta\in\Theta}\left\|\varepsilon_n^{\nu^\mu}(\theta)\right\|\geq\left(K_5\wedge\left(\frac{1}{8\overline{M}_\lambda}\wedge\frac{1}{4K_7}\right)\cdot c\right)\right\}
\end{aligned}\tag{B5.46}$$

Take the ℓ^{th} component $(T_F^\ell(Z,\theta))$ of $T_F(Z,\theta)$ let

$$\mathscr{G}_{6,F}^{\ell} = \left\{ g : \mathcal{S}_Z \to \mathbb{R} : g(z) = T_F^\ell(z,\theta) \;\text{ for some } \theta \in \Theta \right\}$$

By Assumptions SMIM2, SMIM3 and SMIM4, and Lemmas 2.13 and 2.14 in Pakes and Pollard (1989), there exist positive constants A_6 and V_6 such that, for every $F \in \mathcal{F}$, the class $\mathscr{G}_{6,F}^{\ell}$ is Euclidean (A_6, V_6) for an envelope $G_6(z)$ for which $\exists\ \overline{\mu}_{G_6} < \infty$ such that $E_F\left[G_6(Z)^{4q}\right] \leq \overline{\mu}_{G_6}$ (where q is the integer described in Assumption SMIM1). The conditions in Result S1 are satisfied for the integer q described in Assumption SMIM2 and there exists a constant $\overline{M}_5 < \infty$ such that, for all $b > 0$,

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{\theta\in\Theta}\left\|\frac{1}{n}\sum_{i=1}^{n}\Big(T_F(Z_i,\theta) - E_F\left[T_F(Z,\theta)\right]\Big)\right\| > b\right) \leq \frac{\overline{M}_5}{\left(n^{1/2}\cdot b\right)^q} \tag{B5.47}$$

Next, note from Assumption SMIM1 that there exists $\overline{\mu}_\zeta < \infty$ such that $E_F\left[\zeta_F(Z)^{4q}\right] \leq \overline{\mu}_\zeta$ for all $F \in \mathcal{F}$. From here, a straightforward Chebyshev inequality implies that there exists a constant $\overline{M}_6$ such that, for any $b > 0$,

$$\sup_{F\in\mathcal{F}} P_F\left(\left\|\frac{1}{n}\sum_{i=1}^{n}\zeta_F(Z_i)\right\| \geq b\right) \leq \frac{\overline{M}_6}{\left(n^{1/2}\cdot b\right)^q} \tag{B5.48}$$

in both instances (B5.47 and B5.48), q is the integer described in Assumption SMIM1. Take any $c > 0$ and let n_0 be the smallest integer such that

$$s_{2,n} < \left(K_5 \wedge \left(\frac{1}{8\overline{M}_\lambda} \wedge \frac{1}{4K_7}\right)\cdot c\right) \quad \text{and}$$

$$s_{1,n} < \underbrace{\min\left\{\frac{\underline{f}_m}{2}\,,\, D^{\varepsilon^\mu}\cdot\left(\frac{\left(K_5 \wedge \left(\frac{1}{8\overline{M}_\lambda} \wedge \frac{1}{4K_7}\right)\cdot c\right) - s_{2,n}}{3\overline{\phi}}\right)\,,\, D^{\varepsilon^\mu}\cdot\left(\frac{\left(K_5 \wedge \left(\frac{1}{8\overline{M}_\lambda} \wedge \frac{1}{4K_7}\right)\cdot c\right) - s_{2,n}}{3\overline{\phi}}\right)^{1/4}\right\}}_{=\underline{\varphi}^\mu\left(\left(K_5\wedge\left(\frac{1}{8\overline{M}_\lambda}\wedge\frac{1}{4K_7}\right)\cdot c\right)-s_{2,n}\right)\ \text{(see (B5.10))}}$$

