Econometric Supplement for “Testing functional inequalities conditional on estimated functions”

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Abstract

This document includes step-by-step derivations leading to the main econometric results in the paper, along with additional details for the influence-function estimator used in the construction of our test. Every section in this document has the format **SX.X** and every equation has the format **(SX.X.X)**. Any section, assumption or equation that we reference here which does not have this format refers to a section, assumption or equation in the main paper, Appendix A (if it has the format AX.X.X) or Appendix B (if it has the format BX.X.X).

S1 Introduction

Sections S2-S3 describe the step-by-step results leading to Proposition 1 in the paper. Section S4 describes the properties of the influence function estimator used in the construction of our test-statistic, which is discussed in Section 4.5.2 of the paper and in Appendix A.

S2 A useful maximal inequality result

Definition: Euclidean classes of functions

Let $\mathcal{T}$ be a space and d be a pseudometric defined on $\mathcal{T}$. For each $\varepsilon > 0$, define the *packing number* $D(\varepsilon, d, \mathcal{T})$ to be the largest number D for which there exist points $m_1, \ldots, m_D$ in $\mathcal{T}$ such that $d(m_i, m_j) > \varepsilon$ for each $i \neq j$. We say that G is an *envelope* for a class of functions $\mathscr{G}$ if $\sup_{\mathscr{G}} |g(\cdot)| \leq G(\cdot)$. Let μ be a measure on $\mathcal{S}_Z^k$ and denote $\mu h \equiv \int h(z_1, \ldots, z_k) d\mu(z_1, \ldots, z_k)$. We say that the class of functions $\mathscr{G}$ is Euclidean (A, V) for the envelope G if, for any measure μ such that $\mu G^2 < \infty$, we have $D(x, d_\mu, \mathscr{G}) \leq A x^{-V}, \quad 0 < x \leq 1$, where, $d_\mu(g_1, g_2) = \left(\mu |g_1 - g_2|^2 / \mu G^2\right)^{1/2}$. The constants A and V must not depend on μ. Immediate-to-verify criteria for determining the Euclidean property have long been established, for example, in Nolan and Pollard (1987) and Pakes and Pollard (1989).

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S2.1 A maximal inequality for degenerate U-processes

Let $Z_1,\ldots,Z_n$ be i.i.d observations from a distribution F on a set $\mathcal{S}_Z$. Let k be a positive integer and $\mathscr{G}$ a class of real-valued functions on $\mathcal{S}_Z^k = \mathcal{S}_Z \otimes \cdots \otimes \mathcal{S}_Z$ (k factors). For each $g \in \mathscr{G}$, define $U_n^k g = (n)_k^{-1} \sum_{\boldsymbol{i}_k} g(Z_{i_1},\ldots,Z_{i_k})$, where $(n)_k = n(n-1)\cdots(n-k+1)$ and $\sum_{\boldsymbol{i}_k}$ denotes the sum over the $(n)_k$ distinct integers $\{i_1,\ldots,i_k\}$ from the set $\{1,\ldots,n\}$. $U_n^k g$ is a U-statistic of order k and the collection $\{U_n^k g \colon g \in \mathscr{G}\}$ is called a U-process of order k, indexed by $\mathscr{G}$. If every $g \in \mathscr{G}$ is such that

$$\underbrace{E_F\left[g(s_1,\ldots,s_{i-1},Z,s_{i+1},\ldots,s_k)\right] \equiv 0}_{E_F[g(Z_1,\ldots,Z_k)|Z_1=s_1,\ldots,Z_{i-1}=s_{i-1},Z_{i+1}=s_{i+1},\ldots,Z_k=s_k]\equiv 0} \quad , \quad i=1,\ldots,k,$$

$\mathscr{G}$ is an $F-$*degenerate* class of functions on $\mathcal{S}_Z^k$ and $\{U_n^k g \colon g \in \mathscr{G}\}$ is a *degenerate U-process* of order k.

Result S1 ***(Sherman (1994, Corollary 4A))*** *Let $\mathscr{G}$ be a class of $F-$degenerate functions on $\mathcal{S}_Z^k$, $k \geq 1$. Suppose $\mathscr{G}$ is Euclidean (A,V) for an envelope G such that $E_F\left[G(Z_1,\ldots,Z_k)^{4p}\right] < \infty$ for a positive integer p. Then,*

$$E_F\left[\left(\sup_{\mathscr{G}} \left|n^{k/2} U_n^k g\right|\right)^p\right] \leq \Upsilon \cdot \left(E_F\left[G(Z_1,\ldots,Z_k)^{4p}\right]\right)^{1/2} \equiv \overline{M},$$

where Υ is a constant that depends only on p, A, V and $E_F\left[G(Z_1,\ldots,Z_k)^2\right]$. By a Chebyshev inequality, this implies that for each $\varepsilon > 0$,

$$P_F\left(\sup_{\mathscr{G}} \left|n^{k/2} U_n^k g\right| > \varepsilon\right) \leq \frac{\overline{M}}{\varepsilon^p} \quad \textit{and therefore} \quad P_F\left(\sup_{\mathscr{G}} \left|U_n^k g\right| > \varepsilon\right) \leq \frac{\overline{M}}{\left(n^{k/2} \cdot \varepsilon\right)^p}.$$

S3 Details of Proposition 1

S3.1 Asymptotic properties of $\widehat{f_g}(x,\widehat{\theta})$

Using Chebyshev's inequality, the conditions in Assumption 1 yield

$$\sup_{F \in \mathcal{F}} P_F\left(\left\|\frac{1}{n}\sum_{i=1}^n \psi_F^\theta(Z_i)\right\| \geq c\right) = O\left(\frac{1}{\left(n^{1/2} \cdot c\right)^q}\right) \quad \forall\, c > 0,$$

and combined with the assumed properties of ε_n^θ, this yields,

$$\begin{aligned}\sup_{F \in \mathcal{F}} P_F\left(\left\|\widehat{\theta} - \theta_F^*\right\| \geq c\right) &\leq \sup_{F \in \mathcal{F}} P_F\left(\left\|\frac{1}{n}\sum_{i=1}^n \psi_F^\theta(Z_i)\right\| \geq \frac{c}{2}\right) + \sup_{F \in \mathcal{F}} P_F\left(\left\|\varepsilon_n^\theta\right\| \geq \frac{c}{2}\right) \\ &= O\left(\frac{1}{\left(n^{1/2} \cdot c\right)^q}\right) + O\left(\frac{1}{(r_n \cdot c)^q}\right) \quad \forall\, c > 0\end{aligned} \tag{S3.1.1}$$

Therefore, for each $0 < \delta < \overline{\delta}$,

$$\sup_{F\in\mathcal{F}} P_F\left(\left\|\widehat{\theta} - \theta^*_F\right\| \geq c_n\right) = o\left(\frac{1}{n^{1/2+\delta}}\right) \quad \forall\, c_n : \left(n^{1/2} \wedge r_n\right) \cdot n^{-\left(\frac{1+2\delta}{2q}\right)} \cdot c_n \longrightarrow \infty. \tag{S3.1.2}$$

The conditions in Assumption 1 also imply that

$$\left\|\widehat{\theta} - \theta^*_F\right\| = O_p\left(\frac{1}{n^{1/2}}\right) \quad \text{uniformly over } \mathcal{F}, \tag{S3.1.3}$$

meaning that, for every $\delta > 0$, there exists a finite $M_\delta > 0$ and N_δ such that

$$\sup_{F\in\mathcal{F}} P_F\left(n^{1/2} \cdot \left\|\widehat{\theta} - \theta^*_F\right\| \geq M_\delta\right) < \delta \quad \forall\, n \geq N_\delta.$$

For $\{s,d\}$ in $1,\ldots,D$, and $\psi \equiv (\psi_1,\ldots,\psi_D)'$, let

$$\nabla_d K(\psi) \equiv \kappa^{(1)}(\psi_d) \cdot \prod_{\ell\neq d} \kappa(\psi_\ell), \quad \nabla_{ds} K(\psi) \equiv \begin{cases} \kappa^{(1)}(\psi_d)\cdot\kappa^{(1)}(\psi_s)\cdot\prod_{\ell\neq d,s}\kappa(\psi_\ell) & \text{if } d \neq s \\ \kappa^{(2)}(\psi_d)\cdot\prod_{\ell\neq d}\kappa(\psi_\ell) & \text{if } d = s \end{cases} \tag{S3.1.4}$$

Also, as we defined previously, for any $x_1 \in \mathcal{S}_X$, $x_2 \in \mathcal{S}_X$ and $\theta \in \Theta$, let

$$\Delta g_d(x_1,x_2,\theta) \equiv g_d(x_1,\theta) - g_d(x_2,\theta), \quad \text{and} \quad \Delta g(x_1,x_2,\theta) \equiv (\Delta g_1(x_1,x_2,\theta),\ldots,\Delta g_D(x_1,x_2,\theta))'.$$

For a given $x \in \mathcal{S}_X$, $\theta \in \Theta$ and $h > 0$, define

$$\begin{aligned} \Upsilon^{\ell,m}_{f_g}(X_i,x,\theta,h) &\equiv \sum_{d=1}^{D}\sum_{s=1}^{D} \nabla_{ds} K\left(\frac{\Delta g(X_i,x,\theta)}{h}\right)\frac{\partial\Delta g_d(X_i,x,\theta)}{\partial\theta_\ell}\frac{\partial\Delta g_s(X_i,x,\theta)}{\partial\theta_m}, \\ \Phi^{\ell,m}_{f_g}(X_i,x,\theta,h) &\equiv \sum_{d=1}^{D} \nabla_d K\left(\frac{\Delta g(X_i,x,\theta)}{h}\right)\frac{\partial^2\Delta g_d(X_i,x,\theta)}{\partial\theta_m\partial\theta_\ell} \end{aligned} \tag{S3.1.5}$$

By the conditions described in Assumptions 3 and 4, there exist $\overline{\mu}_\Upsilon < \infty$ and $\overline{\mu}_\Phi < \infty$ such that

$$\sup_{\substack{(s,\theta)\in\mathcal{S}_X\times\Theta \\ h>0}} \left|E_F\left[\Upsilon^{\ell,m}_{f_g}(X,s,\theta,h)\right]\right| \leq \overline{\mu}_\Upsilon \quad \text{and} \quad \sup_{\substack{(s,\theta)\in\mathcal{S}_X\times\Theta \\ h>0}} \left|E_F\left[\Phi^{\ell,m}_{f_g}(X,s,\theta,h)\right]\right| \leq \overline{\mu}_\Phi \quad \forall\, F \in \mathcal{F}. \tag{S3.1.6}$$

The conditions in Assumption 3, combined with Lemma 2.13 in Pakes and Pollard (1989) imply that, for each $d = 1,\ldots,D$ and $\ell, m \in 1,\ldots,k$, the classes of functions

$$\mathscr{H}_{1d}^{\ell} = \left\{ m : \mathcal{S}_X \longrightarrow \mathbb{R}\colon\ m(x) = \frac{\partial \Delta g_d(x,s,\theta)}{\partial \theta_\ell} \ \text{ for some } s \in \mathcal{S}_X,\ \theta \in \Theta \right\},$$
$$\mathscr{H}_{2d}^{\ell,m} = \left\{ m : \mathcal{S}_X \longrightarrow \mathbb{R}\colon\ m(x) = \frac{\partial^2 \Delta g_d(x,s,\theta)}{\partial \theta_\ell \partial \theta_m} \ \text{ for some } s \in \mathcal{S}_X,\ \theta \in \Theta \right\}$$

are Euclidean for an envelope $G_1(x)$ that satisfies $E_F\left[G_1(X)^{4q}\right] \leq \overline{\mu}_{G_1} < \infty$ for all $F \in \mathcal{F}$, with q being the integer described in Assumption 1. Now consider the following two classes of functions

$$\mathscr{G}_1^{\ell,m} = \left\{ r : \mathcal{S}_X \longrightarrow \mathbb{R}\colon\ r(x) = \Upsilon_{f_g}^{\ell,m}(x,u,\theta,h) \ \text{ for some } u \in \mathcal{S}_X,\ \theta \in \Theta,\ h > 0 \right\},$$
$$\mathscr{G}_2^{\ell,m} = \left\{ r : \mathcal{S}_X \longrightarrow \mathbb{R}\colon\ r(x) = \Phi_{f_g}^{\ell,m}(x,u,\theta,h) \ \text{ for some } u \in \mathcal{S}_X,\ \theta \in \Theta,\ h > 0 \right\},$$

By the Euclidean properties of $\mathscr{H}_{1d}^{\ell}$ and $\mathscr{H}_{2d}^{\ell,m}$, and by the conditions in Assumptions 3 and 4, the Euclidean-preserving properties described, e.g, in Pakes and Pollard (1989, Lemma 2.14), yield that $\mathscr{G}_1^{\ell,m}$ and $\mathscr{G}_2^{\ell,m}$ are Euclidean classes for an envelope $\overline{G}(\cdot)$ for which $\exists\ \overline{\mu}_{\overline{G}} < \infty$ such that $E_F\left[\overline{G}(X)^{4q}\right] \leq \overline{\mu}_{\overline{G}}\ \forall\ F \in \mathcal{F}$, with q being the integer described in Assumption 1. Next, define

$$\nu_{\Upsilon_f,n}^{\ell,m}(x,\theta,h) = \frac{1}{n}\sum_{i=1}^{n}\left(\Upsilon_{f_g}^{\ell,m}(X_i,x,\theta,h) - E_F\left[\Upsilon_{f_g}^{\ell,m}(X_i,x,\theta,h)\right]\right),$$
$$\nu_{\Phi_f,n}^{\ell,m}(x,\theta,h) = \frac{1}{n}\sum_{i=1}^{n}\left(\Phi_{f_g}^{\ell,m}(X_i,x,\theta,h) - E_F\left[\Phi_{f_g}^{\ell,m}(X_i,x,\theta,h)\right]\right), \tag{S3.1.7}$$

By the Euclidean properties of $\mathscr{G}_1^{\ell,m}$ and $\mathscr{G}_2^{\ell,m}$, and the integrability condition of the corresponding envelope, applying Result S1 we obtain that there exists $\overline{M}_1 < \infty$ such that, for any $b > 0$,

$$\sup_{F \in \mathcal{F}} P_F\left(\sup_{\substack{(x,\theta)\in \mathcal{S}_X \times \Theta \\ h>0}} \left| \nu_{\Upsilon_f,n}^{\ell,m}(x,\theta,h) \right| \geq b \right) \leq \frac{\overline{M}_1}{\left(n^{1/2}\cdot b\right)^q} = O\left(\frac{1}{\left(n^{1/2}\cdot b\right)^q} \right)$$
$$\sup_{F \in \mathcal{F}} P_F\left(\sup_{\substack{(x,\theta)\in \mathcal{S}_X \times \Theta \\ h>0}} \left| \nu_{\Phi_f,n}^{\ell,m}(x,\theta,h) \right| \geq b \right) \leq \frac{\overline{M}_1}{\left(n^{1/2}\cdot b\right)^q} = O\left(\frac{1}{\left(n^{1/2}\cdot b\right)^q} \right) \tag{S3.1.8}$$

for each $\{\ell, m\} \in 1,\ldots,k$. Note that (S3.1.8) implies, in particular, that

$$\begin{aligned}
&\sup_{\substack{(x,\theta)\in\mathcal{S}_X\times\Theta\\ h>0}} \left|\nu^{\ell,m}_{\Upsilon_f,n}(x,\theta,h)\right| = O_p\left(\frac{1}{n^{1/2}}\right) \quad \text{uniformly over } \mathcal{F}\\
&\sup_{\substack{(x,\theta)\in\mathcal{S}_X\times\Theta\\ h>0}} \left|\nu^{\ell,m}_{\Phi_f,n}(x,\theta,h)\right| = O_p\left(\frac{1}{n^{1/2}}\right) \quad \text{uniformly over } \mathcal{F}
\end{aligned} \tag{S3.1.9}$$

Denote

$$\underbrace{\frac{\partial\widehat{f_g}(g(x,\theta))}{\partial\theta}}_{1\times k} = \left(\frac{\partial\widehat{f_g}(g(x,\theta))}{\partial\theta_1},\ldots,\frac{\partial\widehat{f_g}(g(x,\theta))}{\partial\theta_k}\right), \quad \underbrace{\frac{\partial^2\widehat{f_g}(g(x,\theta))}{\partial\theta\partial\theta'}}_{k\times k} = \begin{pmatrix} \frac{\partial^2\widehat{f_g}(g(x,\theta))}{\partial\theta_1^2} & \cdots & \frac{\partial^2\widehat{f_g}(g(x,\theta))}{\partial\theta_1\partial\theta_k}\\ \vdots & \ddots & \vdots \\ \frac{\partial^2\widehat{f_g}(g(x,\theta))}{\partial\theta_k\partial\theta_1} & \cdots & \frac{\partial^2\widehat{f_g}(g(x,\theta))}{\partial\theta_k^2}\end{pmatrix}.$$

For a given $(x,\theta)\in\mathcal{S}_X\times\Theta$ and $F\in\mathcal{F}$, a second-order approximation yields,

$$\widehat{f_g}(g(x,\widehat{\theta})) = \widehat{f_g}(g(x,\theta^*_F)) + \frac{\partial\widehat{f_g}(g(x,\theta^*_F))}{\partial\theta}\left(\widehat{\theta}-\theta^*_F\right) + \frac{1}{2}\left(\widehat{\theta}-\theta^*_F\right)'\frac{\partial^2\widehat{f_g}(g(x,\overline{\theta}_x))}{\partial\theta\partial\theta'}\left(\widehat{\theta}-\theta^*_F\right) \tag{S3.1.10}$$

where $\overline{\theta}_x$ belongs in the line segment connecting $\widehat{\theta}$ and θ^*_F. Thus, since Θ is taken to be convex, we have $\overline{\theta}_x\in\Theta$. Denote

$$\xi^{f_g}_{b,n}(x,\theta) \equiv \left(\widehat{\theta}-\theta^*_F\right)'\frac{\partial^2\widehat{f_g}(g(x,\theta))}{\partial\theta\partial\theta'}\left(\widehat{\theta}-\theta^*_F\right). \tag{S3.1.11}$$

Going back to the definitions in (S3.1.5)-(S3.1.7), we have

$$\begin{aligned}
\frac{\partial^2\widehat{f_g}(g(x,\theta))}{\partial\theta_\ell\partial\theta_m} &= \frac{1}{n\cdot h_n^{D+2}}\sum_{i=1}^n \Upsilon^{\ell,m}_{f_g}(X_i,x,\theta,h_n) + \frac{1}{n\cdot h_n^{D+1}}\sum_{i=1}^n \Phi^{\ell,m}_{f_g}(X_i,x,\theta,h_n)\\
&= \frac{1}{h_n^{D+2}}\cdot E_F\left[\Upsilon^{\ell,m}_{f_g}(X,x,\theta,h_n)\right] + \frac{1}{h_n^{D+2}}\cdot \nu^{\ell,m}_{\Upsilon_f,n}(x,\theta,h_n)\\
&+ \frac{1}{h_n^{D+1}}\cdot E_F\left[\Phi^{\ell,m}_{f_g}(X,x,\theta,h_n)\right] + \frac{1}{h_n^{D+1}}\cdot \nu^{\ell,m}_{\Phi_f,n}(x,\theta,h_n)
\end{aligned} \tag{S3.1.12}$$

From here, we have

$$
\begin{aligned}
\xi_{b,n}^{f_g}(x,\theta) &= \sum_{\ell=1}^{k}\sum_{m=1}^{k} \frac{\partial^2 \widehat{f_g}(g(x,\theta))}{\partial\theta_\ell\theta_m} \cdot \left(\widehat{\theta}_m - \theta_{m,F}^*\right)\cdot\left(\widehat{\theta}_\ell - \theta_{\ell,F}^*\right) \\
&= \sum_{\ell=1}^{k}\sum_{m=1}^{k} \frac{1}{h_n^{D+2}} \cdot E_F\left[\Upsilon_{f_g}^{\ell,m}(X,x,\theta,h_n)\right]\cdot \left(\widehat{\theta}_m - \theta_{m,F}^*\right)\cdot\left(\widehat{\theta}_\ell - \theta_{\ell,F}^*\right) \\
&+ \sum_{\ell=1}^{k}\sum_{m=1}^{k} \frac{1}{h_n^{D+2}} \cdot \nu_{\Upsilon_f,n}^{\ell,m}(x,\theta,h_n)\cdot \left(\widehat{\theta}_m - \theta_{m,F}^*\right)\cdot\left(\widehat{\theta}_\ell - \theta_{\ell,F}^*\right) \\
&+ \sum_{\ell=1}^{k}\sum_{m=1}^{k} \frac{1}{h_n^{D+1}} \cdot E_F\left[\Phi_{f_g}^{\ell,m}(X,x,\theta,h_n)\right]\cdot \left(\widehat{\theta}_m - \theta_{m,F}^*\right)\cdot\left(\widehat{\theta}_\ell - \theta_{\ell,F}^*\right) \\
&+ \sum_{\ell=1}^{k}\sum_{m=1}^{k} \frac{1}{h_n^{D+1}} \cdot \nu_{\Phi_f,n}^{\ell,m}(x,\theta,h_n)\cdot \left(\widehat{\theta}_m - \theta_{m,F}^*\right)\cdot\left(\widehat{\theta}_\ell - \theta_{\ell,F}^*\right)
\end{aligned}
\tag{S3.1.13}
$$

Take any $b > 0$. From (S3.1.13), we have

$$
\begin{aligned}
P_F\left(\sup_{(x,\theta)\in\mathcal{S}_X\times\Theta}\left|\xi_{b,n}^{f_g}(x,\theta)\right| \geq b\right) \leq & \underbrace{\sum_{\ell=1}^{k}\sum_{m=1}^{k} P_F\left(\frac{\overline{\mu}_\Upsilon}{h_n^{D+2}}\cdot\left|\widehat{\theta}_m - \theta_{m,F}^*\right|\cdot\left|\widehat{\theta}_\ell - \theta_{\ell,F}^*\right| \geq \frac{b}{4k^2}\right)}_{(A)} \\
& + \underbrace{\sum_{\ell=1}^{k}\sum_{m=1}^{k} P_F\left(\frac{1}{h_n^{D+2}}\cdot\sup_{\substack{(x,\theta)\in\mathcal{S}_X\times\Theta \\ h>0}}\left|\nu_{\Upsilon_f,n}^{\ell,m}(x,\theta,h)\right|\cdot\left|\widehat{\theta}_m - \theta_{m,F}^*\right|\cdot\left|\widehat{\theta}_\ell - \theta_{\ell,F}^*\right| \geq \frac{b}{4k^2}\right)}_{(B)} \\
& + \underbrace{\sum_{\ell=1}^{k}\sum_{m=1}^{k} P_F\left(\frac{\overline{\mu}_\Phi}{h_n^{D+1}}\cdot\left|\widehat{\theta}_m - \theta_{m,F}^*\right|\cdot\left|\widehat{\theta}_\ell - \theta_{\ell,F}^*\right| \geq \frac{b}{4k^2}\right)}_{(C)} \\
& + \underbrace{\sum_{\ell=1}^{k}\sum_{m=1}^{k} P_F\left(\frac{1}{h_n^{D+1}}\cdot\sup_{\substack{(x,\theta)\in\mathcal{S}_X\times\Theta \\ h>0}}\left|\nu_{\Phi_f,n}^{\ell,m}(x,\theta,h)\right|\cdot\left|\widehat{\theta}_m - \theta_{m,F}^*\right|\cdot\left|\widehat{\theta}_\ell - \theta_{\ell,F}^*\right| \geq \frac{b}{4k^2}\right)}_{(D)}
\end{aligned}
\tag{S3.1.14}
$$

We will describe bounds for each of the terms in (S3.1.14). First, note that for any $c > 0$,

$$P_F\left(\left|\widehat{\theta}_m - \theta^*_{m,F}\right| \cdot \left|\widehat{\theta}_\ell - \theta^*_{\ell,F}\right| \geq c\right) \leq P_F\left(\left\|\widehat{\theta} - \theta^*_F\right\|^2_\infty \geq c\right) = P_F\left(\left\|\widehat{\theta} - \theta^*_F\right\|_\infty \geq c^{1/2}\right)$$
$$\leq P_F\left(\left\|\widehat{\theta} - \theta^*_F\right\| \geq m_k \cdot c^{1/2}\right),$$

where the last inequality follows from the equivalence of norms in Euclidean space and m_k is a constant that depends only on k (the dimension of θ). Using the result in equation (S3.1.1),

$$\sup_{F\in\mathcal{F}} \underbrace{\sum_{\ell=1}^{k}\sum_{m=1}^{k} P_F\left(\frac{\overline{\mu}_\Upsilon}{h_n^{D+2}} \cdot \left|\widehat{\theta}_m - \theta^*_{m,F}\right| \cdot \left|\widehat{\theta}_\ell - \theta^*_{\ell,F}\right| \geq \frac{b}{4k^2}\right)}_{(A)} \leq k^2 \cdot \sup_{F\in\mathcal{F}} P_F\left(\left\|\widehat{\theta} - \theta^*_F\right\| \geq m_k \cdot \left(\frac{h_n^{D+2} \cdot b}{4\overline{\mu}_\Upsilon k^2}\right)^{1/2}\right)$$
$$= O\left(\frac{1}{\left(n^{1/2} \cdot h_n^{\frac{D+2}{2}} \cdot b^{1/2}\right)^q}\right) + O\left(\frac{1}{\left(r_n \cdot h_n^{\frac{D+2}{2}} \cdot b^{1/2}\right)^q}\right),$$

$$\sup_{F\in\mathcal{F}} \underbrace{\sum_{\ell=1}^{k}\sum_{m=1}^{k} P_F\left(\frac{\overline{\mu}_\Phi}{h_n^{D+1}} \cdot \left|\widehat{\theta}_m - \theta^*_{m,F}\right| \cdot \left|\widehat{\theta}_\ell - \theta^*_{\ell,F}\right| \geq \frac{b}{4k^2}\right)}_{(C)} \leq k^2 \cdot \sup_{F\in\mathcal{F}} P_F\left(\left\|\widehat{\theta} - \theta^*_F\right\| \geq \left(\frac{h_n^{D+1} \cdot b}{4\overline{\mu}_\Phi k^2}\right)^{1/2}\right)$$
$$= O\left(\frac{1}{\left(n^{1/2} \cdot h_n^{\frac{D+1}{2}} \cdot b^{1/2}\right)^q}\right) + O\left(\frac{1}{\left(r_n \cdot h_n^{\frac{D+1}{2}} \cdot b^{1/2}\right)^q}\right)$$

Next, for any $c > 0$, equations (S3.1.2) and (S3.1.8) yield,

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{\substack{(x,\theta)\in\mathcal{S}_X\times\Theta\\ h>0}} \left|\nu^{\ell,m}_{\Upsilon_f,n}(x,\theta,h)\right| \cdot \left|\widehat{\theta}_m - \theta^*_{m,F}\right| \cdot \left|\widehat{\theta}_\ell - \theta^*_{\ell,F}\right| \geq c\right)$$
$$\leq \sup_{F\in\mathcal{F}} P_F\left(\sup_{\substack{(x,\theta)\in\mathcal{S}_X\times\Theta\\ h>0}} \left|\nu^{\ell,m}_{\Upsilon_f,n}(x,\theta,h)\right| \geq c^{1/2}\right) + \sup_{F\in\mathcal{F}} P_F\left(\left|\widehat{\theta}_m - \theta^*_{m,F}\right| \cdot \left|\widehat{\theta}_\ell - \theta^*_{\ell,F}\right| \geq c^{1/2}\right)$$
$$\leq \sup_{F\in\mathcal{F}} P_F\left(\sup_{\substack{(x,\theta)\in\mathcal{S}_X\times\Theta\\ h>0}} \left|\nu^{\ell,m}_{\Upsilon_f,n}(x,\theta,h)\right| \geq c^{1/2}\right) + \sup_{F\in\mathcal{F}} P_F\left(\left\|\widehat{\theta} - \theta^*_F\right\| \geq m_k \cdot c^{1/4}\right)$$
$$= O\left(\frac{1}{\left(n^{1/2} \cdot c^{1/2}\right)^q}\right) + O\left(\frac{1}{\left(n^{1/2} \cdot c^{1/4}\right)^q}\right) + O\left(\frac{1}{\left(r_n \cdot c^{1/4}\right)^q}\right)$$

From here we have that, for any $b > 0$,

$$\underbrace{\sup_{F\in\mathcal{F}} \sum_{\ell=1}^{k}\sum_{m=1}^{k} P_F\left(\frac{1}{h_n^{D+2}} \cdot \sup_{\substack{(x,\theta)\in\mathcal{S}_X\times\Theta \\ h>0}} \left|\nu^{\ell,m}_{\Upsilon_f,n}(x,\theta,h)\right| \cdot \left|\widehat{\theta}_m - \theta^*_{m,F}\right| \cdot \left|\widehat{\theta}_\ell - \theta^*_{\ell,F}\right| \geq \frac{b}{4k^2}\right)}_{(B)}$$

$$= O\left(\frac{1}{\left(n^{1/2}\cdot h_n^{\frac{D+2}{2}}\cdot b^{1/2}\right)^q}\right) + O\left(\frac{1}{\left(n^{1/2}\cdot h_n^{\frac{D+2}{4}}\cdot b^{1/4}\right)^q}\right) + O\left(\frac{1}{\left(r_n\cdot h_n^{\frac{D+2}{4}}\cdot b^{1/4}\right)^q}\right)$$

Similarly, the results in equations (S3.1.2) and (S3.1.8) yield for any $b > 0$,

$$\underbrace{\sup_{F\in\mathcal{F}} \sum_{\ell=1}^{k}\sum_{m=1}^{k} P_F\left(\frac{1}{h_n^{D+1}} \cdot \sup_{\substack{(x,\theta)\in\mathcal{S}_X\times\Theta \\ h>0}} \left|\nu^{\ell,m}_{\Phi_f,n}(x,\theta,h)\right| \cdot \left|\widehat{\theta}_m - \theta^*_{m,F}\right| \cdot \left|\widehat{\theta}_\ell - \theta^*_{\ell,F}\right| \geq \frac{b}{4k^2}\right)}_{(D)}$$

$$= O\left(\frac{1}{\left(n^{1/2}\cdot h_n^{\frac{D+1}{2}}\cdot b^{1/2}\right)^q}\right) + O\left(\frac{1}{\left(n^{1/2}\cdot h_n^{\frac{D+1}{4}}\cdot b^{1/4}\right)^q}\right) + O\left(\frac{1}{\left(r_n\cdot h_n^{\frac{D+1}{4}}\cdot b^{1/4}\right)^q}\right)$$

Next note that, since $h_n \longrightarrow 0$, we have

$$\frac{1}{h_n^{\frac{D+1}{2}}} = \frac{h_n^{1/2}}{h_n^{\frac{D+2}{2}}} = o\left(\frac{1}{h_n^{\frac{D+2}{2}}}\right).$$

For n large enough, $h_n^{1/2} < h_n^{1/4}$. From the results above and (S3.1.14) we obtain that, for any $b > 0$,

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{(x,\theta)\in\mathcal{S}_X\times\Theta} \left|\xi^{f_g}_{b,n}(x,\theta)\right| \geq b\right) = O\left(\frac{1}{\left(h_n^{\frac{D+2}{2}}\cdot(n^{1/2}\wedge r_n)\cdot(b^{1/2}\wedge b^{1/4})\right)^q}\right) \tag{S3.1.15}$$

Next, going back to (S3.1.12), and using the results in (S3.1.6) and (S3.1.9), we have

$$\sup_{(x,\theta)\in\mathcal{S}_X\times\Theta}\left|\frac{\partial^2\widehat{f_g}(g(x,\theta))}{\partial\theta_\ell\partial\theta_m}\right| \leq \frac{1}{h_n^{D+2}}\cdot\sup_{(x,\theta)\in\mathcal{S}_X\times\Theta}\left|E_F\left[\Upsilon_{f_g}^{\ell,m}(X,x,\theta,h_n)\right]\right| + \frac{1}{h_n^{D+2}}\cdot\sup_{(x,\theta)\in\mathcal{S}_X\times\Theta}\left|\nu_{\Upsilon_f,n}^{\ell,m}(x,\theta,h_n)\right|$$
$$+\frac{1}{h_n^{D+1}}\cdot\sup_{(x,\theta)\in\mathcal{S}_X\times\Theta}\left|E_F\left[\Phi_{f_g}^{\ell,m}(X,x,\theta,h_n)\right]\right| + \frac{1}{h_n^{D+1}}\cdot\sup_{(x,\theta)\in\mathcal{S}_X\times\Theta}\left|\nu_{\Phi_f,n}^{\ell,m}(x,\theta,h_n)\right|$$
$$\leq \frac{1}{h_n^{D+2}}\cdot\overline{\mu}_\Upsilon + \frac{1}{h_n^{D+2}}\cdot O_p\left(\frac{1}{\sqrt{n}}\right) + \frac{1}{h_n^{D+1}}\cdot\overline{\mu}_\Phi + \frac{1}{h_n^{D+1}}\cdot O_p\left(\frac{1}{\sqrt{n}}\right)$$

uniformly over $\mathcal{F}$. Therefore, under Assmptions 3 and 4,

$$\sup_{(x,\theta)\in\mathcal{S}_X\times\Theta}\left|\frac{\partial^2\widehat{f_g}(g(x,\theta))}{\partial\theta_\ell\partial\theta_m}\right| = O_p\left(\frac{1}{h_n^{D+2}}\right) \quad \text{uniformly over } \mathcal{F}. \tag{S3.1.16}$$

Let us go back to the definition of $\xi_{b,n}^{f_g}(x,\theta)$ in (S3.1.11). Combining (S3.1.16) and (S3.1.3),

$$\sup_{(x,\theta)\in\mathcal{S}_X\times\Theta}\left|\xi_{b,n}^{f_g}(x,\theta)\right| \leq \left\|\widehat{\theta}-\theta_F^*\right\|^2 \times \sup_{(x,\theta)\in\mathcal{S}_X\times\Theta}\left\|\frac{\partial^2\widehat{f_g}(g(x,\theta))}{\partial\theta\partial\theta'}\right\| = O_p\left(\frac{1}{n\cdot h_n^{D+2}}\right) \tag{S3.1.17}$$

uniformly over $\mathcal{F}$. Let us go back to the second-order approximation in (S3.1.10). We have,

$$\widehat{f_g}(g(x,\widehat{\theta})) = \widehat{f_g}(g(x,\theta_F^*)) + \frac{\partial\widehat{f_g}(g(x,\theta_F^*))}{\partial\theta}\left(\widehat{\theta}-\theta_F^*\right) + \underbrace{\frac{1}{2}\left(\widehat{\theta}-\theta_F^*\right)'\frac{\partial^2\widehat{f_g}(g(x,\overline{\theta}_x))}{\partial\theta\partial\theta'}\left(\widehat{\theta}-\theta_F^*\right)}_{=\xi_{b,n}^{f_g}(x,\overline{\theta}_x)}. \tag{S3.1.18}$$

From (S3.1.15) and (S3.1.17),

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{x\in\mathcal{S}_\mathcal{X}}\left|\xi_{b,n}^{f_g}(x,\overline{\theta}_x)\right| \geq b\right) \leq \sup_{F\in\mathcal{F}} P_F\left(\sup_{(x,\theta)\in\mathcal{S}_X\times\Theta}\left|\xi_{b,n}^{f_g}(x,\theta)\right| \geq b\right)$$
$$= O\left(\frac{1}{\left(h_n^{\frac{D+2}{2}}\cdot(n^{1/2}\wedge r_n)\cdot(b^{1/2}\wedge b^{1/4})\right)^q}\right) \quad \forall\, b>0, \quad \text{and}$$
$$\sup_{x\in\mathcal{S}_X}\left|\xi_{b,n}^{f_g}(x,\overline{\theta}_x)\right| \leq \sup_{(x,\theta)\in\mathcal{S}_X\times\Theta}\left|\xi_{b,n}^{f_g}(x,\theta)\right| = O_p\left(\frac{1}{n\cdot h_n^{D+2}}\right) \quad \text{uniformly over } \mathcal{F}. \tag{S3.1.19}$$

Let us now focus on the linear term $\frac{\partial\widehat{f_g}(g(x,\theta_F^*))}{\partial\theta}\left(\widehat{\theta}-\theta_F^*\right)$ in the approximation (S3.1.17). Let $\nabla_d K(\cdot)$ be as defined in (S3.1.4) and for each $\ell = 1,\ldots,k$, and a given $x \in \mathcal{S}_X$, $\theta \in \Theta$ and $h > 0$, let

$$\begin{aligned}\Lambda_{f_g}^{\ell}(X_i,x,\theta,h) &\equiv \sum_{d=1}^{D} \nabla_d K\left(\frac{\Delta g(X_i,x,\theta)}{h}\right)\frac{\partial \Delta g_d(X_i,x,\theta)}{\partial \theta_\ell},\\ \underbrace{\Lambda_{f_g}(X_i,x,\theta,h)}_{1\times k} &\equiv \left(\Lambda_{f_g}^{1}(X_i,x,\theta,h),\ldots,\Lambda_{f_g}^{k}(X_i,x,\theta,h)\right)\end{aligned} \tag{S3.1.20}$$

By the conditions described in Assumptions 3 and 4, there exists $\overline{\mu}_\Lambda < \infty$ such that

$$\sup_{\substack{(x,\theta)\in\mathcal{S}_X\times\Theta\\ h>0}} \left\|E_F\left[\Lambda_{f_g}(X,x,\theta,h)\right]\right\| \le \overline{\mu}_\Lambda \quad \forall\ F \in \mathcal{F}. \tag{S3.1.21}$$

Using the linear representation property of $\widehat{\theta}-\theta_F^*$ described in Assumption 1, we have

$$\begin{aligned}\frac{\partial\widehat{f_g}(g(x,\theta_F^*))}{\partial\theta}\left(\widehat{\theta}-\theta_F^*\right) &= \left(\frac{1}{n\cdot h_n^{D+1}}\sum_{i=1}^{n}\Lambda_{f_g}(X_i,x,\theta_F^*,h_n)\right)\left(\widehat{\theta}-\theta_F^*\right)\\ &= \frac{1}{n^2\cdot h_n^{D+1}}\sum_{i=1}^{n}\sum_{j=1}^{n}\Lambda_{f_g}(X_i,x,\theta_F^*,h_n)\psi_F^{\theta}(Z_j) + \left(\frac{1}{n\cdot h_n^{D+1}}\sum_{i=1}^{n}\Lambda_{f_g}(X_i,x,\theta_F^*,h_n)\right)\cdot\varepsilon_n^{\theta}\end{aligned}$$

Next, define

$$\begin{aligned}\nu_{\Lambda_{f_g},n}^{\ell}(x,\theta,h) &= \frac{1}{n}\sum_{i=1}^{n}\left(\Lambda_{f_g}^{\ell}(X_i,x,\theta,h) - E_F\left[\Lambda_{f_g}^{\ell}(X_i,x,\theta,h)\right]\right),\\ \underbrace{\nu_{\Lambda_{f_g},n}(x,\theta,h)}_{1\times k} &= \left(\nu_{\Lambda_{f_g},n}^{1}(x,\theta,h),\ldots,\nu_{\Lambda_{f_g},n}^{k}(x,\theta,h)\right)\end{aligned}$$

and consider the following class of functions on $\mathcal{S}_X$,

$$\mathscr{G}_3^{\ell} = \left\{m : \mathcal{S}_X \longrightarrow \mathbb{R}\colon\ m(x) = \Lambda_{f_g}^{\ell}(x,s,\theta,h)\ \text{ for some } s\in\mathcal{S}_X,\ \theta\in\Theta,\ h>0\right\}$$

By the conditions in Assumptions 3 and 4 in the paper, and the Euclidean-preserving properties in Pakes and Pollard (1989, Lemma 2.14), $\mathscr{G}_3^{\ell}$ is a Euclidean class for an envelope $\overline{G}_3(\cdot)$ for which $\exists\ \overline{\mu}_{\overline{G}_3} < \infty$ such that $E_F\left[\overline{G}_3(X)^{4q}\right] \le \overline{\mu}_{\overline{G}_3}$ for all $F \in \mathcal{F}$, with q being the integer described in

Assumption 1. Applying Result S1, $\exists\ \overline{M}_3 < \infty$ such that, for any $b > 0$,

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{\substack{(x,\theta)\in\mathcal{S}_X\times\Theta\\ h>0}} \left\|\nu_{\Lambda_{f_g},n}(x,\theta,h)\right\| \geq b\right) \leq \frac{\overline{M}_3}{\left(n^{1/2}\cdot b\right)^q} = O\left(\frac{1}{\left(n^{1/2}\cdot b\right)^q}\right) \tag{S3.1.22}$$

Note that (S3.1.22) implies, in particular, that

$$\sup_{\substack{(x,\theta)\in\mathcal{S}_X\times\Theta\\ h>0}} \left\|\nu_{\Lambda_{f_g},n}(x,\theta,h)\right\| = O_p\left(\frac{1}{n^{1/2}}\right) \quad \text{uniformly over } \mathcal{F} \tag{S3.1.23}$$

Now, consider the following class of functions on $\mathcal{S}_V^2$,

$$\mathscr{G}_{4,F} = \left\{m : \mathcal{S}_V^2 \longrightarrow \mathbb{R}\text{: } m(v_1,v_2) = \Lambda_{f_g}(x_1,s,\theta,h)\psi_F^\theta(z_2) \text{ for some } s\in\mathcal{S}_X,\ \theta\in\Theta,\ h>0\right\}$$

By the conditions in Assumptions 1, 3 and 4, and the Euclidean-preserving properties in Pakes and Pollard (1989, Lemma 2.14), $\mathscr{G}_4$ is a Euclidean class for an envelope $\overline{G}_4(\cdot)$ for which $\exists\ \overline{\mu}_{\overline{G}_4} < \infty$ such that $E_F\left[\overline{G}_4(X_1,Z_2)^{4q}\right] \leq \overline{\mu}_{\overline{G}_4}$ for all $F\in\mathcal{F}$ (with $(X_1,Z_2)\sim F\otimes F$), with q being the integer described in Assumption 1. Define the following U-statistic

$$U_{f_g,n}(x,\theta,h) = \frac{1}{n(n-1)}\sum_{i=1}^n\sum_{j\neq i}\Lambda_{f_g}(X_i,x,\theta,h)\psi_F^\theta(Z_j) \tag{S3.1.24}$$

Since $E_F\left[\psi_F^\theta(Z)\right] = 0$, iterated expectations yields $E_F\left[\Lambda_{f_g}(X_i,x,h)\psi_F^\theta(Z_j)\right] = 0$. Therefore,

$$E_F\left[U_{f_g,n}(x,\theta,h)\right] = 0\ \forall\ x\in\mathcal{S}_X,\ \theta\in\Theta,\ h>0.$$

We will characterize the relevant properties of the U-process $\left\{U_{f_g,n}(x,\theta,h)\text{: } x\in\mathcal{S}_X,\ \theta\in\Theta,\ h>0\right\}$ by first looking at its Hoeffding decomposition (see Serfling (1980, pages 177-178) or Sherman (1994, equations (6)-(7))). For a given $s\in\mathcal{S}_X$, $\theta\in\Theta$ and $h>0$, let

$$\begin{aligned}\varphi_F^{f_g}(Z_i,x,\theta,h) &= E_F\left[\Lambda_{f_g}(X,x,\theta,h)\right]\psi_F^\theta(Z_i),\\ \vartheta_F(V_i,V_j,x,\theta,h) &= \Lambda_{f_g}(X_i,x,h)\psi_F^\theta(Z_j) + \Lambda_{f_g}(X_j,x,h)\psi_F^\theta(Z_i) - \varphi_F^{f_g}(Z_i,x,\theta,h) - \varphi_F^{f_g}(Z_j,x,\theta,h)\end{aligned} \tag{S3.1.25}$$

Note that $\vartheta_F^{f_g}(V_i,V_j,x,\theta,h) = \vartheta_F^{f_g}(V_j,V_i,x,\theta,h)$ and $E_F\left[\vartheta_F^{f_g}(V_i,V_j,x,\theta,h)|V_i\right] = E_F\left[\vartheta_F^{f_g}(V_i,V_j,x,\theta,h)|V_j\right] = 0$. Define the following degenerate U-statistic of order 2,

$$\widetilde{U}_{f_g,n}(x,\theta,h) = \frac{1}{n(n-1)}\sum_{i=1}^n\sum_{j\neq i}\vartheta_F^{f_g}(V_i,V_j,s,\theta,h)$$

From the conditions in Assumptions 1-4, Nolan and Pollard (1987, Lemma 20) (or Lemma 5 in Sherman (1994, Lemma 5)), and Lemma 2.14 in Pakes and Pollard (1989, Lemma 2.14), the class of functions

$$\mathscr{G}_{5,F} = \left\{ m : \mathcal{S}_V^2 \longrightarrow \mathbb{R}\colon\ m(v_1, v_2) = \vartheta_F^{f_g}(v_1, v_2, x, \theta, h) \ \text{ for some } x \in \mathcal{S}_X,\ \theta \in \Theta,\ h > 0 \right\}$$

is Euclidean for an envelope $\overline{G}_5(\cdot)$ s.t $E_F\left[\overline{G}_5(V_1, V_2)^{4q}\right] \leq \overline{\mu}_{\overline{G}_5} < \infty\ \forall\ F \in \mathcal{F}$ (with $(V_1, V_2) \sim F \otimes F$), with q being the integer described in Assumption 1. From here, Result S1 implies that $\exists\ \overline{M}_5 < \infty$ such that, for any $b > 0$,

$$\begin{aligned}
&\sup_{F \in \mathcal{F}} P_F\left(\sup_{\substack{(x,\theta) \in \mathcal{S}_X \times \Theta \\ h > 0}} \left| \widetilde{U}_{f_g,n}(x, \theta, h) \right| \geq b \right) \leq \frac{\overline{M}_5}{(n \cdot b)^q} = O\left(\frac{1}{(n \cdot b)^q} \right), \ \text{ and therefore,} \\
&\sup_{\substack{(x,\theta) \in \mathcal{S}_X \times \Theta \\ h > 0}} \left| \widetilde{U}_{f_g,n}(x, \theta, h) \right| = O_p\left(\frac{1}{n} \right) \quad \text{uniformly over } \mathcal{F}
\end{aligned} \tag{S3.1.26}$$

Let $\varphi_F^{f_g}(Z_i, x, \theta, h)$ be as described in (S3.1.25), and define

$$\nu_{\varphi_{f_g},n}(x, \theta, h) = \frac{1}{n} \sum_{i=1}^{n} \varphi_F^{f_g}(Z_i, x, \theta, h) \tag{S3.1.27}$$

$E_F[\varphi_F^{f_g}(Z_i, x, \theta, h)] = 0$ (and therefore, $E_F[\nu_{\varphi_{f_g},n}(x, \theta, h)] = 0$) for all $x \in \mathcal{S}_X$, $\theta \in \Theta$ and $h > 0$. Again, the conditions in Assumptions 1-4, Nolan and Pollard (1987, Lemma 20) (or Sherman (1994, Lemma 5)), Pakes and Pollard (1989, Lemma 2.14) imply that the class of functions

$$\mathscr{G}_{6,F} = \left\{ m : \mathcal{S}_Z \longrightarrow \mathbb{R}\colon\ m(z) = \varphi_F^{f_g}(z, x, \theta, h) \ \text{ for some } x \in \mathcal{S}_X,\ \theta \in \Theta,\ h > 0 \right\}$$

is Euclidean for an envelope $\overline{G}_6(\cdot)$ s.t $E_F\left[\overline{G}_6(Z)^{4q}\right] \leq \overline{\mu}_{\overline{G}_6} < \infty\ \forall\ F \in \mathcal{F}$, with q being the integer described in Assumption 1. From here, Result S1 implies that $\exists\ \overline{M}_6 < \infty$ such that, for any $b > 0$,

$$\begin{aligned}
&\sup_{F \in \mathcal{F}} P_F\left(\sup_{\substack{(x,\theta) \in \mathcal{S}_X \times \Theta \\ h > 0}} \left| \nu_{\varphi_{f_g},n}(x, \theta, h) \right| \geq b \right) \leq \frac{\overline{M}_6}{\left(n^{1/2} \cdot b\right)^q} = O\left(\frac{1}{\left(n^{1/2} \cdot b\right)^q} \right), \ \text{ and therefore,} \\
&\sup_{\substack{(x,\theta) \in \mathcal{S}_X \times \Theta \\ h > 0}} \left| \nu_{\varphi_{f_g},n}(x, \theta, h) \right| = O_p\left(\frac{1}{n^{1/2}} \right) \quad \text{uniformly over } \mathcal{F}
\end{aligned} \tag{S3.1.28}$$

The Hoeffding decomposition of the U-statistic $U_{f_g,n}(x, \theta, h)$ defined in (S3.1.24) (see Serfling (1980, pages 177-178) or Sherman (1994, equations (6)-(7))) is,

$$U_{f_g,n}(x,\theta,h) = \nu_{\varphi_{f_g},n}(x,\theta,h) + \frac{1}{2}\cdot \widetilde{U}_{f_g,n}(x,\theta,h) \tag{S3.1.29}$$

From (S3.1.26) and (S3.1.28), for any $b>0$ we then have

$$\begin{aligned}
&\sup_{F\in\mathcal{F}} P_F\left(\sup_{\substack{(x,\theta)\in\mathcal{S}_X\times\Theta\\ h>0}}\left|U_{f_g,n}(x,\theta,h)\right|\geq b\right)\\
\leq&\sup_{F\in\mathcal{F}} P_F\left(\sup_{\substack{(x,\theta)\in\mathcal{S}_X\times\Theta\\ h>0}}\left|\nu_{\varphi_{f_g},n}(x,\theta,h)\right|\geq \frac{b}{2}\right)+\sup_{F\in\mathcal{F}} P_F\left(\frac{1}{2}\cdot\sup_{\substack{(x,\theta)\in\mathcal{S}_X\times\Theta\\ h>0}}\left|\widetilde{U}_{f_g,n}(x,\theta,h)\right|\geq \frac{b}{2}\right)\\
=&O\left(\frac{1}{(n\cdot b)^q}\right)+O\left(\frac{1}{\left(n^{1/2}\cdot b\right)^q}\right)=O\left(\frac{1}{\left(n^{1/2}\cdot b\right)^q}\right)
\end{aligned}\tag{S3.1.30}$$

Go back to the second-order approximation in (S3.1.18). For a given $(x,\theta)\in\mathcal{S}_X\times\Theta$ and $F\in\mathcal{F}$, let $\xi^{f_g}_{a,n}(x,\theta)\equiv\frac{\partial\widehat{f_g}(g(x,\theta))}{\partial\theta}\left(\widehat{\theta}-\theta^*_F\right)$. We have,

$$\widehat{f_g}(g(x,\widehat{\theta}))=\widehat{f_g}(g(x,\theta^*_F))+\underbrace{\frac{\partial\widehat{f_g}(g(x,\theta^*_F))}{\partial\theta}\left(\widehat{\theta}-\theta^*_F\right)}_{=\xi^{f_g}_{a,n}(x,\theta^*_F)}+\underbrace{\frac{1}{2}\left(\widehat{\theta}-\theta^*_F\right)'\frac{\partial^2\widehat{f_g}(g(x,\overline{\theta}_x))}{\partial\theta\partial\theta'}\left(\widehat{\theta}-\theta^*_F\right)}_{=\xi^{f_g}_{b,n}(x,\overline{\theta}_x)}. \tag{S3.1.31}$$

The properties of $\xi^{f_g}_{b,n}(x,\overline{\theta}_x)$ were analyzed in (S3.1.15) and (S3.1.17). We can now study the properties of $\xi^{f_g}_{a,n}(x,\theta^*_F)$. Let $m^{f_g}_n(x,\theta,h)\equiv\frac{1}{n}\sum_{i=1}^n\left(\Lambda_{f_g}(X_i,x,\theta,h)\psi^\theta_F(Z_i)-E_F\left[\Lambda_{f_g}(X_i,x,\theta,h)\psi^\theta_F(Z_i)\right]\right)$. Note that, under our assumptions, there exists $\overline{\mu}_m<\infty$ such that

$$\sup_{\substack{(x,\theta)\in\mathcal{S}_X\times\Theta\\ h>0}}\left|E_F\left[\Lambda_{f_g}(X_i,x,\theta,h)\psi^\theta_F(Z_i)\right]\right|\leq\overline{\mu}_m\quad\forall\ F\in\mathcal{F}. \tag{S3.1.32}$$

Our previous arguments and Euclidean properties of $\mathscr{G}_{4,F}$ yield, through Result S1,

$$\begin{aligned}
&\sup_{F\in\mathcal{F}} P_F\left(\sup_{\substack{(x,\theta)\in\mathcal{S}_X\times\Theta\\ h>0}}\left|m^{f_g}_n(x,\theta,h)\right|\geq b\right)=O\left(\frac{1}{\left(n^{1/2}\cdot b\right)^q}\right)\quad\forall\ b>0,\quad\text{and therefore,}\\
&\sup_{\substack{(x,\theta)\in\mathcal{S}_X\times\Theta\\ h>0}}\left|m^{f_g}_n(x,\theta,h)\right|=O_p\left(\frac{1}{n^{1/2}}\right)\quad\text{uniformly over }\mathcal{F}
\end{aligned}\tag{S3.1.33}$$