From (B5.36), (B5.47) and (B5.48), the inequality in (B5.46) implies

$$\sup_{F\in\mathcal{F}} P_F\left(\left\|\varepsilon_n^{\theta}\right\| \geq c\right) \leq \frac{\overline{M}_5 + \overline{M}_6}{\left(n^{1/2}\cdot\left(\left(K_5 \wedge \left(\frac{1}{8\overline{M}_\lambda} \wedge \frac{1}{4K_7}\right)\cdot c\right)\right)\right)^q}$$

$$+ \frac{\overline{M}_3}{\left(n\cdot\sigma_n^d\cdot\left(\frac{\left(K_5\wedge\left(\frac{1}{8\overline{M}_\lambda}\wedge\frac{1}{4K_7}\right)\cdot c\right)-s_{2,n}}{3}\right)\right)^q} + \frac{\overline{M}_4}{\left(n^{3/2}\cdot\sigma_n^d\cdot\left(\frac{\left(K_5\wedge\left(\frac{1}{8\overline{M}_\lambda}\wedge\frac{1}{4K_7}\right)\cdot c\right)-s_{2,n}}{3|K(0)|}\right)\right)^q}$$

$$+ \frac{\overline{M}_1 + \overline{M}_2}{\left(n^{1/2}\cdot\sigma_n^d\cdot\left(\underline{\varphi}^{\mu}\left(\frac{\left(K_5\wedge\left(\frac{1}{8\overline{M}_\lambda}\wedge\frac{1}{4K_7}\right)\cdot c\right)-s_{2,n}}{3\overline{\phi}}\right)-s_{1,n}\right)\right)^q} \quad \forall\ n \geq n_0$$

We can obtain a simplified bound from the previous expression. Take any positive constants A_1, A_2 such that

$$A_1 \leq \frac{1}{3}\cdot\min\left\{\frac{1}{\overline{\phi}}, \frac{1}{|K(0)|}, 1\right\}\times K_5, \quad A_2 \leq \frac{1}{3}\cdot\min\left\{\frac{1}{\overline{\phi}}, \frac{1}{|K(0)|}, 1\right\}\left(\frac{1}{8\overline{M}_\lambda} \wedge \frac{1}{4K_7}\right),$$

and let $M^{SE} \equiv \sum_{j=1}^{7}\overline{M}_j$. Take any positive sequence $c > 0$ and let n_0 be the smallest integer such that

$$s_{2,n} < A_1 \wedge A_2\cdot c, \quad \text{and} \quad s_{1,n} < \min\left\{A_1 - s_{2,n}\,, \left(A_2\cdot c - s_{2,n}\right), \left(A_2\cdot c - s_{2,n}\right)^{1/4}\right\}$$

Let

$$\Lambda_n^{SE}(c) \equiv n^{1/2}\cdot\sigma_n^d\cdot\left(\min\left\{A_1 - s_{2,n}\,, \left(A_2\cdot c - s_{2,n}\right), \left(A_2\cdot c - s_{2,n}\right)^{1/4}\right\} - s_{1,n}\right).$$

Then,

$$\sup_{F\in\mathcal{F}} P_F\left(\left\|\varepsilon_n^{\theta}\right\| \geq c\right) \leq \frac{M^{SE}}{\left(\Lambda_n^{SE}(c)\right)^q} \quad \forall\ n \geq n_0 \tag{B5.49}$$

where q is the integer described in Assumption SMIM1. Recall from Assumption SMIM1 that the bandwidth sequence $\sigma_n \longrightarrow 0$ satisfies $n^{1/2+\Delta}\cdot\sigma_n^L \longrightarrow 0$ and $n^{1/2-\Delta}\cdot\sigma_n^d \longrightarrow \infty$ for some $0 < \Delta < 1/2$. Next recall that the sequences $s_{1,n}$ and $s_{2,n}$ are defined in (B5.17) and (B5.29) as $s_{1,n} \equiv C_B^{\mu_a}\cdot\sigma_n^L$ and $s_{2,n} \equiv \frac{|K(0)|\cdot C_Q}{n\cdot\sigma_n^d} + C_B^{\mu_b}\cdot\sigma_n^L$. Therefore, $n^{1/2+\Delta}\cdot s_{1,n} \longrightarrow 0$ and $n^{1/2+\Delta}\cdot s_{2,n} \longrightarrow 0$ and $\Lambda_n(c) \longrightarrow \infty$ for all $c > 0$, and thus from (B5.49) we have

$$\sup_{F\in\mathcal{F}} P_F\left(\left\|\varepsilon_n^{\theta}\right\| \geq c\right) = O\left(\frac{1}{\left(n^{1/2}\cdot\sigma_n^d\right)^q}\right) \quad \forall\ c > 0$$

Furthermore, recall that Assumption SMIM1 states that Δ and q are such that $q\Delta > 1/2$. From here we have that for any $0 < \delta < q\Delta - \frac{1}{2}$, we have

$$\sup_{F\in\mathcal{F}} P_F\left(\left\|\varepsilon_n^\theta\right\| \geq c\right) = o\left(\frac{1}{n^{1/2+\delta}}\right) \tag{B5.50}$$