We have $\xi^{f_g}_{a,n}(x,\theta) \equiv \left(\frac{1}{n\cdot h_n^{D+1}}\sum_{i=1}^n \Lambda_{f_g}(X_i,x,\theta,h_n)\right)\left(\widehat{\theta}-\theta^*_F\right)$. Therefore,

$$
\begin{aligned}
\xi^{f_g}_{a,n}(x,\theta) &= \frac{1}{n^2\cdot h_n^{D+1}}\sum_{i=1}^n\sum_{j=1}^n \Lambda_{f_g}(X_i,x,\theta,h_n)\psi^\theta_F(Z_j) + \left(\frac{1}{n\cdot h_n^{D+1}}\sum_{i=1}^n \Lambda_{f_g}(X_i,x,\theta,h_n)\right)\varepsilon^\theta_n \\
&= \frac{1}{h_n^{D+1}}\cdot\left(\frac{n-1}{n}\right)\cdot U_{f_g,n}(x,\theta,h_n) + \frac{1}{n\cdot h_n^{D+1}}\cdot E_F\left[\Lambda_{f_g}(X,x,\theta,h_n)\psi^\theta_F(Z)\right] + \frac{1}{n\cdot h_n^{D+1}}\cdot m^{f_g}_n(x,\theta,h_n) \\
&\quad + \frac{1}{h_n^{D+1}}\cdot E_F\left[\Lambda_{f_g}(X,x,\theta,h_n)\right]\varepsilon^\theta_n + \frac{1}{h_n^{D+1}}\cdot \nu_{\Lambda_{f_g},n}(x,\theta,h_n)\varepsilon^\theta_n
\end{aligned}
\tag{S3.1.34}
$$

From (S3.1.21) and (S3.1.32), for any $F\in\mathcal{F}$ we have,

$$
\begin{aligned}
\left|\xi^{f_g}_{a,n}(x,\theta)\right| \leq{}& \frac{1}{h_n^{D+1}}\cdot\left(\frac{n-1}{n}\right)\cdot\left|U_{f_g,n}(x,\theta,h_n)\right| + \frac{1}{n\cdot h_n^{D+1}}\cdot\left|m^{f_g}_n(x,\theta,h_n)\right| + \frac{1}{h_n^{D+1}}\cdot\left\|\nu_{\Lambda_{f_g},n}(x,\theta,h_n)\right\|\cdot\left\|\varepsilon^\theta_n\right\| \\
&+ \frac{\overline{\mu}_\Lambda}{h_n^{D+1}}\cdot\left\|\varepsilon^\theta_n\right\| + \frac{\overline{\mu}_m}{n\cdot h_n^{D+1}}.
\end{aligned}
$$

Thus, for any $b>0$,

$$
\begin{aligned}
\sup_{F\in\mathcal{F}} P_F\left(\sup_{(x,\theta)\in\mathcal{S}_X\times\Theta}\left|\xi^{f_g}_{a,n}(x,\theta)\right|\geq b\right) \leq{}& \underbrace{\sup_{F\in\mathcal{F}} P_F\left(\frac{1}{h_n^{D+1}}\cdot\left(\frac{n-1}{n}\right)\cdot\sup_{\substack{(x,\theta)\in\mathcal{S}_X\times\Theta\\ h>0}}\left|U_{f_g,n}(x,\theta,h)\right|\geq\frac{1}{4}\cdot\left(b-\frac{\overline{\mu}_m}{n\cdot h_n^{D+1}}\right)\right)}_{(A)} \\
&+\underbrace{\sup_{F\in\mathcal{F}} P_F\left(\frac{1}{n\cdot h_n^{D+1}}\cdot\sup_{\substack{(x,\theta)\in\mathcal{S}_X\times\Theta\\ h>0}}\left|m^{f_g}_n(x,\theta,h)\right|\geq\frac{1}{4}\cdot\left(b-\frac{\overline{\mu}_m}{n\cdot h_n^{D+1}}\right)\right)}_{(B)} \\
&+\underbrace{\sup_{F\in\mathcal{F}} P_F\left(\frac{1}{h_n^{D+1}}\cdot\sup_{\substack{(x,\theta)\in\mathcal{S}_X\times\Theta\\ h>0}}\left\|\nu_{\Lambda_{f_g},n}(x,\theta,h)\right\|\cdot\left\|\varepsilon^\theta_n\right\|\geq\frac{1}{4}\cdot\left(b-\frac{\overline{\mu}_m}{n\cdot h_n^{D+1}}\right)\right)}_{(C)} \\
&+\underbrace{\sup_{F\in\mathcal{F}} P_F\left(\frac{\overline{\mu}_\Lambda}{h_n^{D+1}}\cdot\left\|\varepsilon^\theta_n\right\|\geq\frac{1}{4}\cdot\left(b-\frac{\overline{\mu}_m}{n\cdot h_n^{D+1}}\right)\right)}_{(D)}
\end{aligned}
\tag{S3.1.35}
$$

Using our previous results we can analyze each of the terms in (S3.1.35). If $b>0$ is fixed[1], there

[1] What follows is true for any sequence $b_n>0$ such that $b_n\cdot n\cdot h_n^{D+1}\longrightarrow\infty$.

exists n_0 such that $b-\frac{\overline{\mu}_m}{n\cdot h_n^{D+1}}>0\ \forall\ n>n_0$. From (S3.1.30), it follows that

$$\sup_{F\in\mathcal{F}} P_F\left(\frac{1}{h_n^{D+1}}\cdot\left(\frac{n-1}{n}\right)\cdot \sup_{\substack{(x,\theta)\in\mathcal{S}_X\times\Theta\\ h>0}}\left|U_{f_g,n}(x,\theta,h)\right|\geq\frac{1}{4}\cdot\left(b-\frac{\overline{\mu}_m}{n\cdot h_n^{D+1}}\right)\right)$$
$$=\sup_{F\in\mathcal{F}} P_F\left(\sup_{\substack{(x,\theta)\in\mathcal{S}_X\times\Theta\\ h>0}}\left|U_{f_g,n}(x,\theta,h)\right|\geq\frac{1}{4}\cdot\left(\frac{n}{n-1}\right)\cdot h_n^{D+1}\cdot\left(b-\frac{\overline{\mu}_m}{n\cdot h_n^{D+1}}\right)\right)=O\left(\frac{1}{\left(n^{1/2}\cdot h_n^{D+1}\cdot\left(b-\frac{\overline{\mu}_m}{n\cdot h_n^{D+1}}\right)\right)^q}\right) \tag{S3.1.36A}$$

From (S3.1.33),

$$\sup_{F\in\mathcal{F}} P_F\left(\frac{1}{n\cdot h_n^{D+1}}\cdot \sup_{\substack{(x,\theta)\in\mathcal{S}_X\times\Theta\\ h>0}}\left|m_n^{f_g}(x,\theta,h)\right|\geq\frac{1}{4}\cdot\left(b-\frac{\overline{\mu}_m}{n\cdot h_n^{D+1}}\right)\right)$$
$$=\sup_{F\in\mathcal{F}} P_F\left(\sup_{\substack{(x,\theta)\in\mathcal{S}_X\times\Theta\\ h>0}}\left|m_n^{f_g}(x,\theta,h)\right|\geq\frac{1}{4}\cdot n\cdot h_n^{D+1}\cdot\left(b-\frac{\overline{\mu}_m}{n\cdot h_n^{D+1}}\right)\right)=O\left(\frac{1}{\left(n^{3/2}\cdot h_n^{D+1}\cdot\left(b-\frac{\overline{\mu}_m}{n\cdot h_n^{D+1}}\right)\right)^q}\right) \tag{S3.1.36B}$$

Next, from (S3.1.22) and Assumption 1,

$$\begin{aligned}
&\sup_{F\in\mathcal{F}} P_F\left(\frac{1}{h_n^{D+1}}\cdot \sup_{\substack{(x,\theta)\in\mathcal{S}_X\times\Theta\\ h>0}}\left\|\nu_{\Lambda_{f_g},n}(x,\theta,h)\right\|\cdot\left\|\varepsilon_n^{\theta}\right\|\geq\frac{1}{4}\cdot\left(b-\frac{\overline{\mu}_m}{n\cdot h_n^{D+1}}\right)\right)\\
\leq&\sup_{F\in\mathcal{F}} P_F\left(\sup_{\substack{(x,\theta)\in\mathcal{S}_X\times\Theta\\ h>0}}\left\|\nu_{\Lambda_{f_g},n}(x,\theta,h)\right\|\geq\left(\frac{1}{4}\cdot h_n^{D+1}\cdot\left(b-\frac{\overline{\mu}_m}{n\cdot h_n^{D+1}}\right)\right)^{1/2}\right)\\
+&\sup_{F\in\mathcal{F}} P_F\left(\left\|\varepsilon_n^{\theta}\right\|\geq\left(\frac{1}{4}\cdot h_n^{D+1}\cdot\left(b-\frac{\overline{\mu}_m}{n\cdot h_n^{D+1}}\right)\right)^{1/2}\right)\\
=&O\left(\frac{1}{\left(n^{1/2}\cdot h_n^{\frac{D+1}{2}}\cdot\left(b-\frac{\overline{\mu}_m}{n\cdot h_n^{D+1}}\right)^{1/2}\right)^q}\right)+O\left(\frac{1}{\left(r_n\cdot h_n^{\frac{D+1}{2}}\cdot\left(b-\frac{\overline{\mu}_m}{n\cdot h_n^{D+1}}\right)^{1/2}\right)^q}\right)\\
=&O\left(\frac{1}{\left((n^{1/2}\wedge r_n)\cdot h_n^{\frac{D+1}{2}}\cdot\left(b-\frac{\overline{\mu}_m}{n\cdot h_n^{D+1}}\right)^{1/2}\right)^q}\right)
\end{aligned} \tag{S3.1.36C}$$

Finally, also from Assumption 1,

$$\sup_{F\in\mathcal{F}} P_F\left(\frac{\overline{\mu}_\Lambda}{h_n^{D+1}}\cdot\left\|\varepsilon_n^\theta\right\|\geq\frac{1}{4}\cdot\left(b-\frac{\overline{\mu}_m}{n\cdot h_n^{D+1}}\right)\right)=\sup_{F\in\mathcal{F}} P_F\left(\left\|\varepsilon_n^\theta\right\|\geq\frac{h_n^{D+1}}{4\overline{\mu}_m}\cdot\left(b-\frac{\overline{\mu}_m}{n\cdot h_n^{D+1}}\right)\right)$$
$$=O\left(\frac{1}{\left(r_n\cdot h_n^{D+1}\cdot\left(b-\frac{\overline{\mu}_m}{n\cdot h_n^{D+1}}\right)\right)^q}\right) \tag{S3.1.36D}$$

For large enough n, we have[2] $b-\frac{\overline{\mu}_m}{n\cdot h_n^{D+1}}>0$ and $h_n<1$ (and therefore $h_n^{D+1}<h_n^{\frac{D+1}{2}}$). Combining (S3.1.36A)-(S3.1.36D) with (S3.1.35), we obtain,

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{(x,\theta)\in\mathcal{S}_X\times\Theta}\left|\xi_{a,n}^{f_g}(x,\theta)\right|\geq b\right)=O\left(\frac{1}{\left((n^{1/2}\wedge r_n)\cdot h_n^{D+1}\cdot\left(\left(b-\frac{\overline{\mu}_m}{n\cdot h_n^{D+1}}\right)\wedge\left(b-\frac{\overline{\mu}_m}{n\cdot h_n^{D+1}}\right)^{1/2}\right)\right)^q}\right)$$

Going back to the second-order approximation in (S3.1.31),

$$\widehat{f_g}(g(x,\widehat{\theta}))=\widehat{f_g}(g(x,\theta_F^*))+\xi_{a,n}^{f_g}(x,\theta_F^*)+\xi_{b,n}^{f_g}(x,\overline{\theta}_x), \tag{S3.1.37}$$

where, for any $b>0$

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{x\in\mathcal{S}_X}\left|\xi_{a,n}^{f_g}(x,\theta_F^*)\right|\geq b\right)\leq\sup_{F\in\mathcal{F}} P_F\left(\sup_{(x,\theta)\in\mathcal{S}_X\times\Theta}\left|\xi_{a,n}^{f_g}(x,\theta)\right|\geq b\right)$$
$$=O\left(\frac{1}{\left((n^{1/2}\wedge r_n)\cdot h_n^{D+1}\cdot\left(\left(b-\frac{\overline{\mu}_m}{n\cdot h_n^{D+1}}\right)\wedge\left(b-\frac{\overline{\mu}_m}{n\cdot h_n^{D+1}}\right)^{1/2}\right)\right)^q}\right) \tag{S3.1.38}$$

Note that, for any $c>0$, we have $\min\left\{c,c^{1/2},c^{1/4}\right\}=\min\left\{c,c^{1/4}\right\}$. Using this and combining (S3.1.19) with (S3.1.38), we have that for any $b>0$,

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{x\in\mathcal{S}_\mathcal{X}}\left|\xi_{a,n}^{f_g}(x,\theta_F^*)+\xi_{b,n}^{f_g}(x,\overline{\theta}_x)\right|\geq b\right)$$
$$\leq\sup_{F\in\mathcal{F}} P_F\left(\sup_{x\in\mathcal{S}_\mathcal{X}}\left|\xi_{a,n}^{f_g}(x,\theta_F^*)\right|\geq\frac{b}{2}\right)+\sup_{F\in\mathcal{F}} P_F\left(\sup_{x\in\mathcal{S}_\mathcal{X}}\left|\xi_{b,n}^{f_g}(x,\overline{\theta}_x)\right|\geq\frac{b}{2}\right)$$
$$=O\left(\frac{1}{\left((n^{1/2}\wedge r_n)\cdot h_n^{D+1}\cdot\left(\left(\frac{b}{2}-\frac{\overline{\mu}_m}{n\cdot h_n^{D+1}}\right)\wedge\left(\frac{b}{2}-\frac{\overline{\mu}_m}{n\cdot h_n^{D+1}}\right)^{1/4}\right)\right)^q}\right) \tag{S3.1.39}$$

[2]See footnote 1.

For a given $x \in \mathcal{S}_X$, $\theta \in \Theta$ and $h > 0$, let $\nu_n^{f_g}(x,\theta,h) \equiv \frac{1}{n}\sum_{i=1}^n \left(K\left(\frac{\Delta g(X_i,x,\theta)}{h}\right) - E_F\left[K\left(\frac{\Delta g(X_i,x,\theta)}{h}\right)\right]\right)$. Directly from Assumption 4, the class of functions

$$\mathscr{G}_7 = \left\{m : \mathcal{S}_X \longrightarrow \mathbb{R}\colon\, m(x) = K\left(\frac{\Delta g(x,u,\theta)}{h}\right) \text{ for some } u \in \mathcal{S}_X,\, \theta \in \Theta,\, h > 0\right\}$$

is Euclidean for the constant envelope $\overline{K}$. From Result S1, $\exists\ \overline{M}_7 < \infty$ such that, $\forall\ b > 0$,

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{\substack{(x,\theta)\in\mathcal{S}_X\times\Theta\\ h>0}}\left|\nu_n^{f_g}(x,\theta,h)\right| \geq b\right) \leq \frac{\overline{M}_7}{\left(n^{1/2}\cdot b\right)^q} = O\left(\frac{1}{\left(n^{1/2}\cdot b\right)^q}\right), \text{ and therefore,}$$
$$\sup_{\substack{(x,\theta)\in\mathcal{S}_X\times\Theta\\ h>0}}\left|\nu_n^{f_g}(x,\theta,h)\right| = O_p\left(\frac{1}{n^{1/2}}\right) \quad \text{uniformly over } \mathcal{F}. \tag{S3.1.40}$$

For a given $x \in \mathcal{S}_X$ let

$$B_n^{f_g}(x) \equiv \underbrace{\frac{1}{h_n^D}\cdot E_F\left[K\left(\frac{\Delta g(X,x,\theta_F^*)}{h_n}\right)\right] - f_g(g(x,\theta_F^*))}_{\text{bias}}.$$

We have, $\widehat{f_g}(g(x,\theta_F^*)) = f_g(g(x,\theta_F^*)) + \frac{1}{h_n^D}\cdot\nu_n^{f_g}(x,\theta_F^*,h_n) + B_n^{f_g}(x)$. From here, (S3.1.37) yields,

$$\widehat{f_g}(g(x,\widehat{\theta})) = f_g(g(x,\theta_F^*)) + \frac{1}{h_n^D}\cdot\nu_n^{f_g}(x,\theta_F^*,h_n) + B_n^{f_g}(x) + \xi_{a,n}^{f_g}(x,\theta_F^*) + \xi_{b,n}^{f_g}(x,\overline{\theta}_x), \tag{S3.1.41}$$

From Assumptions 4 and the smoothness conditions described in Assumption 2, an M^{th}-order approximation implies that there exists a constant $\overline{B}_{1,f} < \infty$ such that

$$\sup_{x\in\mathcal{X}}\left|B_n^{f_g}(x)\right| \leq \overline{B}_{1,f}\cdot h_n^M \quad \forall\ F \in \mathcal{F}. \tag{S3.1.42}$$

From here,[3]

$$\sup_{x\in\mathcal{X}}\left|\widehat{f_g}(g(x,\widehat{\theta})) - f_g(g(x,\theta_F^*))\right| \leq \frac{1}{h_n^D}\cdot\sup_{x\in\mathcal{X}}\left|\nu_n^{f_g}(x,\theta_F^*,h_n)\right| + \overline{B}_{1,f}\cdot h_n^M + \sup_{x\in\mathcal{X}}\left|\xi_{a,n}^{f_g}(x,\theta_F^*) + \xi_{b,n}^{f_g}(x,\overline{\theta}_x)\right|$$

[3]Note that here we are focusing on our testing range $\mathcal{X}$.

Thus, for any $b > 0$,

$$\begin{aligned}
&\sup_{F\in\mathcal{F}} P_F\left(\sup_{x\in\mathcal{X}} \left|\widehat{f_g}(g(x,\widehat{\theta})) - f_g(g(x,\theta_F^*))\right| \geq b\right)\\
\leq&\sup_{F\in\mathcal{F}} P_F\left(\frac{1}{h_n^D}\cdot\sup_{x\in\mathcal{X}}\left|\nu_n^{f_g}(x,\theta_F^*,h_n)\right| + \sup_{x\in\mathcal{X}}\left|\xi_{a,n}^{f_g}(x,\theta_F^*) + \xi_{b,n}^{f_g}(x,\overline{\theta}_x)\right| \geq b - \overline{B}_{1,f}\cdot h_n^M\right)\\
\leq&\sup_{F\in\mathcal{F}} P_F\left(\frac{1}{h_n^D}\cdot\sup_{x\in\mathcal{X}}\left|\nu_n^{f_g}(x,\theta_F^*,h_n)\right| \geq \left(\frac{b - \overline{B}_{1,f}\cdot h_n^M}{2}\right)\right)\\
+&\sup_{F\in\mathcal{F}} P_F\left(\sup_{x\in\mathcal{X}}\left|\xi_{a,n}^{f_g}(x,\theta_F^*) + \xi_{b,n}^{f_g}(x,\overline{\theta}_x)\right| \geq \left(\frac{b - \overline{B}_{1,f}\cdot h_n^M}{2}\right)\right)
\end{aligned} \tag{S3.1.43}$$

For each[4] $b > 0$, $\exists\ n_0 : \frac{b-\overline{B}_{1,f}\cdot h_n^M}{4} - \frac{\overline{\mu}_m}{n\cdot h_n^D} > 0$, $h_n < 1\ \forall\ n > n_0$. (S3.1.39), (S3.1.40), and (S3.1.43) yield

$$\begin{aligned}
&\sup_{F\in\mathcal{F}} P_F\left(\sup_{x\in\mathcal{X}} \left|\widehat{f_g}(g(x,\widehat{\theta})) - f_g(g(x,\theta_F^*))\right| \geq b\right)\\
&= O\left(\frac{1}{\left(n^{1/2}\cdot h_n^D\cdot\left(\frac{b-\overline{B}_{1,f}\cdot h_n^M}{2}\right)\right)^q}\right) + O\left(\frac{1}{\left((n^{1/2}\wedge r_n)\cdot h_n^{D+1}\cdot\left(\left(\frac{b}{4} - \frac{\overline{B}_{1,f}\cdot h_n^M}{4} - \frac{\overline{\mu}_m}{n\cdot h_n^{D+1}}\right)\wedge\left(\frac{b}{4} - \frac{\overline{B}_{1,f}\cdot h_n^M}{4} - \frac{\overline{\mu}_m}{n\cdot h_n^{D+1}}\right)^{1/4}\right)\right)^q}\right)\\
&= O\left(\frac{1}{\left((n^{1/2}\wedge r_n)\cdot h_n^{D+1}\cdot\left(\left(\frac{b}{4} - \frac{\overline{B}_{1,f}\cdot h_n^M}{4} - \frac{\overline{\mu}_m}{n\cdot h_n^{D+1}}\right)\wedge\left(\frac{b}{4} - \frac{\overline{B}_{1,f}\cdot h_n^M}{4} - \frac{\overline{\mu}_m}{n\cdot h_n^{D+1}}\right)^{1/4}\right)\right)^q}\right)
\end{aligned} \tag{S3.1.44}$$

S3.1.1 A general result for $\sup_{F\in\mathcal{F}} P_F\left(\sup_{x\in\mathcal{X}} \left|\widehat{f_g}(g(x,\widehat{\theta})) - f_g(g(x,\theta_F^*))\right| \geq s_n\right)$

From (S3.1.44), $\exists\ K_1 > 0$, $K_2 > 0$ and $K_3 > 0$ such that, for any sequence $s_n > 0$ (possibly converging to zero) such that $\frac{h_n^M}{s_n} \longrightarrow 0$ and $s_n\cdot n\cdot h_n^{D+1} \longrightarrow \infty$, we have

$$\begin{aligned}
&\sup_{F\in\mathcal{F}} P_F\left(\sup_{x\in\mathcal{X}} \left|\widehat{f_g}(g(x,\widehat{\theta})) - f_g(g(x,\theta_F^*))\right| \geq s_n\right)\\
&= O\left(\frac{1}{\left((n^{1/2}\wedge r_n)\cdot h_n^{D+1}\cdot\left(\left(K_1\cdot s_n - K_2\cdot h_n^M - \frac{K_3}{n\cdot h_n^{D+1}}\right)\wedge\left(K_1\cdot s_n - K_2\cdot h_n^M - \frac{K_3}{n\cdot h_n^{D+1}}\right)^{1/4}\right)\right)^q}\right)
\end{aligned} \tag{S3.1.45}$$

[4]What follows is true more generally if we replace the constant b with a sequence $s_n > 0$ which may converge to zero as long as $h_n^M/s_n \longrightarrow 0$ and $s_n\cdot n\cdot h_n^{D+1} \longrightarrow 0$.

Recall from (S3.1.37) that, $\widehat{f_g}(g(x,\widehat{\theta})) = \widehat{f_g}(g(x,\theta_F^*)) + \xi_{a,n}^{f_g}(x,\theta_F^*) + \xi_{b,n}^{f_g}(x,\overline{\theta}_x)$. From (S3.1.19),

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{x\in\mathcal{S}_\mathcal{X}}\left|\xi_{b,n}^{f_g}(x,\overline{\theta}_x)\right| \geq b\right) = O_p\left(\frac{1}{n\cdot h_n^{D+2}}\right) \quad \text{uniformly over } \mathcal{F}.$$

From (S3.1.34) and the Hoeffding decomposition of $U_{f_g,n}$ in (S3.1.29), we have

$$\begin{aligned}\xi_{a,n}^{f_g}(x,\theta_F^*) =& \left(\frac{n-1}{n}\right)\cdot \underbrace{\left(\frac{1}{h_n^{D+1}}\cdot E_F\left[\Lambda_{f_g}(X,x,\theta_F^*,h_n)\right]\right)\cdot\frac{1}{n}\sum_{i=1}^n \psi_F^\theta(Z_i)}_{=\nu_{\varphi_{f_g,n}(x,\theta_F^*,h_n)} \text{ (see (S3.1.25) and (S3.1.27))}} + \frac{1}{h_n^{D+1}}\cdot E_F\left[\Lambda_{f_g}(X,x,\theta_F^*,h_n)\right]\varepsilon_n^\theta \\ &+ \left(\frac{n-1}{2n}\right)\cdot\frac{1}{h_n^{D+1}}\cdot\widetilde{U}_{f_g,n}(x,\theta_F^*,h_n) + \frac{1}{n\cdot h_n^{D+1}}\cdot E_F\left[\Lambda_{f_g}(X,x,\theta_F^*,h_n)\psi_F^\theta(Z)\right] \\ &+ \frac{1}{n\cdot h_n^{D+1}}\cdot m_n^{f_g}(x,\theta_F^*,h_n) + \frac{1}{h_n^{D+1}}\cdot \nu_{\Lambda_{f_g},n}(x,\theta_F^*,h_n)\varepsilon_n^\theta.\end{aligned} \tag{S3.1.46}$$

Therefore we can express

$$\xi_{a,n}^{f_g}(x,\theta_F^*) = \left(\frac{n-1}{n}\right)\cdot\left(\frac{1}{h_n^{D+1}}\cdot E_F\left[\Lambda_{f_g}(X,x,\theta_F^*,h_n)\right]\right)\cdot\frac{1}{n}\sum_{i=1}^n\psi_F^\theta(Z_i) + \frac{1}{h_n^{D+1}}\cdot E_F\left[\Lambda_{f_g}(X,x,\theta_F^*,h_n)\right]\varepsilon_n^\theta + \varsigma_{1,n}^{f_g}(x) \tag{S3.1.47}$$

where

$$\begin{aligned}\varsigma_{1,n}^{f_g}(x) \equiv& \left(\frac{n-1}{2n}\right)\cdot\frac{1}{h_n^{D+1}}\cdot\widetilde{U}_{f_g,n}(x,\theta_F^*,h_n) + \frac{1}{n\cdot h_n^{D+1}}\cdot E_F\left[\Lambda_{f_g}(X,x,\theta_F^*,h_n)\psi_F^\theta(Z)\right] \\ &+ \frac{1}{n\cdot h_n^{D+1}}\cdot m_n^{f_g}(x,\theta_F^*,h_n) + \frac{1}{h_n^{D+1}}\cdot\nu_{\Lambda_{f_g},n}(x,\theta_F^*,h_n)\varepsilon_n^\theta\end{aligned}$$