More generally, suppose c_n is a sequence such that $n^{1/2-\Delta} \cdot \sigma_n^d \cdot c_n \longrightarrow \infty$. Then, the result in (B5.50) would still hold for c_n. Thus,

$$\sup_{F\in\mathcal{F}} P_F\left(\left\|\varepsilon_n^\theta\right\| \geq c_n\right) = o\left(\frac{1}{n^{1/2+\delta}}\right) \quad \forall\ c_n : n^{1/2-\Delta} \cdot \sigma_n^d \cdot c_n \longrightarrow \infty, \text{ and } 0 < \delta < q\Delta - \frac{1}{2}$$

Next, by Assumptions SMIM1 and SMIM4, there exists $\overline{\mu}_\psi < \infty$ such that $E_F\left[\left(\psi_F^\theta(Z)\right)^{4q}\right] \leq \overline{\mu}_\psi$ for all $F \in \mathcal{F}$. From here, a straightforward Chebyshev inequality implies that there exists a constant $\overline{M}_8$ such that, for any $b > 0$,

$$\sup_{F\in\mathcal{F}} P_F\left(\left\|\frac{1}{n}\sum_{i=1}^n \psi_F^\theta(Z_i)\right\| \geq b\right) \leq \frac{\overline{M}_8}{\left(n^{1/2} \cdot b\right)^q}$$

Thus, going back to the linear representation in (B5.43), we have that for any $c > 0$ there exists n_0 such that

$$\begin{aligned}\sup_{F\in\mathcal{F}} P_F\left(\left\|\widehat{\theta} - \theta^*\right\| \geq c\right) &\leq \sup_{F\in\mathcal{F}} P_F\left(\left\|\frac{1}{n}\sum_{i=1}^n \psi_F^\theta(Z_i)\right\| \geq \frac{c}{2}\right) + \sup_{F\in\mathcal{F}} P_F\left(\left\|\varepsilon_n^\theta\right\| \geq \frac{c}{2}\right) \\ &\leq \frac{\overline{M}_8}{\left(n^{1/2} \cdot c/2\right)^q} + \frac{M^{SE}}{\left(\Lambda_n^{SE}(c/2)\right)^q} \quad \forall\ n \geq n_0.\end{aligned} \tag{B5.51}$$

And so,

$$\sup_{F\in\mathcal{F}} P_F\left(\left\|\widehat{\theta} - \theta^*\right\| \geq c\right) \longrightarrow 0 \quad \forall\ c > 0$$

Thus, $\left\|\widehat{\theta} - \theta^*\right\| = o_p(1)$ uniformly over $\mathcal{F}$. Equipped with the previous expression we can obtain a more precise asymptotic result for ε_n^θ. Recall from (B5.43) that it is defined as

$$\begin{aligned}\varepsilon_n^\theta \equiv &\left(M_k\left(\overline{T}(\overline{\theta})\right) - M_k\left(\lambda_F(\overline{\theta})\right)\right) \cdot \frac{1}{n}\sum_{i=1}^n \zeta_F(Z_i) + \left(M_k\left(\lambda_F(\overline{\theta})\right) - M_k\left(\lambda_F(\theta^*)\right)\right) \cdot \frac{1}{n}\sum_{i=1}^n \zeta_F(Z_i) \\ &- \left(M_k\left(\overline{T}(\overline{\theta})\right) - M_k\left(\lambda_F(\overline{\theta})\right)\right) \cdot \varepsilon_n^{\nu^\mu}(\widehat{\theta}) - M_k\left(\lambda_F(\overline{\theta})\right) \cdot \varepsilon_n^{\nu^\mu}(\widehat{\theta})\end{aligned}$$

Take any $c > 0$. By Assumption SMIM4,

$$\mathbb{1}\left\{\left\|M_k\Big(\overline{T}(\overline{\theta})\Big) - M_k\Big(\lambda_F(\overline{\theta})\Big)\right\| \geq c\right\} \leq \max\left(\mathbb{1}\left\{K_6 \cdot \left\|\overline{T}(\overline{\theta}) - \lambda_F(\overline{\theta})\right\|^{\alpha_1} \geq c\right\}\,,\, \mathbb{1}\left\{\left\|\overline{T}(\overline{\theta}) - \lambda_F(\overline{\theta})\right\| \geq K_5\right\}\right)$$
$$\leq \mathbb{1}\left\{\left\|\overline{T}(\overline{\theta}) - \lambda_F(\overline{\theta})\right\| \geq K_5 \wedge \left(\frac{c}{K_6}\right)^{1/\alpha_1}\right\}$$