Let us analyze $\varsigma_{1,n}^{f_g}(x)$. From (S3.1.26), $\left(\frac{n-1}{2n}\right)\cdot\frac{1}{h_n^{D+1}}\cdot\sup_{x\in\mathcal{S}_X}\left|\widetilde{U}_{f_g,n}(x,\theta_F^*,h_n)\right| = O_p\left(\frac{1}{n\cdot h_n^{D+1}}\right)$. From (S3.1.32), $\frac{1}{n\cdot h_n^{D+1}}\cdot\sup_{x\in\mathcal{S}_X}\left|E_F\left[\Lambda_{f_g}(X,x,\theta_F^*,h_n)\psi_F^\theta(Z)\right]\right| \leq \frac{1}{n\cdot h_n^{D+1}}\cdot\overline{\mu}_m \ \forall\, F\in\mathcal{F}$. From (S3.1.33), $\frac{1}{n\cdot h_n^{D+1}}\cdot\sup_{x\in\mathcal{S}_X}\left|m_n^{f_g}(x,\theta_F^*,h_n)\right| = \frac{1}{n\cdot h_n^{D+1}}\cdot O_p\left(\frac{1}{n^{1/2}}\right) = O_p\left(\frac{1}{n^{3/2}\cdot h_n^{D+1}}\right)$ uniformly over $\mathcal{F}$. Let $\tau > 0$ be the constant described in Assumption 1. From the conditions described there and the result in (S3.1.23),

$$\frac{1}{h_n^{D+1}}\cdot\sup_{x\in\mathcal{S}_X}\left\|\nu_{\Lambda_{f_g},n}(x,\theta_F^*,h_n)\right\|\cdot\left\|\varepsilon_n^\theta\right\| = \frac{1}{h_n^{D+1}}\cdot O_p\left(\frac{1}{n^{1/2}}\right)\cdot o_p\left(\frac{1}{n^{1/2+\tau}}\right) = o_p\left(\frac{1}{n^{1+\tau}\cdot h_n^{D+1}}\right) \quad \text{uniformly over } \mathcal{F}$$

Therefore,

$$\sup_{x\in\mathcal{S}_X}\left|\varsigma_{1,n}^{f_g}(x)\right| = O_p\left(\frac{1}{n\cdot h_n^{D+1}}\right) + O\left(\frac{1}{n\cdot h_n^{D+1}}\right) + O_p\left(\frac{1}{n^{3/2}\cdot h_n^{D+1}}\right) + o_p\left(\frac{1}{n^{1+\tau}\cdot h_n^{D+1}}\right) = O_p\left(\frac{1}{n\cdot h_n^{D+1}}\right), \tag{S3.1.48}$$

uniformly over $\mathcal{F}$. With this result at hand, going back to (S3.1.46) we see that we need to analyze the term $\frac{1}{h_n^{D+1}} \cdot E_F\left[\Lambda_{f_g}(X,x,\theta_F^*,h_n)\right]$. From the definitions in (S3.1.4) and (S3.1.20), we have

$$\frac{1}{h_n^{D+1}} \cdot E_F\left[\Lambda_{f_g}(X,x,\theta_F^*,h_n)\right] = \left(\frac{1}{h_n^{D+1}} \cdot E_F\left[\Lambda^1_{f_g}(X,x,\theta_F^*,h_n)\right], \ldots, \frac{1}{h_n^{D+1}} \cdot E_F\left[\Lambda^k_{f_g}(X,x,\theta_F^*,h_n)\right]\right).$$

For each $\ell = 1,\ldots,k$, we have

$$\frac{1}{h_n^{D+1}} \cdot E_F\left[\Lambda^\ell_{f_g}(X,x,\theta_F^*,h_n)\right] = \sum_{d=1}^{D} \frac{1}{h_n^{D+1}} \cdot E_F\left[\nabla_d K\left(\frac{\Delta g(X,x,\theta_F^*)}{h_n}\right)\frac{\partial \Delta g_d(X,x,\theta)}{\partial \theta_\ell}\right].$$

It suffices to analyze $\frac{1}{h_n^{D+1}} \cdot E_F\left[\nabla_d K\left(\frac{\Delta g(X,x,\theta_F^*)}{h_n}\right)\frac{\partial \Delta g_d(X,x,\theta)}{\partial \theta_\ell}\right]$ for any (d,ℓ). We have,

$$\begin{aligned}
&\frac{1}{h_n^{D+1}} \cdot E_F\left[\nabla_d K\left(\frac{\Delta g(X,x,\theta_F^*)}{h_n}\right)\frac{\partial \Delta g_d(X,x,\theta_F^*)}{\partial \theta_\ell}\right]\\
=&\frac{1}{h_n^{D+1}} \cdot E_F\left[\nabla_d K\left(\frac{g(X,\theta_F^*)-g(x,\theta_F^*)}{h_n}\right)\left(\frac{\partial g_d(X,\theta_F^*)}{\partial \theta_\ell} - \frac{\partial g_d(x,\theta_F^*)}{\partial \theta_\ell}\right)\right]\\
=&\frac{1}{h_n^{D+1}} \cdot E_F\left[\nabla_d K\left(\frac{g(X,\theta_F^*)-g(x,\theta_F^*)}{h_n}\right)\frac{\partial g_d(X,\theta_F^*)}{\partial \theta_\ell}\right] - \frac{1}{h_n^{D+1}} \cdot E_F\left[\nabla_d K\left(\frac{g(X,\theta_F^*)-g(x,\theta_F^*)}{h_n}\right)\right]\frac{\partial g_d(x,\theta_F^*)}{\partial \theta_\ell}
\end{aligned}$$

As we defined in Assumption 2, for a given $g \equiv (g_1,\ldots,g_D) \in \mathcal{S}_{g,F}$, and each (ℓ,d), let

$$\Omega^{d,\ell}_{f_g}(g) = E_F\left[\frac{\partial g_d(X,\theta_F^*)}{\partial \theta_\ell}\middle| g(X,\theta_F^*) = g\right].$$

From the kernel properties described in Assumption 4, we have

$$\int \cdots \int \nabla_d K(\psi_1,\ldots,\psi_D) d\psi_1 \cdots d\psi_D = 0, \quad \int \cdots \int \psi_d \nabla_d K(\psi_1,\ldots,\psi_D) d\psi_1 \cdots d\psi_D = -1,$$

$$\int \cdots \int \psi_1^{j_1} \cdots \psi_D^{j_D} \nabla_d K(\psi_1,\ldots,\psi_D) d\psi_1 \cdots d\psi_D = 0 \quad \forall\ (j_1,\ldots,j_D) : \begin{cases} \sum_{s=1}^{D} j_s \leq M, \\ j_\ell \neq 0 \text{ for some } \ell \neq d, \text{ or } j_d \neq 1 \end{cases}$$

$$\int \cdots \int \left|\psi_1\right|^{j_1} \cdots \left|\psi_D\right|^{j_D} \cdot \left|\nabla_d K(\psi_1,\ldots,\psi_D)\right| d\psi_1 \cdots d\psi_D < \infty \quad \forall\ (j_1,\ldots,j_D) : \sum_{s=1}^{D} j_s = M+1.$$

For a given $x \in \mathcal{X}$ let

$$\Xi_{\ell,f_g}(x,\theta_F^*) \equiv \sum_{d=1}^{D}\left(\frac{\partial g_d(x,\theta_F^*)}{\partial\theta_\ell}\cdot\frac{\partial f_g(g(x,\theta_F^*))}{\partial g_d} - \frac{\partial\left[\Omega_{f_g}^{d,\ell}(g(x,\theta_F^*))\cdot f_g(g(x,\theta_F^*))\right]}{\partial g_d}\right),$$

$$\underbrace{\Xi_{f_g}(x,\theta_F^*)}_{1\times k} \equiv \left(\Xi_{1,f_g}(x,\theta_F^*),\ldots,\Xi_{k,f_g}(x,\theta_F^*)\right)$$

From the smoothness conditions in Assumption 2, an M^{th}-order approximation yields that there exists a constant $\overline{B}_{2,f} < \infty$ such that

$$\frac{1}{h_n^{D+1}}\cdot E_F\left[\Lambda_{f_g}(X,x,\theta_F^*,h_n)\right] = \Xi_{f_g}(x,\theta_F^*) + B_n^{\Xi_{f_g}}(x), \quad \text{where} \quad \sup_{x\in\mathcal{X}}\left\|B_n^{\Xi_{f_g}}(x)\right\| \le \overline{B}_{2,f}\cdot h_n^M \quad \forall\, F\in\mathcal{F}. \tag{S3.1.49}$$

By Assumption 2, $\exists\ \overline{\mu}_{\Xi_{f_g}} < \infty$ such that

$$\sup_{x\in\mathcal{X}}\left\|\Xi_{f_g}(x,\theta_F^*)\right\| \le \overline{\mu}_{\Xi_{f_g}} \quad \forall\, F\in\mathcal{F}. \tag{S3.1.50}$$

Plugging (S3.1.49) into (S3.1.47),

$$\xi_{a,n}^{f_g}(x,\theta_F^*) = \left(\frac{n-1}{n}\right)\cdot\left(\Xi_{f_g}(x,\theta_F^*) + B_n^{\Xi_{f_g}}(x)\right)\frac{1}{n}\sum_{i=1}^{n}\psi_F^\theta(Z_i) + \left(\Xi_{f_g}(x,\theta_F^*) + B_n^{\Xi_{f_g}}(x)\right)\varepsilon_n^\theta + \varsigma_{1,n}^{f_g}(x).$$

From here we can express

$$\xi_{a,n}^{f_g}(x,\theta_F^*) = \Xi_{f_g}(x,\theta_F^*)\frac{1}{n}\sum_{i=1}^{n}\psi_F^\theta(Z_i) + \varsigma_{2,n}^{f_g}(x) + \varsigma_{1,n}^{f_g}(x), \tag{S3.1.51}$$

where $\varsigma_{2,n}^{f_g}(x) \equiv \left(\left(\frac{n-1}{n}\right)\cdot B_n^{\Xi_{f_g}}(x) - \frac{1}{n}\right)\cdot\frac{1}{n}\sum_{i=1}^{n}\psi_F^\theta(Z_i) + \left(\Xi_{f_g}(x,\theta_F^*) + B_n^{\Xi_{f_g}}(x)\right)\varepsilon_n^\theta$. Let $\tau > 0$ be as described in Assumption 1. From the conditions described there, (S3.1.49) and (S3.1.50) yield,

$$\begin{aligned}\sup_{x\in\mathcal{X}}\left|\varsigma_{2,n}^{f_g}(x)\right| &= \left(O\left(h_n^M\right) + O\left(\frac{1}{n}\right)\right)\cdot O_p\left(\frac{1}{n^{1/2}}\right) + \left(O(1) + O\left(h_n^M\right)\right)\cdot o_p\left(\frac{1}{n^{1/2+\tau}}\right)\\ &= O_p\left(\max\left\{\frac{h_n^M}{n^{1/2}}\,,\,\frac{1}{n^{3/2}}\,,\,\frac{1}{n^{1/2+\tau}}\right\}\right) \quad \text{uniformly over } \mathcal{F}\end{aligned} \tag{S3.1.52}$$

Combining (S3.1.48) and (S3.1.52) and defining $\varsigma_n^{f_g}(x) \equiv \varsigma_{1,n}^{f_g}(x) + \varsigma_{2,n}^{f_g}(x)$, (S3.1.51) becomes

$$\begin{aligned}
&\xi_{a,n}^{f_g}(x,\theta_F^*) = \Xi_{f_g}(x,\theta_F^*) \cdot \frac{1}{n}\sum_{i=1}^{n} \psi_F^{\theta}(Z_i) + \varsigma_n^{f_g}(x), \\
&\text{where} \quad \sup_{x\in\mathcal{X}} \left|\varsigma_n^{f_g}(x)\right| = O_p\left(\max\left\{\frac{h_n^M}{n^{1/2}}\,,\,\frac{1}{n^{3/2}}\,,\,\frac{1}{n^{1/2+\tau}}\,,\,\frac{1}{n\cdot h_n^{D+1}}\right\}\right) \quad \text{uniformly over } \mathcal{F}
\end{aligned} \tag{S3.1.53}$$

S3.1.2 A uniform linear representation result for $\widehat{f_g}(g(x,\widehat{\theta})) - f_g(g(x,\theta_F^*))$

Let

$$\psi_F^{f_g}(V_i,x,\theta_F^*,h_n) \equiv \frac{1}{h_n^D}\cdot\left(K\left(\frac{\Delta g(X_i,x,\theta_F^*)}{h_n}\right) - E_F\left[K\left(\frac{\Delta g(X_i,x,\theta_F^*)}{h_n}\right)\right]\right) + \Xi_{f_g}(x,\theta_F^*)\psi_F^{\theta}(Z_i). \tag{S3.1.54}$$

Combining (S3.1.41) and (S3.1.53), we have

$$\begin{aligned}
&\widehat{f_g}(g(x,\widehat{\theta})) = f_g(g(x,\theta_F^*)) + \frac{1}{n}\sum_{i=1}^{n}\psi_F^{f_g}(V_i,x,\theta_F^*,h_n) + \zeta_n^{f_g}(x), \\
&\text{where} \quad \zeta_n^{f_g}(x) \equiv B_n^{f_g}(x) + \varsigma_n^{f_g}(x) + \xi_{b,n}^{f_g}(x,\overline{\theta}_x)
\end{aligned} \tag{S3.1.55}$$

Let $\epsilon > 0$ be as described in Assumption 4. (S3.1.19), (S3.1.42) and (S3.1.53) yield,

$$\begin{aligned}
\sup_{x\in\mathcal{X}} \left|\zeta_n^{f_g}(x)\right| &= O\left(h_n^M\right) + O_p\left(\max\left\{\frac{h_n^M}{n^{1/2}}\,,\,\frac{1}{n^{3/2}}\,,\,\frac{1}{n^{1/2+\tau}}\right\}\right) + O_p\left(\frac{1}{n\cdot h_n^{D+2}}\right) \\
&= O_p\left(\max\left\{h_n^M\,,\,\frac{1}{n^{3/2}}\,,\,\frac{1}{n^{1/2+\tau}}\,,\,\frac{1}{n\cdot h_n^{D+2}}\right\}\right) = o_p\left(\frac{1}{n^{1/2+\epsilon}}\right) \quad \text{uniformly over } \mathcal{F}
\end{aligned} \tag{S3.1.56}$$

By the conditions in Assumptions 1 and 4 along with the results in (S3.1.40) and (S3.1.50), we can once again invoke Result S1 to show that

$$\sup_{x\in\mathcal{X}} \left|\frac{1}{n}\sum_{i=1}^{n}\psi_F^{f_g}(V_i,x,\theta_F^*,h_n)\right| = O_p\left(\frac{1}{n^{1/2}\cdot h_n^D}\right) \quad \text{uniformly over } \mathcal{F}. \tag{S3.1.57}$$

And therefore,

$$\sup_{x\in\mathcal{X}} \left|\widehat{f_g}(g(x,\widehat{\theta})) - f_g(g(x,\theta_F^*))\right| = O_p\left(\frac{1}{n^{1/2}\cdot h_n^D}\right) + o_p\left(\frac{1}{n^{1/2+\epsilon}}\right) = O_p\left(\frac{1}{n^{1/2}\cdot h_n^D}\right) \quad \text{uniformly over } \mathcal{F}. \tag{S3.1.58}$$

From our previous results and Assumption 2, it also follows that

$$\sup_{x\in\mathcal{X}}\left|\frac{1}{\widehat{f}_g(g(x,\widehat{\theta}))}\right| = O_p(1) \quad \text{uniformly over } \mathcal{F}. \tag{S3.1.59}$$

To see why, recall from Assumption 2 that $\inf_{x\in\mathcal{X}} f_g(g(x,\theta_F^*)) \geq \underline{f}_g > 0$ for all $F\in\mathcal{F}$. Take any $\delta\in(0,1)$. Note that, for any $x\in\mathcal{X}$, $\widehat{f}_g(g(x,\widehat{\theta})) < (1-\delta)\cdot\underline{f}_g$ only if $\left|\widehat{f}_g(g(x,\widehat{\theta})) - f_g(g(x,\theta_F^*))\right| > \delta\cdot\underline{f}_g$. Therefore,

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{x\in\mathcal{X}}\left|\frac{1}{\widehat{f}_g(g(x,\widehat{\theta}))}\right| > \frac{1}{(1-\delta)\cdot\underline{f}_g}\right) \leq \sup_{F\in\mathcal{F}} P_F\left(\sup_{x\in\mathcal{X}}\left|\widehat{f}_g(g(x,\widehat{\theta})) - f_g(g(x,\theta_F^*))\right| > \delta\cdot\underline{f}_g\right)$$

From (S3.1.45), $\exists\ \overline{M}_{f_g} > 0$, $K_1>0$, $K_2>0$ and $K_3>0$ s.t, for each $c>0$, $\exists\ n_c$ such that

$$\begin{aligned}&\sup_{F\in\mathcal{F}} P_F\left(\sup_{x\in\mathcal{X}}\left|\widehat{f}_g(g(x,\widehat{\theta})) - f_g(g(x,\theta_F^*))\right| \geq c\right)\\ &\leq \frac{\overline{M}_{f_g}}{\left((n^{1/2}\wedge r_n)\cdot h_n^{D+1}\cdot\left(\left(K_1\cdot c - K_2\cdot h_n^M - \frac{K_3}{n\cdot h_n^{D+1}}\right)\wedge\left(K_1\cdot c - K_2\cdot h_n^M - \frac{K_3}{n\cdot h_n^{D+1}}\right)^{1/4}\right)\right)^q} \quad \forall\ n > n_c\end{aligned}$$

Thus, for any $\delta\in(0,1)$ there exists n_δ such that

$$\begin{aligned}&\sup_{F\in\mathcal{F}} P_F\left(\sup_{x\in\mathcal{X}}\left|\frac{1}{\widehat{f}_g(g(x,\widehat{\theta}))}\right| > \frac{1}{(1-\delta)\cdot\underline{f}_g}\right)\\ &\leq \frac{\overline{M}_{f_g}}{\left((n^{1/2}\wedge r_n)\cdot h_n^{D+1}\cdot\left(\left(K_1\cdot\delta\cdot\underline{f}_g - K_2\cdot h_n^M - \frac{K_3}{n\cdot h_n^{D+1}}\right)\wedge\left(K_1\cdot\delta\cdot\underline{f}_g - K_2\cdot h_n^M - \frac{K_3}{n\cdot h_n^{D+1}}\right)^{1/4}\right)\right)^q} \quad \forall\ n > n_\delta\end{aligned}$$

Now take any $\epsilon>0$ and let $n_{\delta,\epsilon}$ be the smallest integer such that

$$\frac{\overline{M}_{f_g}}{\left((n^{1/2}\wedge r_n)\cdot h_n^{D+1}\cdot\left(\left(K_1\cdot\delta\cdot\underline{f}_g - K_2\cdot h_n^M - \frac{K_3}{n\cdot h_n^{D+1}}\right)\wedge\left(K_1\cdot\delta\cdot\underline{f}_g - K_2\cdot h_n^M - \frac{K_3}{n\cdot h_n^{D+1}}\right)^{1/4}\right)\right)^q} < \epsilon.$$

Thus, for any $\epsilon>0$ and $\delta\in(0,1)$, there exists $n_{\delta,\epsilon}$ such that

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{x\in\mathcal{X}}\left|\frac{1}{\widehat{f}_g(g(x,\widehat{\theta}))}\right| > \frac{1}{(1-\delta)\cdot\underline{f}_g}\right) \leq \epsilon \quad \forall\ n > n_{\delta,\epsilon},$$

which proves (S3.1.59).

S3.2 Asymptotic properties of $\widehat{R}_p(x,t,\widehat{\theta})$

Recall that $\widehat{R}_p(x,t,\theta) = \frac{1}{n\cdot h_n^D}\sum_{i=1}^n S_p(Y_i,t)\omega_p(g(X_i,\theta))K\left(\frac{\Delta g(X_i,x,\theta)}{h_n}\right)$, which is an estimator for the functional $R_{p,F}(x,t,\theta_F^*) \equiv \Gamma_{p,F}(x,t,\theta_F^*)\cdot\omega_p(g(x,\theta_F^*))\cdot f_g(g(x,\theta_F^*))$. The purpose of this section is to derive the type of asymptotic results we obtained in Sections S3.1.1-S3.1.2 for $\widehat{R}_p(x,t,\theta) - R_{p,F}(x,t,\theta_F^*)$. Our analysis will closely follow parallel steps to our study of the asymptotic properties of $\widehat{f_g}(g(x,\widehat{\theta}))$ in Section S3.1. For this reason, **readers who are familiar with the steps we took in Section S3.1 may wish to skip directly to Section S3.3.** As we did in Section S3.1, we begin with a second-order approximation,

$$\widehat{R}_p(x,t,\widehat{\theta}) = \widehat{R}_p(x,t,\theta_F^*) + \frac{\partial\widehat{R}_p(x,t,\theta_F^*)}{\partial\theta}\left(\widehat{\theta}-\theta_F^*\right) + \frac{1}{2}\left(\widehat{\theta}-\theta_F^*\right)'\frac{\partial^2\widehat{R}_p(x,t,\overline{\theta}_x)}{\partial\theta\partial\theta'}\left(\widehat{\theta}-\theta_F^*\right) \qquad \text{(S3.2.1)}$$

where $\overline{\theta}_x$ belongs in the line segment connecting $\widehat{\theta}$ and θ_F^*. Thus, since Θ is taken to be convex, we have $\overline{\theta}_x \in \Theta$. Parallel to the definitions in (S3.1.5), let

$$\begin{aligned}
\Upsilon_{R_p}^{\ell,m}(V_i,x,t,\theta,h) &\equiv S_p(Y_i,t)\omega_p(g(X_i,\theta))\sum_{d=1}^D\sum_{s=1}^D \nabla_{ds}K\left(\frac{\Delta g(X_i,x,\theta)}{h}\right)\frac{\partial\Delta g_d(X_i,x,\theta)}{\partial\theta_\ell}\frac{\partial\Delta g_s(X_i,x,\theta)}{\partial\theta_m},\\
\Phi_{R_p}^{\ell,m}(V_i,x,t,\theta,h) &\equiv S_p(Y_i,t)\sum_{d=1}^D \nabla_d K\left(\frac{\Delta g(X_i,x,\theta)}{h}\right)\times\left\{\omega_p(g(X_i,\theta))\frac{\partial^2\Delta g_d(X_i,x,\theta)}{\partial\theta_m\partial\theta_\ell}\right.\\
&\qquad \left. + \frac{\partial\omega_p(g(X_i,\theta)}{\partial\theta_\ell}\frac{\partial\Delta g_d(X_i,x,\theta)}{\partial\theta_m} + \frac{\partial\omega_p(g(X_i,\theta)}{\partial\theta_m}\frac{\partial\Delta g_d(X_i,x,\theta)}{\partial\theta_\ell}\right\},\\
\rho_{R_p}^{\ell,m}(V_i,x,t,\theta,h) &\equiv S_p(Y_i,t)\frac{\partial^2\omega_p(g(X_i,\theta))}{\partial\theta_m\partial\theta_\ell}K\left(\frac{\Delta g(X_i,x,\theta)}{h_n}\right).
\end{aligned}$$

By the conditions in Assumptions 3 and 4, $\exists\ \overline{\eta}_\Upsilon < \infty$, $\overline{\eta}_\Phi < \infty$ and $\overline{\eta}_\rho < \infty$ such that, $\forall\ F \in \mathcal{F}$,

$$\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}}\left|E_F\left[\Upsilon_{R_p}^{\ell,m}(V,x,t,\theta,h)\right]\right| \le \overline{\eta}_\Upsilon \qquad \sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}}\left|E_F\left[\Phi_{R_p}^{\ell,m}(V,x,t,\theta,h)\right]\right| \le \overline{\eta}_\Phi$$

$$\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}}\left|E_F\left[\rho_{R_p}^{\ell,m}(V,x,t,\theta,h)\right]\right| \le \overline{\eta}_\rho$$

As we stated in the paragraph following Assumption 3, the conditions there, combined with Pakes and Pollard (1989, Lemma 2.13) imply that, for each (ℓ, d), the classes of functions

$$\mathscr{H}_{1d}^{\ell} = \left\{ m : \mathcal{S}_X \longrightarrow \mathbb{R} \colon m(x) = \frac{\partial \Delta g_d(x,s,\theta)}{\partial \theta_\ell} \text{ for some } s \in \mathcal{S}_X,\, \theta \in \Theta \right\},$$
$$\mathscr{H}_{2d}^{\ell,m} = \left\{ m : \mathcal{S}_X \longrightarrow \mathbb{R} \colon m(x) = \frac{\partial^2 \Delta g_d(x,s,\theta)}{\partial \theta_\ell \partial \theta_m} \text{ for some } s \in \mathcal{S}_X,\, \theta \in \Theta \right\},$$
$$\mathscr{H}_{3}^{\ell} = \left\{ m : \mathcal{S}_X \longrightarrow \mathbb{R} \colon m(x) = \frac{\partial \omega_p(x,s,\theta)}{\partial \theta_\ell} \text{ for some } s \in \mathcal{S}_X,\, \theta \in \Theta \right\},$$
$$\mathscr{H}_{4}^{\ell,m} = \left\{ m : \mathcal{S}_X \longrightarrow \mathbb{R} \colon m(x) = \frac{\partial^2 \omega_p(x,s,\theta)}{\partial \theta_\ell \partial \theta_m} \text{ for some } s \in \mathcal{S}_X,\, \theta \in \Theta \right\}$$

are Euclidean for an envelope $G_1(x)$ that satisfies $E_F\left[G_1(X)^{4q}\right] \le \overline{\mu}_{G_1} < \infty$ for all $F \in \mathcal{F}$, with q being the integer described in Assumption 1. Next, consider the following three classes of functions

$$\mathscr{G}_{1,p}^{\ell,m} = \left\{ r : \mathcal{S}_V \longrightarrow \mathbb{R} \colon r(v) = \Upsilon_{R_p}^{\ell,m}(v,u,t,\theta,h) \text{ for some } u \in \mathcal{S}_X,\, t \in \mathcal{T},\, \theta \in \Theta,\, h > 0 \right\},$$
$$\mathscr{G}_{2,p}^{\ell,m} = \left\{ r : \mathcal{S}_V \longrightarrow \mathbb{R} \colon r(v) = \Phi_{R_p}^{\ell,m}(v,u,t,\theta,h) \text{ for some } u \in \mathcal{S}_X,\, t \in \mathcal{T},\, \theta \in \Theta,\, h > 0 \right\},$$
$$\mathscr{G}_{3,p}^{\ell,m} = \left\{ r : \mathcal{S}_V \longrightarrow \mathbb{R} \colon r(v) = \rho_{R_p}^{\ell,m}(v,u,t,\theta,h) \text{ for some } u \in \mathcal{S}_X,\, t \in \mathcal{T},\, \theta \in \Theta,\, h > 0 \right\}.$$

The conditions in Assumptions 3 and 4 and the Euclidean-preserving properties described in Pakes and Pollard (1989, Lemma 2.14), imply that $\mathscr{G}_{1,p}^{\ell,m}$, $\mathscr{G}_{2,p}^{\ell,m}$ and $\mathscr{G}_{3,p}^{\ell,m}$ are Euclidean for an envelope $\overline{G}(\cdot)$ s.t $\exists\ \overline{\eta}_{\overline{G}} < \infty$ s.t $E_F\left[\overline{G}(V)^{4q}\right] \le \overline{\eta}_{\overline{G}}$ for all $F \in \mathcal{F}$. Let

$$\nu_{\Upsilon_{R_p},n}^{\ell,m}(x,t,\theta,h) = \frac{1}{n}\sum_{i=1}^{n}\left(\Upsilon_{R_p}^{\ell,m}(V_i,x,t,\theta,h) - E_F\left[\Upsilon_{R_p}^{\ell,m}(V_i,x,t,\theta,h)\right]\right),$$
$$\nu_{\Phi_{R_p},n}^{\ell,m}(x,t,\theta,h) = \frac{1}{n}\sum_{i=1}^{n}\left(\Phi_{R_p}^{\ell,m}(V_i,x,t,\theta,h) - E_F\left[\Phi_{R_p}^{\ell,m}(V_i,x,t,\theta,h)\right]\right),$$
$$\nu_{\rho_{R_p},n}^{\ell,m}(x,t,\theta,h) = \frac{1}{n}\sum_{i=1}^{n}\left(\rho_{R_p}^{\ell,m}(V_i,x,t,\theta,h) - E_F\left[\rho_{R_p}^{\ell,m}(V_i,x,t,\theta,h)\right]\right)$$

From here, Result S1 implies that $\exists\ \overline{M}_1 < \infty$ s.t, $\forall\ b > 0$,

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}}\left|\nu^{\ell,m}_{\Upsilon_{R_p},n}(x,t,\theta,h)\right|\geq b\right)\leq\frac{\overline{M}_1}{\left(n^{1/2}\cdot b\right)^q}=O\left(\frac{1}{\left(n^{1/2}\cdot b\right)^q}\right)$$

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}}\left|\nu^{\ell,m}_{\Phi_{R_p},n}(x,t,\theta,h)\right|\geq b\right)\leq\frac{\overline{M}_1}{\left(n^{1/2}\cdot b\right)^q}=O\left(\frac{1}{\left(n^{1/2}\cdot b\right)^q}\right),\tag{S3.2.2}$$

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}}\left|\nu^{\ell,m}_{\rho_{R_p},n}(x,t,\theta,h)\right|\geq b\right)\leq\frac{\overline{M}_1}{\left(n^{1/2}\cdot b\right)^q}=O\left(\frac{1}{\left(n^{1/2}\cdot b\right)^q}\right)$$

Note that (S3.2.2) implies, in particular, that

$$\left.\begin{array}{l}\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}}\left|\nu^{\ell,m}_{\Upsilon_{R_p},n}(x,t,\theta,h)\right|=O_p\left(\frac{1}{n^{1/2}}\right)\\ \sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}}\left|\nu^{\ell,m}_{\Phi_{R_p},n}(x,t,\theta,h)\right|=O_p\left(\frac{1}{n^{1/2}}\right)\\ \sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}}\left|\nu^{\ell,m}_{\rho_{R_p},n}(x,t,\theta,h)\right|=O_p\left(\frac{1}{n^{1/2}}\right)\end{array}\right\}\text{ uniformly over }\mathcal{F}.\tag{S3.2.3}$$

Denote $\xi^{R_p}_{b,n}(x,t,\theta)\equiv\left(\widehat{\theta}-\theta^*_F\right)'\frac{\partial^2\widehat{R}_p(x,t,\theta)}{\partial\theta\partial\theta'}\left(\widehat{\theta}-\theta^*_F\right)$. We have

$$\begin{aligned}\xi^{R_p}_{b,n}(x,t,\theta)&=\sum_{\ell=1}^k\sum_{m=1}^k\frac{\partial^2\widehat{R}_p(x,t,\theta)}{\partial\theta_\ell\theta_m}\cdot\left(\widehat{\theta}_m-\theta^*_{m,F}\right)\cdot\left(\widehat{\theta}_\ell-\theta^*_{\ell,F}\right)\\
&=\sum_{\ell=1}^k\sum_{m=1}^k\frac{1}{h_n^{D+2}}\cdot E_F\left[\Upsilon^{\ell,m}_{R_p}(V,x,t,\theta,h_n)\right]\cdot\left(\widehat{\theta}_m-\theta^*_{m,F}\right)\cdot\left(\widehat{\theta}_\ell-\theta^*_{\ell,F}\right)+\sum_{\ell=1}^k\sum_{m=1}^k\frac{1}{h_n^{D+2}}\cdot\nu^{\ell,m}_{\Upsilon_{R_p},n}(x,t,\theta,h_n)\cdot\left(\widehat{\theta}_m-\theta^*_{m,F}\right)\cdot\left(\widehat{\theta}_\ell-\theta^*_{\ell,F}\right)\\
&+\sum_{\ell=1}^k\sum_{m=1}^k\frac{1}{h_n^{D+1}}\cdot E_F\left[\Phi^{\ell,m}_{R_p}(V,x,t,\theta,h_n)\right]\cdot\left(\widehat{\theta}_m-\theta^*_{m,F}\right)\cdot\left(\widehat{\theta}_\ell-\theta^*_{\ell,F}\right)+\sum_{\ell=1}^k\sum_{m=1}^k\frac{1}{h_n^{D+1}}\cdot\nu^{\ell,m}_{\Phi_{R_p},n}(x,t,\theta,h_n)\cdot\left(\widehat{\theta}_m-\theta^*_{m,F}\right)\cdot\left(\widehat{\theta}_\ell-\theta^*_{\ell,F}\right)\\
&+\sum_{\ell=1}^k\sum_{m=1}^k\frac{1}{h_n^{D}}\cdot E_F\left[\rho^{\ell,m}_{R_p}(V,x,t,\theta,h_n)\right]\cdot\left(\widehat{\theta}_m-\theta^*_{m,F}\right)\cdot\left(\widehat{\theta}_\ell-\theta^*_{\ell,F}\right)+\sum_{\ell=1}^k\sum_{m=1}^k\frac{1}{h_n^{D}}\cdot\nu^{\ell,m}_{\rho_{R_p},n}(x,t,\theta,h_n)\cdot\left(\widehat{\theta}_m-\theta^*_{m,F}\right)\cdot\left(\widehat{\theta}_\ell-\theta^*_{\ell,F}\right).\end{aligned}\tag{S3.2.4}$$

Take any $b>0$. From (S3.2.4), using the results in (S3.2.2) and (S3.2.3) and the conditions in Assumption 1, parallel steps to those we used in equations (S3.1.14)-(S3.1.17) lead us to the following result, which is analogous to that in (S3.1.19)

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{(x,t)\in\mathcal{S}_X\times\mathcal{T}}\left|\xi^{R_p}_{b,n}(x,t,\overline{\theta}_x)\right|\geq b\right)\leq \sup_{F\in\mathcal{F}} P_F\left(\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}}}\left|\xi^{R_p}_{b,n}(x,t,\theta)\right|\geq b\right)$$
$$= O\left(\frac{1}{\left(h_n^{\frac{D+2}{2}}\cdot(n^{1/2}\wedge r_n)\cdot(b^{1/2}\wedge b^{1/4})\right)^q}\right)\quad\forall\, b>0,\quad\text{and}$$
$$\sup_{(x,t)\in\mathcal{S}_X\times\mathcal{T}}\left|\xi^{R_p}_{b,n}(x,t,\overline{\theta}_x)\right|\leq \sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}}}\left|\xi^{R_p}_{b,n}(x,t,\theta)\right| = O_p\left(\frac{1}{n\cdot h_n^{D+2}}\right)\quad\text{uniformly over }\mathcal{F}. \tag{S3.2.5}$$

Denote $\xi^{R_p}_{a,n}(x,t,\theta)\equiv\frac{\partial\widehat{R}_p(x,t,\theta)}{\partial\theta}\left(\widehat{\theta}-\theta^*_F\right)$. The decomposition of $\xi^{R_p}_{a,n}(x,t,\theta)$ will be described in detail in equation (S3.2.18). First, we need to introduce each one of the terms that appear there, as well as their relevant asymptotic properties. For each $p=1,\ldots,P$ define

$$\Lambda^{\ell}_{R_p}(V_i,x,t,\theta,h)\equiv S_p(Y_i,t)\omega_p(g(X_i,\theta))\sum_{d=1}^{D}\nabla_d K\left(\frac{\Delta g(X_i,x,\theta)}{h}\right)\frac{\partial\Delta g_d(X_i,x,\theta)}{\partial\theta_\ell},$$
$$\beta^{\ell}_{R_p}(V_i,x,t,\theta,h)\equiv S_p(Y_i,t)\frac{\partial\omega_p(g(X_i,\theta))}{\partial\theta_\ell}K\left(\frac{\Delta g(X_i,x,\theta)}{h}\right),$$
$$\underbrace{\Lambda_{R_p}(V_i,x,t,\theta,h)}_{1\times k}\equiv\left(\Lambda^1_{R_p}(V_i,x,t,\theta,h),\ldots,\Lambda^k_{R_p}(V_i,x,t,\theta,h)\right),$$
$$\underbrace{\beta_{R_p}(V_i,x,t,\theta,h)}_{1\times k}\equiv\left(\beta^1_{R_p}(V_i,x,t,\theta,h),\ldots,\beta^k_{R_p}(V_i,x,t,\theta,h)\right).$$

By the conditions in Assumptions 3 and 4, there exist $\overline{\eta}_\Lambda<\infty$ and $\overline{\eta}_\beta<\infty$ such that, $\forall\, F\in\mathcal{F}$,

$$\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}}\left\|E_F\left[\Lambda_{R_p}(V,x,t,\theta,h)\right]\right\|\leq\overline{\eta}_\Lambda\qquad\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}}\left\|E_F\left[\beta_{R_p}(V,x,t,\theta,h)\right]\right\|\leq\overline{\eta}_\beta. \tag{S3.2.6}$$

From the Euclidean properties of $\mathscr{G}^{\ell,m}_{1,p}$, $\mathscr{G}^{\ell,m}_{2,p}$ and $\mathscr{G}^{\ell,m}_{3,p}$, the following classes are also Euclidean,

$$\mathscr{M}^{\ell}_{4,p}=\left\{r:\mathcal{S}_V\longrightarrow\mathbb{R}: r(v)=\Lambda^{\ell}_{R_p}(v,u,t,\theta,h)\ \text{ for some } u\in\mathcal{S}_X,\, t\in\mathcal{T},\,\theta\in\Theta,\, h>0\right\},$$
$$\mathscr{M}^{\ell}_{5,p}=\left\{r:\mathcal{S}_V\longrightarrow\mathbb{R}: r(v)=\beta^{\ell}_{R_p}(v,u,t,\theta,h)\ \text{ for some } u\in\mathcal{S}_X,\, t\in\mathcal{T},\,\theta\in\Theta,\, h>0\right\}$$

for an envelope $\overline{M}_1(\cdot)$ such that $E_F\left[\overline{M}_1(V)^{4q}\right] \leq \overline{\eta}_{\overline{M}_1}$ for all $F \in \mathcal{F}$. From here, if we define

$$\begin{aligned}
\nu^{\ell}_{\Lambda_{R_p},n}(x,t,\theta,h) &= \frac{1}{n}\sum_{i=1}^{n}\left(\Lambda^{\ell}_{R_p}(V_i,x,t,\theta,h) - E_F\left[\Lambda^{\ell}_{R_p}(V_i,x,t,\theta,h)\right]\right),\\
\nu^{\ell}_{\beta_{R_p},n}(x,t,\theta,h) &= \frac{1}{n}\sum_{i=1}^{n}\left(\Lambda^{\ell}_{R_p}(V_i,x,t,\theta,h) - E_F\left[\Lambda^{\ell}_{R_p}(V_i,x,t,\theta,h)\right]\right),\\
\underbrace{\nu_{\Lambda_{R_p},n}(x,t,\theta,h)}_{1\times k} &= \left(\nu^{1}_{\Lambda_{R_p},n}(x,t,\theta,h),\ldots,\nu^{k}_{\Lambda_{R_p},n}(x,t,\theta,h)\right),\\
\underbrace{\nu_{\beta_{R_p},n}(x,t,\theta,h)}_{1\times k} &= \left(\nu^{1}_{\beta_{R_p},n}(x,t,\theta,h),\ldots,\nu^{k}_{\beta_{R_p},n}(x,t,\theta,h)\right),
\end{aligned}$$

then, Result S1 yields that, for any $b > 0$,

$$\begin{aligned}
\sup_{F\in\mathcal{F}} P_F\left(\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}}\left\|\nu_{\Lambda_{R_p},n}(x,t,\theta,h)\right\| \geq b\right) &= O\left(\frac{1}{\left(n^{1/2}\cdot b\right)^q}\right),\\
\sup_{F\in\mathcal{F}} P_F\left(\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}}\left\|\nu_{\beta_{R_p},n}(x,t,\theta,h)\right\| \geq b\right) &= O\left(\frac{1}{\left(n^{1/2}\cdot b\right)^q}\right), \quad \text{and}
\end{aligned} \tag{S3.2.7}$$

$$\left.\begin{aligned}
\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}}\left\|\nu_{\Lambda_{R_p},n}(x,t,\theta,h)\right\| &= O_p\left(\tfrac{1}{n^{1/2}}\right)\\
\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}}\left\|\nu_{\beta_{R_p},n}(x,t,\theta,h)\right\| &= O_p\left(\tfrac{1}{n^{1/2}}\right)
\end{aligned}\right\}\ \text{uniformly over } \mathcal{F}. \tag{S3.2.8}$$

Note that, under our assumptions, there exist $\overline{\eta}_{m,\Lambda} < \infty$ and $\overline{\eta}_{m,\beta} < \infty$ such that, for all $F \in \mathcal{F}$,

$$\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}}\left|E_F\left[\Lambda_{R_p}(V_i,x,t,\theta,h)\psi^{\theta}_F(Z_i)\right]\right| \leq \overline{\eta}_{m,\Lambda}, \qquad \sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}}\left|E_F\left[\Lambda_{R_p}(V_i,x,t,\theta,h)\psi^{\theta}_F(Z_i)\right]\right| \leq \overline{\eta}_{m,\beta} \tag{S3.2.9}$$

Recall that we have defined $v \equiv (y,x,z)$. Take the classes of functions

$$\mathscr{M}^p_{6,F} = \Big\{ m : \mathcal{S}_V \longrightarrow \mathbb{R}\colon\ m(v_1) = \Lambda_{R_p}(v_1,u,t,\theta,h)\psi^\theta_F(z_1) \text{ for some } u \in \mathcal{S}_X,\ t \in \mathcal{T},\ \theta \in \Theta,\ h > 0 \Big\},$$
$$\mathscr{M}^p_{7,F} = \Big\{ m : \mathcal{S}_V \longrightarrow \mathbb{R}\colon\ m(v_1) = \beta_{R_p}(v_1,u,t,\theta,h)\psi^\theta_F(z_1) \text{ for some } u \in \mathcal{S}_X,\ t \in \mathcal{T},\ \theta \in \Theta,\ h > 0 \Big\}$$

By the properties of the classes $\mathscr{M}^\ell_{4,p}$ and $\mathscr{M}^\ell_{5,p}$, and the integrability properties of the influence function $\psi^\theta_F(Z)$ in Assumption 1, Pakes and Pollard (1989, Lemma 2.14) implies that both $\mathscr{M}^p_{6,F}$ and $\mathscr{M}^p_{7,F}$ are Euclidean for an envelope $\overline{M}_2(\cdot)$ s.t $E_F\left[\overline{M}_2(V)^{4q}\right] \le \overline{\eta}_{\overline{M}_2} < \infty\ \forall\ F \in \mathcal{F}$. Next, let

$$m^{R_p}_{\Lambda,n}(x,t,\theta,h) = \frac{1}{n}\sum_{i=1}^n \Big(\Lambda_{R_p}(V_i,x,t,\theta,h)\psi^\theta_F(Z_i) - E_F\Big[\Lambda_{R_p}(V_i,x,t,\theta,h)\psi^\theta_F(Z_i)\Big]\Big),$$
$$m^{R_p}_{\beta,n}(x,t,\theta,h) = \frac{1}{n}\sum_{i=1}^n \Big(\beta_{R_p}(V_i,x,t,\theta,h)\psi^\theta_F(Z_i) - E_F\Big[\beta_{R_p}(V_i,x,t,\theta,h)\psi^\theta_F(Z_i)\Big]\Big),$$

By the Euclidean properties of $\mathscr{M}^p_{6,F}$ and $\mathscr{M}^p_{7,F}$, Result S1 yields that, for any $b > 0$,

$$\begin{aligned}
&\sup_{F\in\mathcal{F}} P_F\left(\sup_{\substack{\theta\in\Theta \\ (x,t)\in\mathcal{S}_X\times\mathcal{T} \\ h>0}} \left| m^{R_p}_{\Lambda,n}(x,t,\theta,h)\right| \ge b \right) = O\left(\frac{1}{\left(n^{1/2}\cdot b\right)^q}\right), \\
&\sup_{F\in\mathcal{F}} P_F\left(\sup_{\substack{\theta\in\Theta \\ (x,t)\in\mathcal{S}_X\times\mathcal{T} \\ h>0}} \left| m^{R_p}_{\beta,n}(x,t,\theta,h)\right| \ge b \right) = O\left(\frac{1}{\left(n^{1/2}\cdot b\right)^q}\right), \quad \text{and} \\
&\left.\begin{aligned}
\sup_{\substack{\theta\in\Theta \\ (x,t)\in\mathcal{S}_X\times\mathcal{T} \\ h>0}} \left| m^{R_p}_{\Lambda,n}(x,t,\theta,h)\right| &= O_p\left(\tfrac{1}{n^{1/2}}\right) \\
\sup_{\substack{\theta\in\Theta \\ (x,t)\in\mathcal{S}_X\times\mathcal{T} \\ h>0}} \left| m^{R_p}_{\beta,n}(x,t,\theta,h)\right| &= O_p\left(\tfrac{1}{n^{1/2}}\right)
\end{aligned}\right\} \text{ uniformly over } \mathcal{F}.
\end{aligned} \tag{S3.2.10}$$

Consider the following classes of functions on $\mathcal{S}^2_V$,

$$\mathscr{G}^p_{4,F} = \Big\{ m : \mathcal{S}^2_V \longrightarrow \mathbb{R}\colon\ m(v_1,v_2) = \Lambda_{R_p}(v_1,s,t,\theta,h)\psi^\theta_F(z_2) \text{ for some } s \in \mathcal{S}_X,\ \theta \in \Theta,\ h > 0 \Big\},$$
$$\mathscr{G}^p_{5,F} = \Big\{ m : \mathcal{S}^2_V \longrightarrow \mathbb{R}\colon\ m(v_1,v_2) = \beta_{R_p}(v_1,s,t,\theta,h)\psi^\theta_F(z_2) \text{ for some } s \in \mathcal{S}_X,\ \theta \in \Theta,\ h > 0 \Big\}.$$

Similar to the case of $\mathscr{M}^p_{6,F}$ and $\mathscr{M}^p_{7,F}$, the Euclidean properties of $\mathscr{M}^\ell_{4,p}$ and $\mathscr{M}^\ell_{5,p}$, the integrability properties of $\psi_F^\theta(Z)$ in Assumption 1, and Pakes and Pollard (1989, Lemma 2.14) imply that $\mathscr{G}_{4,F}^p$ and $\mathscr{G}_{5,F}^p$ are Euclidean for an envelope $\overline{G}_4(\cdot)$ s.t $E_F\left[\overline{G}_4(V_1,Z_2)^{4q}\right] \leq \overline{\eta}_{\overline{G}_4} < \infty \ \forall\ F \in \mathcal{F}$ (with $(X_1,Z_2) \sim F \otimes F$), where q is the integer described in Assumption 1. Next, define the following U-statistics,

$$\begin{aligned} U^{\Lambda}_{R_p,n}(x,t,\theta,h) &= \frac{1}{n(n-1)}\sum_{i=1}^{n}\sum_{j\neq i}\Lambda_{R_p}(V_i,x,t,\theta,h)\psi_F^\theta(Z_j), \\ U^{\beta}_{R_p,n}(x,t,\theta,h) &= \frac{1}{n(n-1)}\sum_{i=1}^{n}\sum_{j\neq i}\beta_{R_p}(V_i,x,t,\theta,h)\psi_F^\theta(Z_j). \end{aligned} \tag{S3.2.11}$$

Let

$$\begin{aligned} \varphi^{R_p}_{\Lambda,F}(Z_i,x,t,\theta,h) &= E_F\left[\Lambda_{R_p}(V,x,t,\theta,h)\right]\psi_F^\theta(Z_i), \\ \vartheta^{R_p}_{\Lambda,F}(V_i,V_j,x,t,\theta,h) &= \Lambda_{R_p}(V_i,x,t,h)\psi_F^\theta(Z_j) + \Lambda_{R_p}(V_j,x,t,h)\psi_F^\theta(Z_i) - \varphi^{R_p}_{\Lambda,F}(Z_i,x,t,\theta,h) - \varphi^{R_p}_{\Lambda,F}(Z_j,x,t,\theta,h), \\ \varphi^{R_p}_{\beta,F}(Z_i,x,t,\theta,h) &= E_F\left[\beta_{R_p}(V,x,t,\theta,h)\right]\psi_F^\theta(Z_i), \\ \vartheta^{R_p}_{\beta,F}(V_i,V_j,x,t,\theta,h) &= \beta_{R_p}(V_i,x,t,h)\psi_F^\theta(Z_j) + \beta_{R_p}(V_j,x,t,h)\psi_F^\theta(Z_i) - \varphi^{R_p}_{\beta,F}(Z_i,x,t,\theta,h) - \varphi^{R_p}_{\beta,F}(Z_j,x,t,\theta,h), \end{aligned} \tag{S3.2.12}$$

Note that $\vartheta^{R_p}_{\Lambda,F}(v_1,v_2,x,t,\theta,h) = \vartheta^{R_p}_{\Lambda,F}(v_2,v_1,x,t,\theta,h)$ and $\vartheta^{R_p}_{\beta,F}(v_1,v_2,x,t,\theta,h) = \vartheta^{R_p}_{\beta,F}(v_2,v_1,x,t,\theta,h)$ (they are both symmetric in their first two arguments), and

$$\begin{aligned} E_F\left[\vartheta^{R_p}_{\Lambda,F}(V_i,V_j,x,t,\theta,h)\middle|V_i\right] &= E_F\left[\vartheta^{R_p}_{\Lambda,F}(V_i,V_j,x,t,\theta,h)\middle|V_j\right] = 0, \\ E_F\left[\vartheta^{R_p}_{\beta,F}(V_i,V_j,x,t,\theta,h)\middle|V_i\right] &= E_F\left[\vartheta^{R_p}_{\beta,F}(V_i,V_j,x,t,\theta,h)\middle|V_j\right] = 0 \end{aligned}$$

From the functionals in (S3.2.12), define the following degenerate U-statistics

$$\begin{aligned} \widetilde{U}^{\Lambda}_{R_p,n}(x,t,\theta,h) &= \frac{1}{n(n-1)}\sum_{i=1}^{n}\sum_{j\neq i}\vartheta^{R_p}_{\Lambda,F}(V_i,V_j,x,t,\theta,h), \\ \widetilde{U}^{\beta}_{R_p,n}(x,t,\theta,h) &= \frac{1}{n(n-1)}\sum_{i=1}^{n}\sum_{j\neq i}\vartheta^{R_p}_{\beta,F}(V_i,V_j,x,t,\theta,h) \end{aligned}$$

From the Euclidean properties of $\mathscr{G}_{4,F}^p$ and $\mathscr{G}_{5,F}^p$, Nolan and Pollard (1987, Lemma 20) (or Sherman (1994, Lemma 5)) along with Pakes and Pollard (1989, Lemma 2.14) imply that

$$\begin{aligned} \mathscr{G}_{6,F}^p &= \left\{m : \mathcal{S}_V^2 \longrightarrow \mathbb{R}\text{: } m(v_1,v_2) = \vartheta^{R_p}_{\Lambda,F}(v_1,v_2,x,t,\theta,h) \text{ for some } x\in\mathcal{S}_X,\ t\in\mathcal{T},\ \theta\in\Theta,\ h>0\right\}, \\ \mathscr{G}_{7,F}^p &= \left\{m : \mathcal{S}_V^2 \longrightarrow \mathbb{R}\text{: } m(v_1,v_2) = \vartheta^{R_p}_{\beta,F}(v_1,v_2,x,t,\theta,h) \text{ for some } x\in\mathcal{S}_X,\ t\in\mathcal{T},\ \theta\in\Theta,\ h>0\right\} \end{aligned}$$

are Euclidean for an envelope $\overline{G}_5(\cdot)$ s.t $E_F\left[\overline{G}_5(V_1,V_2)^{4q}\right] \leq \overline{\eta}_{\overline{G}_5} < \infty \ \forall\ F \in \mathcal{F}$ (with $(V_1,V_2) \sim F \otimes F$)

and q being the integer described in Assumption 1. From here, Result S1 implies that, $\forall\ b > 0$,

$$\begin{aligned}
&\sup_{F\in\mathcal{F}} P_F\left(\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}} \left|\widetilde{U}^{\Lambda}_{R_p,n}(x,t,\theta,h)\right| \geq b\right) = O\left(\frac{1}{(n\cdot b)^q}\right),\\
&\sup_{F\in\mathcal{F}} P_F\left(\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}} \left|\widetilde{U}^{\beta}_{R_p,n}(x,t,\theta,h)\right| \geq b\right) = O\left(\frac{1}{(n\cdot b)^q}\right), \quad \text{and} \qquad\qquad \text{(S3.2.13)}\\
&\left.\begin{aligned}
\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}} \left|\widetilde{U}^{\Lambda}_{R_p,n}(x,t,\theta,h)\right| &= O_p\left(\tfrac{1}{n}\right)\\
\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}} \left|\widetilde{U}^{\beta}_{R_p,n}(x,t,\theta,h)\right| &= O_p\left(\tfrac{1}{n}\right)
\end{aligned}\right\} \text{ uniformly over } \mathcal{F}
\end{aligned}$$