Thus, from (B5.47), for any $c > 0$ there exists n_0 such that

$$\sup_{F\in\mathcal{F}} P_F\left(\left\|M_k\Big(\overline{T}(\overline{\theta})\Big) - M_k\Big(\lambda_F(\overline{\theta})\Big)\right\| \geq c\right) \leq \sup_{F\in\mathcal{F}} P_F\left(\sup_{\theta\in\Theta}\left\|\frac{1}{n}\sum_{i=1}^{n}\Big(T_F(Z_i,\theta) - E_F\left[T_F(Z,\theta)\right]\Big)\right\| \geq K_5 \wedge \left(\frac{c}{K_6}\right)^{1/\alpha_1}\right)$$
$$\leq \frac{\overline{M}_5}{\left(n^{1/2}\cdot\left[K_5 \wedge \left(\frac{c}{K_6}\right)^{1/\alpha_1}\right]\right)^q} \quad \forall\, n \geq n_0$$

Take any $c > 0$ and $\tau > 0$. Then, there exists n_* such that $K_5 \wedge \left(\frac{n^{-\tau}\cdot c}{K_6}\right)^{1/\alpha_1} = \left(\frac{n^{-\tau}\cdot c}{K_6}\right)^{1/\alpha_1}$ for all $n \geq n_*$. Therefore, there exists n_0 such that

$$\sup_{F\in\mathcal{F}} P_F\left(\left\|M_k\Big(\overline{T}(\overline{\theta})\Big) - M_k\Big(\lambda_F(\overline{\theta})\Big)\right\| \geq n^{-\tau}\cdot c\right) \leq \frac{\overline{M}_5}{\left(n^{1/2-\tau/\alpha_1}\cdot (c/K_6)^{1/\alpha_1}\right)^q} \quad \forall\, n \geq n_0$$

Therefore,

$$\sup_{F\in\mathcal{F}} P_F\left(\left\|M_k\Big(\overline{T}(\overline{\theta})\Big) - M_k\Big(\lambda_F(\overline{\theta})\Big)\right\| \geq n^{-\tau}\cdot c\right) \longrightarrow 0 \quad \forall\, c > 0,\, \tau < \frac{\alpha_1}{2}$$

which means,

$$\left\|M_k\Big(\overline{T}(\overline{\theta})\Big) - M_k\Big(\lambda_F(\overline{\theta})\Big)\right\| = o_p\left(n^{-\tau}\right) \quad \forall\, \tau < \frac{\alpha_1}{2}, \quad \text{uniformly over } \mathcal{F}. \tag{B5.52}$$

Take any $c > 0$. Once again from Assumption SMIM4, we have

$$\mathbb{1}\left\{\left\|M_k\left(\lambda_F(\overline{\theta})\right) - M_k\left(\lambda_F(\theta^*)\right)\right\| \geq c\right\} \leq \mathbb{1}\left\{\left\|\lambda_F(\overline{\theta}) - \lambda_F(\theta^*)\right\| \geq K_5 \wedge \left(\frac{c}{K_6}\right)^{1/\alpha_1}\right\}$$
$$\leq \max\left(\mathbb{1}\left\{K_9\cdot\left\|\overline{\theta} - \theta^*\right\|^{\alpha_2} \geq K_5 \wedge \left(\frac{c}{K_6}\right)^{1/\alpha_1}\right\}\,,\, \mathbb{1}\left\{\left\|\overline{\theta} - \theta^*\right\| \geq K_8\right\}\right)$$
$$\leq \mathbb{1}\left\{\left\|\overline{\theta} - \theta^*\right\| \geq \left[\frac{1}{K_9}\cdot\left(K_5 \wedge \left(\frac{c}{K_6}\right)^{1/\alpha_1}\right)\right]^{1/\alpha_2} \wedge K_8\right\}$$

Recall that $\overline{\theta}$ belongs in the line segment connecting $\widehat{\theta}$ and θ^* and therefore $\|\overline{\theta} - \theta^*\| \leq \|\widehat{\theta} - \theta^*\|$.