Note from the definitions in (S3.2.12) that $E_F[\varphi^{R_p}_{\Lambda,F}(Z_i,x,t,\theta,h)] = E_F[\varphi^{R_p}_{\beta,F}(Z_i,x,t,\theta,h)] = 0$. Define $\nu^{\Lambda}_{\varphi_{R_p},n}(x,t,\theta,h) \equiv \frac{1}{n}\sum_{i=1}^{n}\varphi^{R_p}_{\Lambda,F}(Z_i,x,t,\theta,h)$ and $\nu^{\beta}_{\varphi_{R_p},n}(x,t,\theta,h) \equiv \frac{1}{n}\sum_{i=1}^{n}\varphi^{R_p}_{\beta,F}(Z_i,x,t,\theta,h)$. Let

$$\begin{aligned}
\mathscr{G}^{p}_{8,F} &= \left\{m:\mathcal{S}_Z \longrightarrow \mathbb{R}\colon\ m(z) = \varphi^{R_p}_{\Lambda,F}(z,x,t,\theta,h) \ \text{ for some } x\in\mathcal{S}_X,\ t\in\mathcal{T},\ \theta\in\Theta,\ h>0\right\},\\
\mathscr{G}^{p}_{9,F} &= \left\{m:\mathcal{S}_Z \longrightarrow \mathbb{R}\colon\ m(z) = \varphi^{R_p}_{\beta,F}(z,x,t,\theta,h) \ \text{ for some } x\in\mathcal{S}_X,\ t\in\mathcal{T},\ \theta\in\Theta,\ h>0\right\},
\end{aligned}$$

By the same arguments used for $\mathscr{G}^{p}_{6,F}$ and $\mathscr{G}^{p}_{7,F}$, both $\mathscr{G}^{p}_{8,F}$ and $\mathscr{G}^{p}_{9,F}$ are Euclidean classes of functions for an envelope $\overline{G}_6(\cdot)$ s.t $E_F\left[\overline{G}_6(Z)^{4q}\right] \leq \overline{\eta}_{\overline{G}_6} < \infty\ \forall\ F\in\mathcal{F}$, with q being the integer described in

Assumption 1. From here, Result S1 yields that, for any $b > 0$,

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}} \left|\nu^{\Lambda}_{\varphi_{R_p},n}(x,t,\theta,h)\right| \geq b\right) = O\left(\frac{1}{\left(n^{1/2}\cdot b\right)^q}\right),$$

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}} \left|\nu^{\beta}_{\varphi_{R_p},n}(x,t,\theta,h)\right| \geq b\right) = O\left(\frac{1}{\left(n^{1/2}\cdot b\right)^q}\right), \quad \text{and} \tag{S3.2.14}$$

$$\left.\begin{aligned}\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}} \left|\nu^{\Lambda}_{\varphi_{R_p},n}(x,t,\theta,h)\right| &= O_p\left(\tfrac{1}{n^{1/2}}\right)\\ \sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}} \left|\nu^{\beta}_{\varphi_{R_p},n}(x,t,\theta,h)\right| &= O_p\left(\tfrac{1}{n^{1/2}}\right)\end{aligned}\right\} \text{ uniformly over } \mathcal{F}$$

The Hoeffding decompositions of the U-statistics $U^{\Lambda}_{R_p,n}(x,t,\theta,h)$ and $U^{\beta}_{R_p,n}(x,t,\theta,h)$ defined in (S3.2.11) are given by (see Serfling (1980, pages 177-178) or Sherman (1994, equations (6)-(7))),

$$\begin{aligned}U^{\Lambda}_{R_p,n}(x,t,\theta,h) &= \nu^{\Lambda}_{\varphi_{R_p},n}(x,t,\theta,h) + \frac{1}{2}\cdot\widetilde{U}^{\Lambda}_{R_p,n}(x,t,\theta,h),\\ U^{\beta}_{R_p,n}(x,t,\theta,h) &= \nu^{\beta}_{\varphi_{R_p},n}(x,t,\theta,h) + \frac{1}{2}\cdot\widetilde{U}^{\beta}_{R_p,n}(x,t,\theta,h).\end{aligned} \tag{S3.2.15}$$

Therefore, from (S3.2.13) and (S3.2.14), for any $b > 0$ we have

$$\begin{aligned}&\sup_{F\in\mathcal{F}} P_F\left(\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}} \left|U^{\Lambda}_{R_p,n}(x,t,\theta,h)\right| \geq b\right)\\ \leq&\sup_{F\in\mathcal{F}} P_F\left(\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}} \left|\nu^{\Lambda}_{\varphi_{R_p,n}}(x,t,\theta,h)\right| \geq \frac{b}{2}\right) + \sup_{F\in\mathcal{F}} P_F\left(\frac{1}{2}\cdot\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}} \left|\widetilde{U}^{\Lambda}_{R_p,n}(x,t,\theta,h)\right| \geq \frac{b}{2}\right)\\ =&O\left(\frac{1}{(n\cdot b)^q}\right) + O\left(\frac{1}{\left(n^{1/2}\cdot b\right)^q}\right) = O\left(\frac{1}{\left(n^{1/2}\cdot b\right)^q}\right)\end{aligned} \tag{S3.2.16}$$

and

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}}\left|U^{\beta}_{R_p,n}(x,t,\theta,h)\right|\geq b\right)$$
$$\leq\sup_{F\in\mathcal{F}} P_F\left(\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}}\left|\nu^{\beta}_{\varphi_{R_p,n}}(x,t,\theta,h)\right|\geq\frac{b}{2}\right)+\sup_{F\in\mathcal{F}} P_F\left(\frac{1}{2}\cdot\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}}\left|\widetilde{U}^{\beta}_{R_p,n}(x,t,\theta,h)\right|\geq\frac{b}{2}\right) \tag{S3.2.17}$$
$$=O\left(\frac{1}{(n\cdot b)^q}\right)+O\left(\frac{1}{\left(n^{1/2}\cdot b\right)^q}\right)=O\left(\frac{1}{\left(n^{1/2}\cdot b\right)^q}\right)$$

We are ready to analyze the components of $\xi^{R_p}_{a,n}(x,t,\theta)$. We have,

$$\begin{aligned}
\xi^{R_p}_{a,n}(x,t,\theta) &\equiv \frac{\partial\widehat{R}_p(x,t,\theta)}{\partial\theta}\left(\widehat{\theta}-\theta^*_F\right)\\
&= \frac{1}{n^2\cdot h_n^{D+1}}\sum_{i=1}^n\sum_{j=1}^n \Lambda_{R_p}(V_i,x,t,\theta,h_n)\psi^{\theta}_F(Z_j)+\left(\frac{1}{n\cdot h_n^{D+1}}\sum_{i=1}^n\Lambda_{R_p}(V_i,x,t,\theta,h_n)\right)\varepsilon^{\theta}_n\\
&+ \frac{1}{n^2\cdot h_n^{D}}\sum_{i=1}^n\sum_{j=1}^n \beta_{R_p}(V_i,x,t,\theta,h_n)\psi^{\theta}_F(Z_j)+\left(\frac{1}{n\cdot h_n^{D}}\sum_{i=1}^n\beta_{R_p}(V_i,x,t,\theta,h_n)\right)\varepsilon^{\theta}_n\\
&= \frac{1}{h_n^{D+1}}\cdot\left(\frac{n-1}{n}\right)\cdot U^{\Lambda}_{R_p,n}(x,t,\theta,h_n)+\frac{1}{h_n^{D}}\cdot\left(\frac{n-1}{n}\right)\cdot U^{\beta}_{R_p,n}(x,t,\theta,h_n)\\
&+ \frac{1}{n\cdot h_n^{D+1}}\cdot E_F\left[\Lambda_{R_p}(V,x,t,\theta,h_n)\psi^{\theta}_F(Z)\right]+\frac{1}{n\cdot h_n^{D+1}}\cdot m^{R_p}_{\Lambda,n}(x,t,\theta,h_n)\\
&+ \frac{1}{h_n^{D+1}}\cdot E_F\left[\Lambda_{R_p}(V,x,t,\theta,h_n)\right]\varepsilon^{\theta}_n+\frac{1}{h_n^{D+1}}\cdot\nu_{\Lambda_{R_p},n}(x,t,\theta,h_n)\varepsilon^{\theta}_n\\
&+ \frac{1}{n\cdot h_n^{D}}\cdot E_F\left[\beta_{R_p}(V,x,t,\theta,h_n)\psi^{\theta}_F(Z)\right]+\frac{1}{n\cdot h_n^{D}}\cdot m^{R_p}_{\beta,n}(x,t,\theta,h_n)\\
&+ \frac{1}{h_n^{D}}\cdot E_F\left[\beta_{R_p}(V,x,t,\theta,h_n)\right]\varepsilon^{\theta}_n+\frac{1}{h_n^{D}}\cdot\nu_{\beta_{R_p},n}(x,t,\theta,h_n)\varepsilon^{\theta}_n
\end{aligned} \tag{S3.2.18}$$

Thus, from (S3.2.6), (S3.2.9) and (S3.2.18),

$$\begin{aligned}
&\left|\xi^{R_p}_{a,n}(x,t,\theta)\right|\leq\frac{1}{h_n^{D+1}}\cdot\left(\frac{n-1}{n}\right)\cdot\left|U^{\Lambda}_{R_p,n}(x,t,\theta,h_n)\right|+\frac{1}{h_n^{D}}\cdot\left(\frac{n-1}{n}\right)\cdot\left|U^{\beta}_{R_p,n}(x,t,\theta,h_n)\right|\\
&+\frac{1}{n\cdot h_n^{D+1}}\cdot\left|m^{R_p}_{\Lambda,n}(x,t,\theta,h_n)\right|+\frac{1}{h_n^{D+1}}\cdot\left\|\nu_{\Lambda_{R_p},n}(x,t,\theta,h_n)\right\|\cdot\left\|\varepsilon^{\theta}_n\right\|+\frac{\overline{\eta}_{\Lambda}}{h_n^{D+1}}\cdot\left\|\varepsilon^{\theta}_n\right\|+\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}},\\
&+\frac{1}{n\cdot h_n^{D}}\cdot\left|m^{R_p}_{\beta,n}(x,t,\theta,h_n)\right|+\frac{1}{h_n^{D}}\cdot\left\|\nu_{\beta_{R_p},n}(x,t,\theta,h_n)\right\|\cdot\left\|\varepsilon^{\theta}_n\right\|+\frac{\overline{\eta}_{\beta}}{h_n^{D}}\cdot\left\|\varepsilon^{\theta}_n\right\|+\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^{D}}
\end{aligned}$$

Thus, for any $b > 0$,

$$\begin{aligned}
&\sup_{F\in\mathcal{F}} P_F\left(\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}}}\left|\xi^{R_p}_{a,n}(x,t,\theta)\right|\geq b\right)\\
\leq&\sup_{F\in\mathcal{F}} P_F\left(\frac{1}{h_n^{D+1}}\cdot\left(\frac{n-1}{n}\right)\cdot\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}}\left|U^{\Lambda}_{R_p,n}(x,t,\theta,h)\right|\geq\frac{1}{8}\cdot\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^{D}}\right)\right)\\
&+\sup_{F\in\mathcal{F}} P_F\left(\frac{1}{h_n^{D}}\cdot\left(\frac{n-1}{n}\right)\cdot\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}}\left|U^{\beta}_{R_p,n}(x,t,\theta,h)\right|\geq\frac{1}{8}\cdot\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^{D}}\right)\right)\\
&+\sup_{F\in\mathcal{F}} P_F\left(\frac{1}{n\cdot h_n^{D+1}}\cdot\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}}\left|m^{R_p}_{\Lambda,n}(x,t,\theta,h)\right|\geq\frac{1}{8}\cdot\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^{D}}\right)\right)\\
&+\sup_{F\in\mathcal{F}} P_F\left(\frac{1}{h_n^{D+1}}\cdot\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}}\left\|\nu_{\Lambda_{R_p},n}(x,t,\theta,h)\right\|\cdot\left\|\varepsilon^{\theta}_n\right\|\geq\frac{1}{8}\cdot\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^{D}}\right)\right)\\
&+\sup_{F\in\mathcal{F}} P_F\left(\frac{\overline{\eta}_{\Lambda}}{h_n^{D+1}}\cdot\left\|\varepsilon^{\theta}_n\right\|\geq\frac{1}{8}\cdot\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^{D}}\right)\right)\\
&+\sup_{F\in\mathcal{F}} P_F\left(\frac{1}{n\cdot h_n^{D}}\cdot\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}}\left|m^{R_p}_{\beta,n}(x,t,\theta,h)\right|\geq\frac{1}{8}\cdot\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^{D}}\right)\right)\\
&+\sup_{F\in\mathcal{F}} P_F\left(\frac{1}{h_n^{D}}\cdot\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}}\left\|\nu_{\beta_{R_p},n}(x,t,\theta,h)\right\|\cdot\left\|\varepsilon^{\theta}_n\right\|\geq\frac{1}{8}\cdot\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^{D}}\right)\right)\\
&+\sup_{F\in\mathcal{F}} P_F\left(\frac{\overline{\eta}_{\beta}}{h_n^{D}}\cdot\left\|\varepsilon^{\theta}_n\right\|\geq\frac{1}{8}\cdot\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^{D}}\right)\right)
\end{aligned}\tag{S3.2.19}$$

Using our previous results we can analyze each of the terms on the right hand side of (S3.2.19). If $b > 0$ is fixed[5], there exists n_0 such that $b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^{D}} > 0 \;\forall\; n > n_0$. From (S3.2.16), it follows that

[5] As before, what follows is true for any sequence $b_n > 0$ such that $b_n\cdot n\cdot h_n^{D+1}\longrightarrow\infty$.

$$\sup_{F\in\mathcal{F}} P_F\left(\frac{1}{h_n^{D+1}}\cdot\left(\frac{n-1}{n}\right)\cdot\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}}\left|U^{\Lambda}_{R_p,n}(x,t,\theta,h)\right|\geq\frac{1}{8}\cdot\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^{D}}\right)\right)$$

$$=\sup_{F\in\mathcal{F}} P_F\left(\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}}\left|U^{\Lambda}_{R_p,n}(x,t,\theta,h)\right|\geq\frac{1}{8}\cdot\left(\frac{n}{n-1}\right)\cdot h_n^{D+1}\cdot\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^{D}}\right)\right)=O\left(\frac{1}{\left(n^{1/2}\cdot h_n^{D+1}\cdot\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^{D}}\right)\right)^{q}}\right) \tag{S3.2.20A}$$

From (S3.2.17),

$$\sup_{F\in\mathcal{F}} P_F\left(\frac{1}{h_n^{D}}\cdot\left(\frac{n-1}{n}\right)\cdot\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}}\left|U^{\beta}_{R_p,n}(x,t,\theta,h)\right|\geq\frac{1}{8}\cdot\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^{D}}\right)\right)$$

$$=\sup_{F\in\mathcal{F}} P_F\left(\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}}\left|U^{\beta}_{R_p,n}(x,t,\theta,h)\right|\geq\frac{1}{8}\cdot\left(\frac{n}{n-1}\right)\cdot h_n^{D+1}\cdot\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^{D}}\right)\right)=O\left(\frac{1}{\left(n^{1/2}\cdot h_n^{D}\cdot\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^{D}}\right)\right)^{q}}\right) \tag{S3.2.20B}$$

From (S3.2.10),

$$\sup_{F\in\mathcal{F}} P_F\left(\frac{1}{n\cdot h_n^{D+1}}\cdot\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}}\left|m^{R_p}_{\Lambda,n}(x,t,\theta,h)\right|\geq\frac{1}{8}\cdot\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^{D}}\right)\right)$$

$$=\sup_{F\in\mathcal{F}} P_F\left(\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}}\left|m^{R_p}_{\Lambda,n}(x,t,\theta,h)\right|\geq\frac{1}{8}\cdot n\cdot h_n^{D+1}\cdot\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^{D}}\right)\right)=O\left(\frac{1}{\left(n^{3/2}\cdot h_n^{D+1}\cdot\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^{D}}\right)\right)^{q}}\right) \tag{S3.2.20C}$$

Next, from (S3.2.7) and Assumption 1,

$$\begin{aligned}
&\sup_{F\in\mathcal{F}} P_F\left(\frac{1}{h_n^{D+1}}\cdot\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}}\left\|\nu_{\Lambda_{R_p},n}(x,t,\theta,h)\right\|\cdot\left\|\varepsilon_n^{\theta}\right\|\geq\frac{1}{8}\cdot\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^{D}}\right)\right)\\
\leq&\sup_{F\in\mathcal{F}} P_F\left(\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}}\left\|\nu_{\Lambda_{R_p},n}(x,t,\theta,h)\right\|\geq\left(\frac{1}{8}\cdot h_n^{D+1}\cdot\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^{D}}\right)\right)^{1/2}\right)+\sup_{F\in\mathcal{F}} P_F\left(\left\|\varepsilon_n^{\theta}\right\|\geq\left(\frac{1}{8}\cdot h_n^{D+1}\cdot\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^{D}}\right)\right)^{1/2}\right)\\
=&O\left(\frac{1}{\left(n^{1/2}\cdot h_n^{\frac{D+1}{2}}\cdot\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^{D}}\right)^{1/2}\right)^q}\right)+O\left(\frac{1}{\left(r_n\cdot h_n^{\frac{D+1}{2}}\cdot\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^{D}}\right)^{1/2}\right)^q}\right)\\
=&O\left(\frac{1}{\left((n^{1/2}\wedge r_n)\cdot h_n^{\frac{D+1}{2}}\cdot\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^{D}}\right)^{1/2}\right)^q}\right)
\end{aligned}\tag{S3.2.20D}$$

Next, also from Assumption 1,

$$\begin{aligned}
&\sup_{F\in\mathcal{F}} P_F\left(\frac{\overline{\eta}_{\Lambda}}{h_n^{D+1}}\cdot\left\|\varepsilon_n^{\theta}\right\|\geq\frac{1}{4}\cdot\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^{D}}\right)\right)=\sup_{F\in\mathcal{F}} P_F\left(\left\|\varepsilon_n^{\theta}\right\|\geq\frac{h_n^{D+1}}{4\overline{\eta}_{\Lambda}}\cdot\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^{D}}\right)\right)\\
=&O\left(\frac{1}{\left(r_n\cdot h_n^{D+1}\cdot\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^{D}}\right)\right)^q}\right)
\end{aligned}\tag{S3.2.20E}$$

From (S3.2.10),

$$\begin{aligned}
&\sup_{F\in\mathcal{F}} P_F\left(\frac{1}{n\cdot h_n^{D}}\cdot\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}}\left|m_{\beta,n}^{R_p}(x,t,\theta,h)\right|\geq\frac{1}{8}\cdot\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^{D}}\right)\right)\\
=&\sup_{F\in\mathcal{F}} P_F\left(\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}}\left|m_{\beta,n}^{R_p}(x,t,\theta,h)\right|\geq\frac{1}{8}\cdot n\cdot h_n^{D}\cdot\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^{D}}\right)\right)=O\left(\frac{1}{\left(n^{3/2}\cdot h_n^{D}\cdot\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^{D}}\right)\right)^q}\right)
\end{aligned}\tag{S3.2.20F}$$

From (S3.2.7) and Assumption 1,

$$\sup_{F\in\mathcal{F}} P_F\left(\frac{1}{h_n^D}\cdot \sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}} \left\|\nu_{\beta_{R_p},n}(x,t,\theta,h)\right\|\cdot\left\|\varepsilon_n^{\theta}\right\| \geq \frac{1}{8}\cdot\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^{D}}\right)\right)$$

$$\leq \sup_{F\in\mathcal{F}} P_F\left(\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}} \left\|\nu_{\beta_{R_p},n}(x,t,\theta,h)\right\| \geq \left(\frac{1}{8}\cdot h_n^D\cdot\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^{D}}\right)\right)^{1/2}\right)+\sup_{F\in\mathcal{F}} P_F\left(\left\|\varepsilon_n^{\theta}\right\|\geq\left(\frac{1}{8}\cdot h_n^D\cdot\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^{D}}\right)\right)^{1/2}\right)$$

$$=O\left(\frac{1}{\left(n^{1/2}\cdot h_n^{\frac{D}{2}}\cdot\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^{D}}\right)^{1/2}\right)^q}\right)+O\left(\frac{1}{\left(r_n\cdot h_n^{\frac{D}{2}}\cdot\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^{D}}\right)^{1/2}\right)^q}\right)$$

$$=O\left(\frac{1}{\left((n^{1/2}\wedge r_n)\cdot h_n^{\frac{D}{2}}\cdot\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^{D}}\right)^{1/2}\right)^q}\right) \tag{S3.2.20G}$$

Also from Assumption 1,

$$\sup_{F\in\mathcal{F}} P_F\left(\frac{\overline{\eta}_{\Lambda}}{h_n^D}\cdot\left\|\varepsilon_n^{\theta}\right\|\geq\frac{1}{4}\cdot\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^{D}}\right)\right)=\sup_{F\in\mathcal{F}} P_F\left(\left\|\varepsilon_n^{\theta}\right\|\geq\frac{h_n^D}{4\overline{\eta}_{\Lambda}}\cdot\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^{D}}\right)\right)$$
$$=O\left(\frac{1}{\left(r_n\cdot h_n^D\cdot\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^{D}}\right)\right)^q}\right) \tag{S3.2.20H}$$

For large enough n, we have[6] $b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^{D}}>0$ and $h_n<1$ (and therefore $h_n^{D+1}<h_n^{\frac{D+1}{2}}<h_n^{\frac{D}{2}}$). Combining (S3.2.20A)-(S3.2.20H) with (S3.2.19) we obtain,

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}}}\left|\xi_{a,n}^{R_p}(x,t,\theta)\right|\geq b\right)=O\left(\frac{1}{\left((n^{1/2}\wedge r_n)\cdot h_n^{D+1}\cdot\left(\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^{D}}\right)\wedge\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^{D}}\right)^{1/2}\right)\right)^q}\right)$$

Going back to the second-order approximation in (S3.2.1),

$$\widehat{R}_p(x,t,\widehat{\theta})=\widehat{R}_p(x,t,\theta_F^*)+\xi_{a,n}^{R_p}(x,t,\theta_F^*)+\xi_{b,n}^{R_p}(x,t,\overline{\theta}_x), \tag{S3.2.21}$$

[6]See footnote 5.

where, for any $b > 0$

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{(x,t)\in\mathcal{S}_X\times\mathcal{T}}\left|\xi^{R_p}_{a,n}(x,t,\theta^*_F)\right| \geq b\right) \leq \sup_{F\in\mathcal{F}} P_F\left(\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}}}\left|\xi^{R_p}_{a,n}(x,t,\theta)\right| \geq b\right)$$
$$= O\left(\frac{1}{\left((n^{1/2}\wedge r_n)\cdot h_n^{D+1}\cdot\left(\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^{D}}\right)\wedge\left(b-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^{D}}\right)^{1/2}\right)\right)^q}\right) \tag{S3.2.22}$$

As we have done before, note that for any $c > 0$, we have $\min\left\{c,\, c^{1/2}\,,\, c^{1/4}\right\} = \min\left\{c\,,\, c^{1/4}\right\}$. Using this and combining (S3.2.5) with (S3.2.22), we have that for any $b > 0$,

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{(x,t)\in\mathcal{S}_\mathcal{X}\times\mathcal{T}}\left|\xi^{R_p}_{a,n}(x,t,\theta^*_F)+\xi^{R_p}_{b,n}(x,t,\overline{\theta}_x)\right| \geq b\right)$$
$$\leq \sup_{F\in\mathcal{F}} P_F\left(\sup_{(x,t)\in\mathcal{S}_\mathcal{X}\times\mathcal{T}}\left|\xi^{R_p}_{a,n}(x,t,\theta^*_F)\right| \geq \frac{b}{2}\right)+\sup_{F\in\mathcal{F}} P_F\left(\sup_{(x,t)\in\mathcal{S}_\mathcal{X}\times\mathcal{T}}\left|\xi^{R_p}_{b,n}(x,t,\overline{\theta}_x)\right| \geq \frac{b}{2}\right)$$
$$= O\left(\frac{1}{\left((n^{1/2}\wedge r_n)\cdot h_n^{D+1}\cdot\left(\left(\frac{b}{2}-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^{D}}\right)\wedge\left(\frac{b}{2}-\frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}}-\frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^{D}}\right)^{1/4}\right)\right)^q}\right) \tag{S3.2.23}$$

Now let us analyze $\widehat{R}_p(x,t,\theta^*_F)$. For a given $(x,t)\in\mathcal{S}_X\times\mathcal{T}$, $\theta\in\Theta$ and $h>0$ denote

$$\nu^{R_p}_n(x,t,\theta,h) \equiv \frac{1}{n}\sum_{i=1}^{n}\left(S_p(Y_i,t)\omega_p(g(X_i,\theta))K\left(\frac{\Delta g(X_i,x,\theta)}{h}\right)-E_F\left[S_p(Y_i,t)\omega_p(g(X_i,\theta))K\left(\frac{\Delta g(X_i,x,\theta)}{h}\right)\right]\right)$$

From Assumptions 3 and 4, the class of functions

$$\mathcal{G}^p_{10} = \left\{m:\mathcal{S}_V\longrightarrow\mathbb{R}\colon m(v)=S_p(y,u,t)\omega_p(g(x,\theta))K\left(\frac{\Delta g(x,u,\theta)}{h}\right)\ \text{for some } u\in\mathcal{S}_X,\, t\in\mathcal{T},\, \theta\in\Theta,\, h>0\right\}$$

is Euclidean for an envelope $\overline{\omega}\cdot\overline{K}\cdot\overline{G}_7(Y)$ s.t $E_F\left[\overline{G}_7(Y)^{4q}\right]\leq\overline{\eta}_{\overline{G}_7}<\infty\ \forall\ F\in\mathcal{F}$, with q being the integer described in Assumption 1. From here, Result S1 yields that, for any $b > 0$,

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}} \left|\nu_n^{R_p}(x,t,\theta,h)\right| \geq b\right) = O\left(\frac{1}{\left(n^{1/2}\cdot b\right)^q}\right), \text{ and therefore,}$$
$$\sup_{\substack{\theta\in\Theta\\(x,t)\in\mathcal{S}_X\times\mathcal{T}\\h>0}} \left|\nu_n^{R_p}(x,t,\theta,h)\right| = O_p\left(\frac{1}{n^{1/2}}\right) \quad \text{uniformly over } \mathcal{F}. \tag{S3.2.24}$$

Recall that we have defined $R_{p,F}(x,t,\theta_F^*) \equiv \Gamma_{p,F}(x,t,\theta_F^*)\cdot\omega_p(g(x,\theta_F^*))\cdot f_g(g(x,\theta_F^*))$. Let

$$B_n^{R_p}(x,t) \equiv \underbrace{\frac{1}{h_n^D}\cdot E_F\left[S_p(Y,t)\omega_p(g(X,\theta))K\left(\frac{\Delta g(X,x,\theta_F^*)}{h_n}\right)\right] - R_{p,F}(x,t,\theta_F^*)}_{\text{bias}}.$$

We have, $\widehat{R}_p(x,t,\theta_F^*) = R_{p,F}(x,t,\theta_F^*) + \frac{1}{h_n^D}\cdot\nu_n^{R_p}(x,t,\theta_F^*,h_n) + B_n^{R_p}(x,t)$. From here, (S3.2.21) yields,

$$\widehat{R}_p(x,t,\widehat{\theta}) = R_{p,F}(x,t,\theta_F^*) + \frac{1}{h_n^D}\cdot\nu_n^{R_p}(x,t,\theta_F^*,h_n) + B_n^{R_p}(x,t) + \xi_{a,n}^{R_p}(x,t,\theta_F^*) + \xi_{b,n}^{R_p}(x,t,\overline{\theta}_x), \tag{S3.2.25}$$

From Assumptions 4 and the smoothness conditions described in Assumption 2, an M^{th}-order approximation implies that there exists a constant $\overline{B}_{1,f} < \infty$ such that

$$\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left|B_n^{R_p}(x,t)\right| \leq \overline{B}_{1,R}\cdot h_n^M \quad \forall\ F\in\mathcal{F}. \tag{S3.2.26}$$

Thus,

$$\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left|\widehat{R}_p(x,t,\widehat{\theta}) - R_{p,F}(x,t,\theta_F^*)\right| \leq \frac{1}{h_n^D}\cdot\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\nu_n^{R_p}(x,t,\theta_F^*,h_n)\right| + \overline{B}_{1,R}\cdot h_n^M$$
$$+ \sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\xi_{a,n}^{R_p}(x,t,\theta_F^*) + \xi_{b,n}^{R_p}(x,t,\overline{\theta}_x)\right|$$

Thus, for any $b > 0$,

$$\begin{aligned}
&\sup_{F\in\mathcal{F}} P_F\left(\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left|\widehat{R}_p(x,t,\widehat{\theta}) - R_{p,F}(x,t,\theta_F^*)\right| \geq b\right)\\
\leq &\sup_{F\in\mathcal{F}} P_F\left(\frac{1}{h_n^D}\cdot \sup_{(x,t)\in\in\mathcal{X}\times\mathcal{T}} \left|\nu_n^{R_p}(x,t,\theta_F^*,h_n)\right| + \sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left|\xi_{a,n}^{R_p}(x,t,\theta_F^*) + \xi_{b,n}^{R_p}(x,t,\overline{\theta}_x)\right| \geq b - \overline{B}_{1,R}\cdot h_n^M\right)\\
\leq &\sup_{F\in\mathcal{F}} P_F\left(\frac{1}{h_n^D}\cdot \sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left|\nu_n^{R_p}(x,t,\theta_F^*,h_n)\right| \geq \left(\frac{b-\overline{B}_{1,R}\cdot h_n^M}{2}\right)\right)\\
+&\sup_{F\in\mathcal{F}} P_F\left(\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left|\xi_{a,n}^{R_p}(x,t,\theta_F^*) + \xi_{b,n}^{R_p}(x,t,\overline{\theta}_x)\right| \geq \left(\frac{b-\overline{B}_{1,R}\cdot h_n^M}{2}\right)\right)
\end{aligned} \tag{S3.2.27}$$

For any given $b > 0$, there exists an n_0 such that[7] $\frac{b}{4} - \frac{\overline{B}_{1,R}}{4}\cdot h_n^M - \frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}} - \frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^D} > 0$, and $h_n < 1 \;\forall\; n > n_0$. From (S3.2.23) and (S3.2.24), the inequality in (S3.2.27) yields

$$\begin{aligned}
&\sup_{F\in\mathcal{F}} P_F\left(\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left|\widehat{R}_p(x,t,\widehat{\theta}) - R_{p,F}(x,t,\theta_F^*)\right| \geq b\right)\\
&= O\left(\frac{1}{\left(n^{1/2}\cdot h_n^D\cdot\left(\frac{b}{2} - \frac{\overline{B}_{1,R}}{2}\cdot h_n^M\right)\right)^q}\right)\\
&+ O\left(\frac{1}{\left((n^{1/2}\wedge r_n)\cdot h_n^{D+1}\cdot\left(\left(\frac{b}{4} - \frac{\overline{B}_{1,R}}{4}\cdot h_n^M - \frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}} - \frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^D}\right)\wedge\left(\frac{b}{4} - \frac{\overline{B}_{1,R}}{4}\cdot h_n^M - \frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}} - \frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^D}\right)^{1/4}\right)\right)^q}\right)\\
&= O\left(\frac{1}{\left((n^{1/2}\wedge r_n)\cdot h_n^{D+1}\cdot\left(\left(\frac{b}{4} - \frac{\overline{B}_{1,R}}{2}\cdot h_n^M - \frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}} - \frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^D}\right)\wedge\left(\frac{b}{4} - \frac{\overline{B}_{1,R}}{2}\cdot h_n^M - \frac{\overline{\eta}_{m,\Lambda}}{n\cdot h_n^{D+1}} - \frac{\overline{\eta}_{m,\beta}}{n\cdot h_n^D}\right)^{1/4}\right)\right)^q}\right)
\end{aligned} \tag{S3.2.28}$$

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[7] What follows is true more generally if we replace the constant b with a sequence $s_n > 0$ which may converge to zero as long as $h_n^M/s_n \longrightarrow 0$ and $s_n\cdot n\cdot h_n^{D+1} \longrightarrow 0$. See footnote 4.

S3.2.1 A general result for $\sup_{F\in\mathcal{F}} P_F\left(\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left|\widehat{R}_p(x,t,\widehat{\theta}) - R_{p,F}(x,t,\theta_F^*)\right| \geq s_n\right)$

From (S3.2.28), $\exists$ $K_4 > 0$, $K_5 > 0$, $K_6 > 0$ and $C_3 > 0$ such that, for any sequence $s_n > 0$ (possibly converging to zero) such that $\frac{h_n^M}{s_n} \longrightarrow 0$ and $s_n \cdot n \cdot h_n^{D+1} \longrightarrow \infty$,

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left|\widehat{R}_p(x,t,\widehat{\theta}) - R_{p,F}(x,t,\theta_F^*)\right| \geq s_n\right)$$
$$= O\left(\frac{1}{\left((n^{1/2} \wedge r_n)\cdot h_n^{D+1} \cdot \left(\left(K_4 \cdot s_n - K_5 \cdot h_n^M - \frac{K_6}{n\cdot h_n^{D+1}} - \frac{C_3}{n\cdot h_n^D}\right) \wedge \left(K_4 \cdot s_n - K_5 \cdot h_n^M - \frac{K_6}{n\cdot h_n^{D+1}} - \frac{C_3}{n\cdot h_n^D}\right)^{1/4}\right)\right)^q}\right) \tag{S3.2.29}$$

S3.2.2 A uniform linear representation result for $\widehat{R}_p(x,t,\widehat{\theta}) - R_{p,F}(x,t,\theta_F^*)$

Recall that $\widehat{R}_p(x,t,\widehat{\theta}) = \widehat{R}_p(x,t,\theta_F^*) + \xi_{a,n}^{R_p}(x,t,\theta_F^*) + \xi_{b,n}^{R_p}(x,t,\overline{\theta}_x)$. From (S3.2.5),

$$\sup_{(x,t)\in\mathcal{S}_X\times\mathcal{T}} \left|\xi_{b,n}^{R_p}(x,t,\overline{\theta}_x)\right| = O_p\left(\frac{1}{n\cdot h_n^{D+2}}\right) \quad \text{uniformly over } \mathcal{F}.$$

Let us focus on the term $\xi_{a,n}^{R_p}(x,t,\theta_F^*)$. From (S3.2.18) and the Hoeffding decompositions of $U_{R_p,n}^{\Lambda}$ and $U_{R_p,n}^{\beta}$ in (S3.2.15) we have,

$$\begin{aligned}
\xi_{a,n}^{R_p}(x,t,\theta_F^*) =& \left(\frac{n-1}{n}\right)\cdot\left(\frac{1}{h_n^{D+1}}\cdot E_F\left[\Lambda_{R_p}(V,x,t,\theta_F^*,h_n)\right] + \frac{1}{h_n^D}\cdot E_F\left[\beta_{R_p}(V,x,t,\theta_F^*,h_n)\right]\right)\cdot\frac{1}{n}\sum_{i=1}^n \psi_F^\theta(Z_i)\\
&+\left(\frac{1}{h_n^{D+1}}\cdot E_F\left[\Lambda_{R_p}(V,x,t,\theta_F^*,h_n)\right] + \frac{1}{h_n^{D+1}}\cdot E_F\left[\beta_{R_p}(V,x,t,\theta_F^*,h_n)\right]\right)\varepsilon_n^\theta\\
&+\left(\frac{n-1}{2n}\right)\cdot\left(\frac{1}{h_n^{D+1}}\cdot\widetilde{U}_{R_p,n}^{\Lambda}(x,t,\theta_F^*,h_n) + \frac{1}{h_n^D}\cdot\widetilde{U}_{R_p,n}^{\beta}(x,t,\theta_F^*,h_n)\right)\\
&+\frac{1}{n}\cdot\left(\frac{1}{h_n^{D+1}}\cdot E_F\left[\Lambda_{R_p}(V,x,t,\theta_F^*,h_n)\psi_F^\theta(Z)\right] + \frac{1}{h_n^D}\cdot E_F\left[\beta_{R_p}(V,x,t,\theta_F^*,h_n)\psi_F^\theta(Z)\right]\right)\\
&+\frac{1}{n\cdot h_n^{D+1}}\cdot m_{\Lambda,n}^{R_p}(x,t,\theta_F^*,h_n) + \frac{1}{h_n^{D+1}}\cdot\nu_{\Lambda_{R_p},n}(x,t,\theta_F^*,h_n)\varepsilon_n^\theta\\
&+\frac{1}{n\cdot h_n^D}\cdot m_{\beta,n}^{R_p}(x,t,\theta_F^*,h_n) + \frac{1}{h_n^D}\cdot\nu_{\beta_{R_p},n}(x,t,\theta_F^*,h_n)\varepsilon_n^\theta
\end{aligned} \tag{S3.2.30}$$

Therefore we can express

$$\xi_{a,n}^{R_p}(x,t,\theta_F^*) = \left(\frac{n-1}{n}\right)\cdot\left(\frac{1}{h_n^{D+1}}\cdot E_F\left[\Lambda_{R_p}(V,x,t,\theta_F^*,h_n)\right] + \frac{1}{h_n^D}\cdot E_F\left[\beta_{R_p}(V,x,t,\theta_F^*,h_n)\right]\right)\cdot\frac{1}{n}\sum_{i=1}^n \psi_F^\theta(Z_i)$$
$$+\left(\frac{1}{h_n^{D+1}}\cdot E_F\left[\Lambda_{R_p}(V,x,t,\theta_F^*,h_n)\right] + \frac{1}{h_n^{D+1}}\cdot E_F\left[\beta_{R_p}(V,x,t,\theta_F^*,h_n)\right]\right)\varepsilon_n^\theta + \varsigma_{1,n}^{R_p}(x,t), \quad \text{where} \tag{S3.2.31}$$

$$\varsigma_{1,n}^{R_p}(x,t) \equiv \left(\frac{n-1}{2n}\right)\cdot\left(\frac{1}{h_n^{D+1}}\cdot\widetilde{U}_{R_p,n}^{\Lambda}(x,t,\theta_F^*,h_n) + \frac{1}{h_n^D}\cdot\widetilde{U}_{R_p,n}^{\beta}(x,t,\theta_F^*,h_n)\right)$$
$$+\frac{1}{n}\cdot\left(\frac{1}{h_n^{D+1}}\cdot E_F\left[\Lambda_{R_p}(V,x,t,\theta_F^*,h_n)\psi_F^\theta(Z)\right] + \frac{1}{h_n^D}\cdot E_F\left[\beta_{R_p}(V,x,t,\theta_F^*,h_n)\psi_F^\theta(Z)\right]\right)$$
$$+\frac{1}{n\cdot h_n^{D+1}}\cdot m_{\Lambda,n}^{R_p}(x,t,\theta_F^*,h_n) + \frac{1}{h_n^{D+1}}\cdot\nu_{\Lambda_{R_p},n}(x,t,\theta_F^*,h_n)\varepsilon_n^\theta + \frac{1}{n\cdot h_n^D}\cdot m_{\beta,n}^{R_p}(x,t,\theta_F^*,h_n) + \frac{1}{h_n^D}\cdot\nu_{\beta_{R_p},n}(x,t,\theta_F^*,h_n)\varepsilon_n^\theta$$

Let us analyze each of the components of $\varsigma_{1,n}^{R_p}(x,t)$. From (S3.2.13),

$$\left(\frac{n-1}{2n}\right)\cdot\left(\frac{1}{h_n^{D+1}}\cdot\sup_{(x,t)\in\mathcal{S}_X\times\mathcal{T}}\left|\widetilde{U}_{R_p,n}^{\Lambda}(x,t,\theta_F^*,h_n)\right| + \frac{1}{h_n^D}\cdot\sup_{(x,t)\in\mathcal{S}_X\times\mathcal{T}}\left|\widetilde{U}_{R_p,n}^{\beta}(x,t,\theta_F^*,h_n)\right|\right)$$
$$= O_p\left(\frac{1}{n\cdot h_n^{D+1}}\right) + O_p\left(\frac{1}{n\cdot h_n^D}\right) = O_p\left(\frac{1}{n\cdot h_n^{D+1}}\right).$$

From (S3.2.9),

$$\left.\begin{aligned}&\tfrac{1}{n\cdot h_n^{D+1}}\cdot\sup_{(x,t)\in\mathcal{S}_X\times\mathcal{T}}\left|E_F\left[\Lambda_{R_p}(V,x,t,\theta_F^*,h_n)\psi_F^\theta(Z)\right]\right| \le \tfrac{1}{n\cdot h_n^{D+1}}\cdot\overline{\eta}_{m,\Lambda}\\ &\tfrac{1}{n\cdot h_n^{D}}\cdot\sup_{(x,t)\in\mathcal{S}_X\times\mathcal{T}}\left|E_F\left[\beta_{R_p}(V,x,t,\theta_F^*,h_n)\psi_F^\theta(Z)\right]\right| \le \tfrac{1}{n\cdot h_n^{D}}\cdot\overline{\eta}_{m,\beta}\end{aligned}\right\}\quad \forall\ F\in\mathcal{F}$$

From (S3.2.10),

$$\left.\begin{aligned}&\tfrac{1}{n\cdot h_n^{D+1}}\cdot\sup_{(x,t)\in\mathcal{S}_X\times\mathcal{T}}\left|m_{\Lambda,n}^{R_p}(x,t,\theta_F^*,h_n)\right| = \tfrac{1}{n\cdot h_n^{D+1}}\cdot O_p\left(\tfrac{1}{n^{1/2}}\right) = O_p\left(\tfrac{1}{n^{3/2}\cdot h_n^{D+1}}\right)\\ &\tfrac{1}{n\cdot h_n^{D}}\cdot\sup_{(x,t)\in\mathcal{S}_X\times\mathcal{T}}\left|m_{\beta,n}^{R_p}(x,t,\theta_F^*,h_n)\right| = \tfrac{1}{n\cdot h_n^{D}}\cdot O_p\left(\tfrac{1}{n^{1/2}}\right) = O_p\left(\tfrac{1}{n^{3/2}\cdot h_n^{D}}\right)\end{aligned}\right\}\quad\text{uniformly over }\mathcal{F}$$

Let $\tau > 0$ be the constant in Assumption 1. From the conditions described there and (S3.2.8),

$$\left.\begin{aligned}&\tfrac{1}{h_n^{D+1}}\cdot\sup_{(x,t)\in\mathcal{S}_X\times\mathcal{T}}\left\|\nu_{\Lambda_{R_p},n}(x,t,\theta_F^*,h_n)\right\|\cdot\left\|\varepsilon_n^\theta\right\| = \tfrac{1}{h_n^{D+1}}\cdot O_p\left(\tfrac{1}{n^{1/2}}\right)\cdot o_p\left(\tfrac{1}{n^{1/2+\tau}}\right) = o_p\left(\tfrac{1}{n^{1+\tau}\cdot h_n^{D+1}}\right)\\ &\tfrac{1}{h_n^{D}}\cdot\sup_{(x,t)\in\mathcal{S}_X\times\mathcal{T}}\left\|\nu_{\beta_{R_p},n}(x,t,\theta_F^*,h_n)\right\|\cdot\left\|\varepsilon_n^\theta\right\| = \tfrac{1}{h_n^{D}}\cdot O_p\left(\tfrac{1}{n^{1/2}}\right)\cdot o_p\left(\tfrac{1}{n^{1/2+\tau}}\right) = o_p\left(\tfrac{1}{n^{1+\tau}\cdot h_n^{D}}\right)\end{aligned}\right\}\quad\text{uniformly over }\mathcal{F}$$

Therefore,

$$\begin{aligned}
\sup_{(x,t)\in\mathcal{S}_X\times\mathcal{T}} \left|\varsigma_{1,n}^{R_p}(x,t)\right| &= O_p\left(\frac{1}{n\cdot h_n^{D+1}}\right) + O\left(\frac{1}{n\cdot h_n^{D+1}}\right) + O_p\left(\frac{1}{n^{3/2}\cdot h_n^{D+1}}\right) + o_p\left(\frac{1}{n^{1+\tau}\cdot h_n^{D+1}}\right)\\
&\quad + O_p\left(\frac{1}{n\cdot h_n^{D}}\right) + O\left(\frac{1}{n\cdot h_n^{D}}\right) + O_p\left(\frac{1}{n^{3/2}\cdot h_n^{D}}\right) + o_p\left(\frac{1}{n^{1+\tau}\cdot h_n^{D}}\right) \qquad \text{(S3.2.32)}\\
&= O_p\left(\frac{1}{n\cdot h_n^{D+1}}\right) \quad \text{uniformly over } \mathcal{F}
\end{aligned}$$

With this result at hand, going back to (S3.2.30) we see that we need to analyze the term

$$\frac{1}{h_n^{D+1}}\cdot E_F\left[\Lambda_{R_p}(V,x,t,\theta_F^*,h_n)\right] + \frac{1}{h_n^{D}}\cdot E_F\left[\beta_{R_p}(V,x,t,\theta_F^*,h_n)\right].$$

As defined in Assumption 2, for each $p=1,\ldots,P$, $\ell=1,\ldots,k$ and $d=1,\ldots,D$, let

$$\begin{aligned}
\Omega_{R_p,1}^{d,\ell}(g,t) &= E_F\left[S_p(Y,t)\omega_p(g(X,\theta_F^*))\frac{\partial g_d(X,\theta_F^*)}{\partial\theta_\ell}\middle| g(X,\theta_F^*)=g\right],\\
\Omega_{R_p,2}(g,t) &= E_F\left[S_p(Y,t)\omega_p(g(X,\theta_F^*))\middle| g(X,\theta_F^*)=g\right],\\
\Omega_{R_p,3}^{\ell}(g,t) &= E_F\left[S_p(Y,t)\frac{\partial\omega_p(g(X,\theta_F^*))}{\partial\theta_\ell}\middle| g(X,\theta_F^*)=g\right].
\end{aligned}$$

As we defined in equation (A1), for a given $F\in\mathcal{F}$ and $(x,t)\in\mathcal{X}\times\mathcal{T}$ let

$$\begin{aligned}
\Xi_{\ell,R_p}(x,t,\theta_F^*) &\equiv \sum_{d=1}^{D}\Bigg(\frac{\partial\left[\Omega_{R_p,2}(g(x,\theta_F^*),t)f_g(g(x,\theta_F^*))\right]}{\partial g_d}\cdot\frac{\partial g_d(x,\theta_F^*)}{\partial\theta_\ell} - \frac{\partial\left[\Omega_{R_p,1}^{d,\ell}(g(x,\theta_F^*),t)f_g(g(x,\theta_F^*))\right]}{\partial g_d}\\
&\qquad +\Omega_{R_p,3}^{\ell}(g(x,\theta_F^*),t)\cdot f_g(g(x,\theta_F^*))\Bigg),\\
\underbrace{\Xi_{R_p}(x,t,\theta_F^*)}_{1\times k} &\equiv \left(\Xi_{1,R_p}(x,t,\theta_F^*),\ldots,\Xi_{k,R_p}(x,t,\theta_F^*)\right)
\end{aligned}$$

From the smoothness conditions described in Assumption 2 and M^{th} order approximation yields the existence of a constant $\overline{B}_{2,R}<\infty$ such that,

$$\begin{aligned}
&\frac{1}{h_n^{D+1}}\cdot E_F\left[\Lambda_{R_p}(V,x,t,\theta_F^*,h_n)\right] + \frac{1}{h_n^{D}}\cdot E_F\left[\beta_{R_p}(V,x,t,\theta_F^*,h_n)\right] = \Xi_{R_p}(x,t,\theta_F^*) + B_n^{\Xi_{R_p}}(x,t), \quad \text{where}\\
&\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left\|B_n^{\Xi_{R_p}}(x,t)\right\| \le \overline{B}_{2,R}\cdot h_n^M \quad \forall\, F\in\mathcal{F}.
\end{aligned} \qquad \text{(S3.2.33)}$$

By Assumption 2, $\exists\ \overline{\eta}_{\Xi_{R_p}} < \infty$ such that

$$\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left\|\Xi_{R_p}(x,t,\theta_F^*)\right\| \leq \overline{\eta}_{\Xi_{R_p}} \quad \forall\ F\in\mathcal{F}. \tag{S3.2.34}$$

Plugging (S3.2.33) into (S3.2.31),

$$\xi_{a,n}^{R_p}(x,t,\theta_F^*) = \left(\frac{n-1}{n}\right)\cdot\left(\Xi_{R_p}(x,t,\theta_F^*) + B_n^{\Xi_{R_p}}(x,t)\right)\frac{1}{n}\sum_{i=1}^{n}\psi_F^{\theta}(Z_i) + \left(\Xi_{R_p}(x,t,\theta_F^*) + B_n^{\Xi_{R_p}}(x,t)\right)\varepsilon_n^{\theta} + \varsigma_{1,n}^{R_p}(x,t).$$

From here we can express

$$\xi_{a,n}^{R_p}(x,t,\theta_F^*) = \Xi_{R_p}(x,t,\theta_F^*)\frac{1}{n}\sum_{i=1}^{n}\psi_F^{\theta}(Z_i) + \varsigma_{2,n}^{R_p}(x,t) + \varsigma_{1,n}^{R_p}(x,t), \tag{S3.2.35}$$

where $\varsigma_{2,n}^{R_p}(x,t) \equiv \left(\left(\frac{n-1}{n}\right)\cdot B_n^{\Xi_{R_p}}(x,t) - \frac{1}{n}\right)\cdot\frac{1}{n}\sum_{i=1}^{n}\psi_F^{\theta}(Z_i) + \left(\Xi_{R_p}(x,t,\theta_F^*) + B_n^{\Xi_{R_p}}(x,t)\right)\varepsilon_n^{\theta}$. Let $\tau > 0$ be the constant in Assumption 1. From the conditions described there, (S3.2.33) and (S3.2.34) yield,

$$\begin{aligned}\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left|\varsigma_{2,n}^{R_p}(x,t)\right| &= \left(O\left(h_n^M\right) + O\left(\frac{1}{n}\right)\right)\cdot O_p\left(\frac{1}{n^{1/2}}\right) + \left(O(1) + O\left(h_n^M\right)\right)\cdot o_p\left(\frac{1}{n^{1/2+\tau}}\right)\\ &= O_p\left(\max\left\{\frac{h_n^M}{n^{1/2}}\ ,\ \frac{1}{n^{3/2}}\ ,\ \frac{1}{n^{1/2+\tau}}\right\}\right) \quad \text{uniformly over } \mathcal{F}\end{aligned}$$

Combining this with (S3.2.32), and defining $\varsigma_n^{R_p}(x,t) \equiv \varsigma_{1,n}^{R_p}(x,t) + \varsigma_{2,n}^{R_p}(x,t)$, (S3.2.35) becomes

$$\begin{aligned}&\xi_{a,n}^{R_p}(x,t,\theta_F^*) = \Xi_{R_p}(x,t,\theta_F^*)\cdot\frac{1}{n}\sum_{i=1}^{n}\psi_F^{\theta}(Z_i) + \varsigma_n^{R_p}(x,t),\\ &\text{where} \quad \sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left|\varsigma_n^{R_p}(x,t)\right| = O_p\left(\max\left\{\frac{h_n^M}{n^{1/2}}\ ,\ \frac{1}{n^{3/2}}\ ,\ \frac{1}{n^{1/2+\tau}}\ ,\ \frac{1}{n\cdot h_n^{D+1}}\right\}\right) \quad \text{uniformly over } \mathcal{F}\end{aligned} \tag{S3.2.36}$$

Let

$$\begin{aligned}&\psi_F^{R_p}(V_i,x,t,\theta_F^*,h_n)\\ &\equiv \frac{1}{h_n^D}\cdot\left(S_p(Y_i,t)\omega_p(g(X_i,\theta_F^*))K\left(\frac{\Delta g(X_i,x,\theta_F^*)}{h_n}\right) - E_F\left[S_p(Y_i,t)\omega_p(g(X_i,\theta_F^*))K\left(\frac{\Delta g(X_i,x,\theta_F^*)}{h_n}\right)\right]\right)\\ &\quad + \Xi_{R_p}(x,t,\theta_F^*)\psi_F^{\theta}(Z_i).\end{aligned} \tag{S3.2.37}$$

Combining (S3.2.25) and (S3.2.36), we have

$$\begin{aligned}&\widehat{R_p}(x,t,\widehat{\theta}) = R_{p,F}(x,t,\theta_F^*) + \frac{1}{n}\sum_{i=1}^{n}\psi_F^{R_p}(V_i,x,t,\theta_F^*,h_n) + \zeta_n^{R_p}(x,t),\\ &\text{where}\quad \zeta_n^{R_p}(x,t) \equiv B_n^{R_p}(x,t) + \varsigma_n^{R_p}(x,t) + \xi_{b,n}^{R_p}(x,t,\overline{\theta}_x)\end{aligned} \tag{S3.2.38}$$

Let $\epsilon > 0$ be the constant described in Assumption 4. Applying the results from (S3.2.5), (S3.2.26) and (S3.2.36),

$$\begin{aligned}\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\zeta_n^{R_p}(x,t)\right| &= O\left(h_n^M\right) + O_p\left(\max\left\{\frac{h_n^M}{n^{1/2}}, \frac{1}{n^{3/2}}, \frac{1}{n^{1/2+\tau}}\right\}\right) + O_p\left(\frac{1}{n\cdot h_n^{D+2}}\right)\\ &= O_p\left(\max\left\{h_n^M, \frac{1}{n^{3/2}}, \frac{1}{n^{1/2+\tau}}, \frac{1}{n\cdot h_n^{D+2}}\right\}\right) = o_p\left(\frac{1}{n^{1/2+\epsilon}}\right) \quad \text{uniformly over } \mathcal{F}\end{aligned} \tag{S3.2.39}$$

By the conditions in Assumptions 1, 2, 3 and 4 along with the results in (S3.2.24) and (S3.2.34), we can once again invoke Result S1 to show that

$$\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\frac{1}{n}\sum_{i=1}^{n}\psi_F^{R_p}(V_i,x,t,\theta_F^*,h_n)\right| = O_p\left(\frac{1}{n^{1/2}\cdot h_n^D}\right) \quad \text{uniformly over } \mathcal{F}. \tag{S3.2.40}$$

And therefore,

$$\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\widehat{R_p}(x,t,\widehat{\theta}) - R_{p,F}(x,t,\theta_F^*)\right| = O_p\left(\frac{1}{n^{1/2}\cdot h_n^D}\right) + o_p\left(\frac{1}{n^{1/2+\epsilon}}\right) = O_p\left(\frac{1}{n^{1/2}\cdot h_n^D}\right) \quad \text{uniformly over } \mathcal{F}. \tag{S3.2.41}$$

S3.3 Asymptotic properties of $\widehat{Q_p}(x,t,\widehat{\theta})$

Recall that $\widehat{Q_p}(x,t,\widehat{\theta}) \equiv \frac{\widehat{R_p}(x,t,\widehat{\theta})}{\widehat{f_g}(g(x,\widehat{\theta}))}$, which is an estimator for the functional,

$$Q_{p,F}(x,t,\theta_F^*) \equiv \Gamma_{p,F}(x,t,\theta_F^*)\cdot\omega_p(g(x,\theta_F^*)) = \frac{R_{p,F}(x,t,\theta_F^*)}{f_g(g(x,\theta_F^*))}.$$

Combining our results for $\widehat{f_g}(g(x,\widehat{\theta}))$ and $\widehat{R_p}(x,t,\widehat{\theta})$ we can obtain the relevant asymptotic properties of $\widehat{Q_p}(x,t,\widehat{\theta})$. These are summarized in the following result.