Thus, from the above result and (B5.51) we have that for each $c > 0$, there exists n_0 such that

$$\begin{aligned}
&\sup_{F\in\mathcal{F}} P_F\left(\left\|M_k\left(\lambda_F(\overline{\theta})\right) - M_k\left(\lambda_F(\theta^*)\right)\right\| \geq c\right) \\
&\leq \sup_{F\in\mathcal{F}} P_F\left(\left\|\widehat{\theta} - \theta^*\right\| \geq \left[\frac{1}{K_9}\cdot\left(K_5 \wedge \left(\frac{c}{K_6}\right)^{1/\alpha_1}\right)\right]^{1/\alpha_2} \wedge K_8\right) \\
&\leq \frac{\overline{M}_8}{\left(n^{1/2}\cdot\frac{1}{2}\cdot\left(\left[\frac{1}{K_9}\cdot\left(K_5\wedge\left(\frac{c}{K_6}\right)^{1/\alpha_1}\right)\right]^{1/\alpha_2}\wedge K_8\right)\right)^q} + \frac{M^{SE}}{\left(\Lambda_n^{SE}\left(\frac{1}{2}\cdot\left(\left[\frac{1}{K_9}\cdot\left(K_5\wedge\left(\frac{c}{K_6}\right)^{1/\alpha_1}\right)\right]^{1/\alpha_2}\wedge K_8\right)\right)\right)^q} \\
&\quad \forall\ n \geq n_0.
\end{aligned}$$

Recall that

$$\Lambda_n^{SE}(c) \equiv n^{1/2}\cdot\sigma_n^d\cdot\left(\min\left\{A_1 - s_{2,n}\ ,\ \left(A_2\cdot c - s_{2,n}\right), \left(A_2\cdot c - s_{2,n}\right)^{1/4}\right\} - s_{1,n}\right).$$

And recall from Assumption SMIM1 that the bandwidth sequence $\sigma_n \longrightarrow 0$ satisfies $n^{1/2+\Delta}\cdot\sigma_n^L \longrightarrow 0$ and $n^{1/2-\Delta}\cdot\sigma_n^d \longrightarrow \infty$ for some $0 < \Delta < 1/2$. Take any $c > 0$ and $\tau > 0$. Then, there exists n_* such that

$$\begin{aligned}
&\frac{1}{2}\cdot\left(\left[\frac{1}{K_9}\cdot\left(K_5\wedge\left(\frac{c}{K_6}\right)^{1/\alpha_1}\right)\right]^{1/\alpha_2}\wedge K_8\right) = \frac{1}{2\cdot K_9^{1/\alpha_2}}\cdot\left(\frac{c}{K_6}\right)^{1/(\alpha_1\cdot\alpha_2)}\cdot n^{-\tau/(\alpha_1\cdot\alpha_2)}, \\
&\Lambda_n^{SE}\left(\frac{1}{2}\cdot\left(\left[\frac{1}{K_9}\cdot\left(K_5\wedge\left(\frac{n^{-\tau}\cdot c}{K_6}\right)^{1/\alpha_1}\right)\right]^{1/\alpha_2}\wedge K_8\right)\right) = n^{1/2-\tau/(\alpha_1\cdot\alpha_2)}\cdot\sigma_n^d\cdot\frac{A_2}{2\cdot K_9^{1/\alpha_2}}\cdot\left(\frac{c}{K_6}\right)^{1/(\alpha_1\cdot\alpha_2)} + o(1) \\
&\forall\ n \geq n_*
\end{aligned}$$

Therefore, for each $c > 0$ there exists n_0 such that

$$\begin{aligned}
&\sup_{F\in\mathcal{F}} P_F\left(\left\|M_k\left(\lambda_F(\overline{\theta})\right) - M_k\left(\lambda_F(\theta^*)\right)\right\| \geq n^{-\tau}\cdot c\right) \\
&\leq \frac{\overline{M}_8}{\left(n^{1/2-\tau/(\alpha_1\cdot\alpha_2)}\cdot\frac{1}{2\cdot K_9^{1/\alpha_2}}\cdot\left(\frac{c}{K_6}\right)^{1/(\alpha_1\cdot\alpha_2)}\right)^q} + \frac{M^{SE}}{\left(n^{1/2-\tau/(\alpha_1\cdot\alpha_2)}\cdot\sigma_n^d\cdot\frac{A_2}{2\cdot K_9^{1/\alpha_2}}\cdot\left(\frac{c}{K_6}\right)^{1/(\alpha_1\cdot\alpha_2)} + o(1)\right)^q} \\
&\quad \forall\ n \geq n_0.
\end{aligned}$$

Therefore,

$$\sup_{F\in\mathcal{F}} P_F\left(\left\|M_k\left(\lambda_F(\overline{\theta})\right) - M_k\left(\lambda_F(\theta^*)\right)\right\| \geq n^{-\tau}\cdot c\right) \longrightarrow 0 \quad \forall\ c > 0,\ \tau < \alpha_1\cdot\alpha_2\cdot\Delta.$$

which means,

$$\left\|M_k\left(\lambda_F(\overline{\theta})\right)-M_k\left(\lambda_F(\theta^*)\right)\right\|=o_p\left(n^{-\tau}\right)\quad \forall\ \tau<\alpha_1\cdot\alpha_2\cdot\Delta,\quad \text{uniformly over } \mathcal{F}. \tag{B5.53}$$

Next, recall from (B5.35) that, uniformly over $\mathcal{F}$,

$$\sup_{\theta\in\Theta}\left\|\varepsilon_n^{\nu^\mu}(\theta)\right\|=O_p\left(\frac{1}{n\cdot\sigma_n^d}\right)+O_p\left(\frac{1}{n^{3/2}\cdot\sigma_n^d}\right)+O_p\left(\left(\frac{1}{n^{1/2}\cdot\sigma_n^d}+s_{1,n}\right)^2\right)+s_{2,n}=o_p\left(\frac{1}{n^{1/2+\Delta}}\right)$$

and from (B5.48) we also have that, uniformly over $\mathcal{F}$,

$$\left\|\frac{1}{n}\sum_{i=1}^n\zeta_F(Z_i)\right\|=O_p\left(\frac{1}{n^{1/2}}\right).$$

From here, (B5.52) and (B5.53) we have that, uniformly over $\mathcal{F}$, and for any $0<\tau<\left(\frac{\alpha_1}{2}\right)\wedge(\alpha_1\cdot\alpha_2\cdot\Delta)$,

$$\begin{aligned}\left\|\varepsilon_n^\theta\right\|\le&\underbrace{\left\|M_k\left(\overline{T}(\overline{\theta})\right)-M_k\left(\lambda_F(\overline{\theta})\right)\right\|}_{=o_p(n^{-\tau})}\cdot\underbrace{\left\|\frac{1}{n}\sum_{i=1}^n\zeta_F(Z_i)\right\|}_{=O_p(n^{-1/2})}+\underbrace{\left\|M_k\left(\lambda_F(\overline{\theta})\right)-M_k\left(\lambda_F(\theta^*)\right)\right\|}_{=o_p(n^{-\tau})}\cdot\underbrace{\left\|\frac{1}{n}\sum_{i=1}^n\zeta_F(Z_i)\right\|}_{=O_p(n^{-1/2})}\\&+\underbrace{\left\|M_k\left(\overline{T}(\overline{\theta})\right)-M_k\left(\lambda_F(\overline{\theta})\right)\right\|}_{=o_p(n^{-\tau})}\cdot\underbrace{\sup_{\theta\in\Theta}\left\|\varepsilon_n^{\nu^\mu}(\theta)\right\|}_{=o_p(n^{-1/2-\Delta})}+\overline{M}_\lambda\cdot\underbrace{\sup_{\theta\in\Theta}\left\|\varepsilon_n^{\nu^\mu}(\theta)\right\|}_{=o_p(n^{-1/2-\Delta})}\end{aligned}$$

Therefore, for any $0<\tau<\min\left\{\left(\frac{\alpha_1}{2}\right),\ (\alpha_1\cdot\alpha_2\cdot\Delta),\ \Delta\right\}$,

$$\left\|\varepsilon_n^\theta\right\|=o_p\left(\frac{1}{n^{1/2+\tau}}\right),\quad \text{uniformly over } \mathcal{F}. \tag{B5.54}$$

Together, (B5.42), (B5.49) and (B5.54) show that the conditions in Assumption 1 of the paper are satisfied, with $\psi_F^\theta(Z_i)=M_k\left(\lambda_F(\theta^*)\right)\cdot\zeta_F(Z_i)$, $r_n=n^{1/2}\cdot\sigma_n^d$, $0<\tau<\min\left\{\left(\frac{\alpha_1}{2}\right),\ (\alpha_1\cdot\alpha_2\cdot\Delta),\ \Delta\right\}$, and $0<\overline{\delta}<q\Delta-\frac{1}{2}$. This proves Result SMIM. ∎

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