Proposition S1 *Under Assumptions 1-4, the following results hold.*

(i) There exist finite constants $A_1 > 0$, $A_2 > 0$, $C_1 > 0$, $C_2 > 0$ and $C_3 > 0$ such that, for any sequence

$s_n > 0$ *such that* $s_n \longrightarrow 0$, *with*

$$\frac{h_n^M}{s_n} \longrightarrow 0 \quad \text{and} \quad s_n \cdot n \cdot h_n^{D+1} \longrightarrow \infty,$$

we have,

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left\|\widehat{Q}(x,t,\widehat{\theta}) - Q_F(x,t,\theta_F^*)\right\| \geq s_n\right)$$
$$= O\left(\frac{1}{\left((n^{1/2}\wedge r_n)\cdot h_n^{D+1}\cdot\left(A_1\cdot s_n - C_1\cdot h_n^M - \frac{C_2}{n\cdot h_n^{D+1}} - \frac{C_3}{n\cdot h_n^D}\right)\right)^q}\right)$$

(ii) Let $\Xi_{f_g}(x,\theta_F^*)$ *and* $\Xi_{R_p}(x,t,\theta_F^*)$ *be as described in* (A1) *and, for each p, define*

$$\underbrace{\Xi_{Q_p}(x,t,\theta_F^*)}_{1\times k} \equiv \frac{\Xi_{R_p}(x,t,\theta_F^*) - Q_{p,F}(x,t,\theta_F^*)\cdot\Xi_{f_g}(x,\theta_F^*)}{f_g(g(x,\theta_F^*))} \tag{S3.3.1}$$

and

$$\begin{aligned}\psi_F^{Q_p}(V_i,x,t,\theta_F^*,h_n) \equiv &\frac{1}{h_n^D}\Bigg\{\left(\frac{S_p(Y_i,t)-\Gamma_{p,F}(x,t,\theta_F^*)}{f_g(g(x,\theta_F^*))}\right)\cdot\omega_p(g(X_i,\theta_F^*))\cdot K\left(\frac{\Delta g(X_i,x,\theta_F^*)}{h_n}\right)\\ &-E_F\left[\left(\frac{S_p(Y_i,t)-\Gamma_{p,F}(x,t,\theta_F^*)}{f_g(g(x,\theta_F^*))}\right)\cdot\omega_p(g(X_i,\theta_F^*))\cdot K\left(\frac{\Delta g(X_i,x,\theta_F^*)}{h_n}\right)\right]\Bigg\}\\ &+\Xi_{Q_p}(x,t,\theta_F^*)\psi_F^\theta(Z_i).\end{aligned}$$

And let

$$\psi_F^Q(V_i,x,t,\theta_F^*,h_n) \equiv \left(\psi_F^{Q_1}(V_i,x,t,\theta_F^*,h_n),\ldots,\psi_F^{Q_P}(V_i,x,t,\theta_F^*,h_n)\right)'.$$

We have

$$\widehat{Q}(x,t,\widehat{\theta}) = Q_F(x,t,\theta_F^*) + \frac{1}{n}\sum_{i=1}^n \psi_F^Q(V_i,x,t,\theta_F^*,h_n) + \zeta_n^Q(x,t), \quad \text{where}$$
$$\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left\|\zeta_n^Q(x,t)\right\| = o_p\left(\frac{1}{n^{1/2+\epsilon}}\right) \quad \text{uniformly over } \mathcal{F}.$$

where $\epsilon > 0$ *is the constant described in Assumption 4. And we have,*

$$\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left\|\widehat{Q}(x,t,\widehat{\theta}) - Q_F(x,t,\theta_F^*)\right\| = o_p\left(\frac{1}{n^{1/4+\epsilon/2}}\right) \quad \text{uniformly over } \mathcal{F}.$$

S3.3.1 Proof of part (i) of Proposition S1

Let K_1, K_2 and K_3 be the constants described in (S3.1.45) and let K_4, K_5, K_6 and C_3 be the constants described in (S3.2.29), and define $C_0 \equiv K_1 \wedge K_4$, $C_1 \equiv K_2 \vee K_5$ and $C_2 \equiv K_3 \vee K_6$. Then, for any sequence $s_n > 0$ (possibly converging to zero) such that $\frac{h_n^M}{s_n} \longrightarrow 0$ and $s_n \cdot n \cdot h_n^{D+1} \longrightarrow \infty$, we have

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{x\in\mathcal{X}}\left|\widehat{f_g}(g(x,\widehat{\theta})) - f_g(g(x,\theta_F^*))\right| \geq s_n\right) + \sup_{F\in\mathcal{F}} P_F\left(\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\widehat{R}_{p,F}(x,t,\widehat{\theta}) - R_{p,F}(x,t,\theta_F^*)\right| \geq s_n\right)$$
$$= O\left(\frac{1}{\left((n^{1/2}\wedge r_n)\cdot h_n^{D+1}\cdot\left(\left(C_0\cdot s_n - C_1\cdot h_n^M - \frac{C_2}{n\cdot h_n^{D+1}} - \frac{C_3}{n\cdot h_n^D}\right)\wedge\left(C_0\cdot s_n - C_1\cdot h_n^M - \frac{C_2}{n\cdot h_n^{D+1}} - \frac{C_3}{n\cdot h_n^D}\right)^{1/4}\right)\right)^q}\right) \tag{S3.3.2}$$

We have

$$\widehat{Q}_p(x,t,\widehat{\theta}) = Q_{p,F}(x,t,\theta_F^*) + Q_{p,F}(x,t,\theta_F^*)\cdot\frac{\left(f_g(g(x,\theta_F^*)) - \widehat{f_g}(g(x,\widehat{\theta}))\right)}{\widehat{f_g}(g(x,\widehat{\theta}))} + \frac{\left(\widehat{R}_p(x,t,\widehat{\theta}) - R_{p,F}(x,t,\theta_F^*)\right)}{\widehat{f_g}(g(x,\widehat{\theta}))}$$

Recall from Assumption 2 that $\exists\ \underline{f}_g > 0$, $\overline{\Gamma} < \infty$ such that, $\forall\ F \in \mathcal{F}$, $\inf_{x\in\mathcal{X}} f_g(g(x,\theta_F^*)) \geq \underline{f}_g$, and $\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\Gamma_{p,F}(x,t,\theta_F^*)\right| \leq \overline{\Gamma}$, for each $p = 1,\ldots,P$. Also recall that the nonnegative weight function $\omega_p(\cdot)$ is bounded above by $\overline{\omega}$. Therefore, $\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|Q_{p,F}(x,t,\theta_F^*)\right| \leq \overline{\Gamma}\cdot\overline{\omega} \equiv \overline{Q}$. Thus,

$$\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\widehat{Q}_p(x,t,\widehat{\theta}) - Q_{p,F}(x,t,\theta_F^*)\right| \leq \frac{\overline{Q}\cdot\sup_{x\in\mathcal{X}}\left|\widehat{f_g}(g(x,\widehat{\theta})) - f_g(g(x,\theta_F^*))\right|}{\inf_{x\in\mathcal{X}}\left|\widehat{f_g}(g(x,\widehat{\theta}))\right|} + \frac{\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\widehat{R}_p(x,t,\widehat{\theta}) - R_{p,F}(x,t,\theta_F^*)\right|}{\inf_{x\in\mathcal{X}}\left|\widehat{f_g}(g(x,\widehat{\theta}))\right|}.$$

Recall from (S3.1.59) that $\sup_{x\in\mathcal{X}}\left|\frac{1}{\widehat{f_g}(g(x,\widehat{\theta}))}\right| = \frac{1}{\inf_{x\in\mathcal{X}}\left|\widehat{f_g}(g(x,\widehat{\theta}))\right|} = O_p(1)$ uniformly over $\mathcal{F}$. Take any $s > 0$. From here and the previous expression,

$$\mathbb{1}\left\{\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\widehat{Q}_p(x,t,\widehat{\theta}) - Q_{p,F}(x,t,\theta_F^*)\right| \geq s\right\} \leq \underbrace{\mathbb{1}\left\{\sup_{x\in\mathcal{X}}\left|\widehat{f_g}(g(x,\widehat{\theta})) - f_g(g(x,\theta_F^*))\right| \geq \left(\frac{s}{2\overline{Q}}\right)\cdot\inf_{x\in\mathcal{X}}\left|\widehat{f_g}(g(x,\widehat{\theta}))\right|\right\}}_{(A)}$$
$$+ \underbrace{\mathbb{1}\left\{\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\widehat{R}_p(x,t,\widehat{\theta}) - R_{p,F}(x,t,\theta_F^*)\right| \geq \left(\frac{s}{2}\right)\cdot\inf_{x\in\mathcal{X}}\left|\widehat{f_g}(g(x,\widehat{\theta}))\right|\right\}}_{(B)} \tag{S3.3.3}$$

Let us analyze the indicator function (A) in (S3.3.3). We have

$$\begin{aligned}
&\mathbb{1}\left\{\sup_{x\in\mathcal{X}}\left|\widehat{f_g}(g(x,\widehat{\theta}))-f_g(g(x,\theta_F^*))\right|\geq\left(\frac{s}{2\overline{Q}}\right)\cdot\inf_{x\in\mathcal{X}}\left|\widehat{f_g}(g(x,\widehat{\theta}))\right|\right\}\\
&=\underbrace{\mathbb{1}\left\{\sup_{x\in\mathcal{X}}\left|\widehat{f_g}(g(x,\widehat{\theta}))-f_g(g(x,\theta_F^*))\right|\geq\left(\frac{s}{2\overline{Q}}\right)\cdot\inf_{x\in\mathcal{X}}\left|\widehat{f_g}(g(x,\widehat{\theta}))\right|\right\}\cdot\mathbb{1}\left\{\inf_{x\in\mathcal{X}}\left|\widehat{f_g}(g(x,\widehat{\theta}))\right|\geq\frac{1}{2}\cdot\underline{f}_g\right\}}_{\leq\mathbb{1}\left\{\sup_{x\in\mathcal{X}}\left|\widehat{f_g}(g(x,\widehat{\theta}))-f_g(g(x,\theta_F^*))\right|\geq\left(\frac{\underline{f}_g}{4\overline{Q}}\right)\cdot s\right\}}\\
&+\underbrace{\mathbb{1}\left\{\sup_{x\in\mathcal{X}}\left|\widehat{f_g}(g(x,\widehat{\theta}))-f_g(g(x,\theta_F^*))\right|\geq\left(\frac{s}{2\overline{Q}}\right)\cdot\inf_{x\in\mathcal{X}}\left|\widehat{f_g}(g(x,\widehat{\theta}))\right|\right\}\cdot\mathbb{1}\left\{\inf_{x\in\mathcal{X}}\left|\widehat{f_g}(g(x,\widehat{\theta}))\right|<\frac{1}{2}\cdot\underline{f}_g\right\}}_{\leq 1\times\mathbb{1}\left\{\sup_{x\in\mathcal{X}}\left|\widehat{f_g}(g(x,\widehat{\theta}))-f_g(g(x,\theta_F^*))\right|\geq\frac{1}{2}\cdot\underline{f}_g\right\}}\\
&\leq\mathbb{1}\left\{\sup_{x\in\mathcal{X}}\left|\widehat{f_g}(g(x,\widehat{\theta}))-f_g(g(x,\theta_F^*))\right|\geq\left(\frac{\underline{f}_g}{4\overline{Q}}\cdot s\right)\wedge\frac{\underline{f}_g}{2}\right\}.
\end{aligned}\tag{S3.3.4A}$$

Next, the indicator function (B) in (S3.3.3). We have

$$\begin{aligned}
&\mathbb{1}\left\{\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\widehat{R}_p(x,t,\widehat{\theta})-R_{p,F}(x,t,\theta_F^*)\right|\geq\left(\frac{s}{2}\right)\cdot\inf_{x\in\mathcal{X}}\left|\widehat{f_g}(g(x,\widehat{\theta}))\right|\right\}\\
&=\underbrace{\mathbb{1}\left\{\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\widehat{R}_p(x,t,\widehat{\theta})-R_{p,F}(x,t,\theta_F^*)\right|\geq\left(\frac{s}{2}\right)\cdot\inf_{x\in\mathcal{X}}\left|\widehat{f_g}(g(x,\widehat{\theta}))\right|\right\}\cdot\mathbb{1}\left\{\inf_{x\in\mathcal{X}}\left|\widehat{f_g}(g(x,\widehat{\theta}))\right|\geq\frac{1}{2}\cdot\underline{f}_g\right\}}_{\mathbb{1}\left\{\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\widehat{R}_p(x,t,\widehat{\theta})-R_{p,F}(x,t,\theta_F^*)\right|\geq\left(\frac{\underline{f}_g}{4}\right)\cdot s\right\}}\\
&+\underbrace{\mathbb{1}\left\{\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\widehat{R}_p(x,t,\widehat{\theta})-R_{p,F}(x,t,\theta_F^*)\right|\geq\left(\frac{s}{2}\right)\cdot\inf_{x\in\mathcal{X}}\left|\widehat{f_g}(g(x,\widehat{\theta}))\right|\right\}\cdot\mathbb{1}\left\{\inf_{x\in\mathcal{X}}\left|\widehat{f_g}(g(x,\widehat{\theta}))\right|<\frac{1}{2}\cdot\underline{f}_g\right\}}_{\leq 1\times\mathbb{1}\left\{\sup_{x\in\mathcal{X}}\left|\widehat{f_g}(g(x,\widehat{\theta}))-f_g(g(x,\theta_F^*))\right|\geq\frac{1}{2}\cdot\underline{f}_g\right\}}\\
&\leq\mathbb{1}\left\{\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\widehat{R}_p(x,t,\widehat{\theta})-R_{p,F}(x,t,\theta_F^*)\right|\geq\left(\frac{\underline{f}_g}{4}\right)\cdot s\right\}\vee\mathbb{1}\left\{\sup_{x\in\mathcal{X}}\left|\widehat{f_g}(g(x,\widehat{\theta}))-f_g(g(x,\theta_F^*))\right|\geq\frac{1}{2}\cdot\underline{f}_g\right\}
\end{aligned}\tag{S3.3.4B}$$

Let $D_1\equiv\frac{\underline{f}_g}{4}\wedge\frac{\underline{f}_g}{4\overline{Q}}$ and $D_2\equiv\frac{\underline{f}_g}{2}$. Combining (S3.3.4A) and (S3.3.4B) with (S3.3.3), $\forall\ s>0$,

$$\begin{aligned}
\mathbb{1}\left\{\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\widehat{Q}_p(x,t,\widehat{\theta})-Q_{p,F}(x,t,\theta_F^*)\right|\geq s\right\}&\leq\mathbb{1}\left\{\sup_{x\in\mathcal{X}}\left|\widehat{f_g}(g(x,\widehat{\theta}))-f_g(g(x,\theta_F^*))\right|\geq(D_1\cdot s)\wedge D_2\right\}\\
&+\mathbb{1}\left\{\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\widehat{R}_p(x,t,\widehat{\theta})-R_{p,F}(x,t,\theta_F^*)\right|\geq(D_1\cdot s)\wedge D_2\right\}
\end{aligned}$$

Let $\widetilde{A}_1 \equiv C_0 \cdot D_1$ and $A_2 \equiv C_0 \cdot D_2$. From here and (S3.3.2), we have that for any $s > 0$,

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left|\widehat{Q}_p(x,t,\widehat{\theta}) - Q_{p,F}(x,t,\theta_F^*)\right| \geq s\right)$$
$$= O\left(\frac{1}{\left((n^{1/2}\wedge r_n)\cdot h_n^{D+1}\cdot\left(\left(\left(\left(\widetilde{A}_1\cdot s\right)\wedge A_2\right) - C_1\cdot h_n^M - \frac{C_2}{n\cdot h_n^{D+1}} - \frac{C_3}{n\cdot h_n^D}\right)\wedge\left(\left(\left(\widetilde{A}_1\cdot s\right)\wedge A_2\right) - C_1\cdot h_n^M - \frac{C_2}{n\cdot h_n^{D+1}} - \frac{C_3}{n\cdot h_n^D}\right)^{1/4}\right)\right)^q}\right)$$

Next, let us stack $\widehat{Q}(x,t,\widehat{\theta}) \equiv \left(\widehat{Q}_1(x,t,\widehat{\theta}),\ldots,\widehat{Q}_P(x,t,\widehat{\theta})\right)'$, and $Q_F(x,t,\theta_F^*) \equiv \left(Q_{1,F}(x,t,\theta_F^*),\ldots,Q_{P,F}(x,t,\theta_F^*)\right)'$. By the equivalence of norms in Euclidean space, for some constant $m > 0$ that depends only on P,

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left\|\widehat{Q}(x,t,\widehat{\theta}) - Q_F(x,t,\theta_F^*)\right\| \geq s\right) \leq \sum_{p=1}^{P} \sup_{F\in\mathcal{F}} P_F\left(\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left|\widehat{Q}_p(x,t,\widehat{\theta}) - Q_{p,F}(x,t,\theta_F^*)\right| \geq m\cdot s\right)$$

for any $s > 0$. Therefore, if we let $A_1 \equiv m\cdot\widetilde{A}_1$, the above result yields,

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left\|\widehat{Q}(x,t,\widehat{\theta}) - Q_F(x,t,\theta_F^*)\right\| \geq s\right)$$
$$= O\left(\frac{1}{\left((n^{1/2}\wedge r_n)\cdot h_n^{D+1}\cdot\left(\left(((A_1\cdot s)\wedge A_2) - C_1\cdot h_n^M - \frac{C_2}{n\cdot h_n^{D+1}} - \frac{C_3}{n\cdot h_n^D}\right)\wedge\left(((A_1\cdot s)\wedge A_2) - C_1\cdot h_n^M - \frac{C_2}{n\cdot h_n^{D+1}} - \frac{C_3}{n\cdot h_n^D}\right)^{1/4}\right)\right)^q}\right)$$

Take any sequence $s_n > 0$ such that $s_n \longrightarrow 0$ and $\frac{h_n^M}{s_n} \longrightarrow 0$ and $s_n\cdot n\cdot h_n^{D+1} \longrightarrow \infty$. Since $s_n \longrightarrow 0$, $\exists\ n_0$ such that

$$A_1\cdot s_n < A_2 \quad\text{and}\quad A_1\cdot s_n - C_1\cdot h_n^M - \frac{C_2}{n\cdot h_n^{D+1}} - \frac{C_3}{n\cdot h_n^D} < 1 \quad \forall\ n > n_0.$$

Note that for any such s_n,

$$\left(((A_1\cdot s_n)\wedge A_2) - C_1\cdot h_n^M - \frac{C_2}{n\cdot h_n^{D+1}} - \frac{C_3}{n\cdot h_n^D}\right)\wedge\left(((A_1\cdot s_n)\wedge A_2) - C_1\cdot h_n^M - \frac{C_2}{n\cdot h_n^{D+1}} - \frac{C_3}{n\cdot h_n^D}\right)^{1/4}$$
$$= A_1\cdot s_n - C_1\cdot h_n^M - \frac{C_2}{n\cdot h_n^{D+1}} - \frac{C_3}{n\cdot h_n^D} \quad \forall\ n > n_0.$$

Therefore, for any such sequence s_n we have

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left\|\widehat{Q}(x,t,\widehat{\theta}) - Q_F(x,t,\theta_F^*)\right\| \geq s_n\right)$$
$$= O\left(\frac{1}{\left(\left(n^{1/2}\wedge r_n\right)\cdot h_n^{D+1}\cdot\left(A_1\cdot s_n - C_1\cdot h_n^M - \frac{C_2}{n\cdot h_n^{D+1}} - \frac{C_3}{n\cdot h_n^{D}}\right)\right)^q}\right) \tag{S3.3.5}$$

This proves part (i) of Proposition S1, ■

S3.4 Proof of part (ii) of Proposition S1

A second-order approximation yields

$$\widehat{Q}_p(x,t,\widehat{\theta}) = Q_{p,F}(x,t,\theta_F^*) + \frac{1}{f_g(g(x,\theta_F^*)} \cdot \left(\widehat{R}_p(x,t,\widehat{\theta}) - R_{p,F}(x,t,\theta_F^*)\right) - \frac{Q_{p,F}(x,t,\theta_F^*)}{f_g(g(x,\theta_F^*))}\cdot\left(\widehat{f}_g(g(x,\widehat{\theta})) - f_g(x,\theta_F^*)\right)$$
$$- \frac{\left(\widehat{R}_p(x,t,\widehat{\theta}) - R_{p,F}(x,t,\theta_F^*)\right)\cdot\left(\widehat{f}_g(g(x,\widehat{\theta})) - f_g(x,\theta_F^*)\right)}{\widetilde{f}_g(x)^2} + \frac{\widetilde{R}_p(x,t)\cdot\left(\widehat{f}_g(g(x,\widehat{\theta})) - f_g(x,\theta_F^*)\right)^2}{\widetilde{f}_g(x)^3} \tag{S3.4.1}$$

where $\widetilde{f}_g(x)$ is an intermediate point between $\widehat{f}_g(g(x,\widehat{\theta}))$ and $f_g(x,\theta_F^*)$, and $\widetilde{R}_p(x,t)$ is an intermediate point between $\widehat{R}_p(x,t,\widehat{\theta})$ and $R_{p,F}(x,t,\theta_F^*)$. In (S3.1.59) we showed that $\sup_{x\in\mathcal{X}}\left|\frac{1}{\widehat{f}_g(g(x,\widehat{\theta}))}\right| = O_p(1)$ uniformly over $\mathcal{F}$. We showed that this follows because, for any $\delta\in(0,1)$,

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{x\in\mathcal{X}}\left|\frac{1}{\widehat{f}_g(g(x,\widehat{\theta}))}\right| > \frac{1}{(1-\delta)\cdot \underline{f}_g}\right) \leq \sup_{F\in\mathcal{F}} P_F\left(\sup_{x\in\mathcal{X}}\left|\widehat{f}_g(g(x,\widehat{\theta})) - f_g(g(x,\theta_F^*))\right| > \delta\cdot\underline{f}_g\right)$$

and, from (S3.1.45), for any $\delta\in(0,1)$ and $\epsilon>0$, $\exists\ n_{\delta,\epsilon}$ such that

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{x\in\mathcal{X}}\left|\widehat{f}_g(g(x,\widehat{\theta})) - f_g(g(x,\theta_F^*))\right| > \delta\cdot\underline{f}_g\right) < \epsilon \quad \forall\ n > n_{\delta,\epsilon}.$$

Similarly, we have

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{x\in\mathcal{X}}\left|\frac{1}{\widetilde{f}_g(g(x)}\right| > \frac{1}{(1-\delta)\cdot \underline{f}_g}\right) \leq \sup_{F\in\mathcal{F}} P_F\left(\sup_{x\in\mathcal{X}}\left|\widetilde{f}_g(x) - f_g(g(x,\theta_F^*))\right| > \delta\cdot\underline{f}_g\right).$$

Since $\widetilde{f_g}(x)$ is an intermediate value between $\widehat{f_g}(g(x,\widehat{\theta}))$ and $f_g(x,\theta^*_F)$, we have

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{x\in\mathcal{X}}\left|\frac{1}{\widetilde{f_g}(g(x)}\right| > \frac{1}{(1-\delta)\cdot \underline{f}_g}\right) \le \sup_{F\in\mathcal{F}} P_F\left(\sup_{x\in\mathcal{X}}\left|\widehat{f_g}(g(x,\widehat{\theta})) - f_g(g(x,\theta^*_F))\right| > \delta\cdot \underline{f}_g\right)$$

and thus,

$$\sup_{x\in\mathcal{X}}\left|\frac{1}{\widetilde{f_g}(x)}\right| = O_p(1) \quad \text{uniformly over } \mathcal{F}. \tag{S3.4.2}$$

From Assumption 2, $\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|R_{p,F}(x,t,\theta^*_F)\right| \le \overline{\Gamma}\cdot\overline{\omega}\cdot\overline{f}_g \equiv \overline{R}$, and from (S3.2.41),

$$\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\widehat{R}_p(x,t,\widehat{\theta}) - R_{p,F}(x,t,\theta^*_F)\right| = O_p\left(\frac{1}{n^{1/2}\cdot h_n^D}\right) \quad \text{uniformly over } \mathcal{F}.$$

Therefore, since $\widetilde{R}_p(x,t)$ is an intermediate point between $\widehat{R}_p(x,t,\widehat{\theta})$ and $R_{p,F}(x,t,\theta^*_F)$,

$$\begin{aligned}
\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\widetilde{R}_p(x,t)\right| &\le \overline{R} + \sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\widetilde{R}_p(x,t) - R_{p,F}(x,t,\theta^*_F)\right| \\
&\le \overline{R} + \sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\widehat{R}_p(x,t,\widehat{\theta}) - R_{p,F}(x,t,\theta^*_F)\right| \\
&= O(1) + O_p\left(\frac{1}{n^{1/2}\cdot h_n^D}\right) = O_p(1) \quad \text{uniformly over } \mathcal{F}.
\end{aligned} \tag{S3.4.3}$$

Let $\psi_F^{f_g}(V_i,x,\theta^*_F,h_n)$ and $\psi_F^{R_p}(V_i,x,t,\theta^*_F,h_n)$ be as described in (S3.1.54) and (S3.2.37) and define

$$\begin{aligned}
\psi_F^{Q_p}(V_i,x,t,\theta^*_F,h_n) &\equiv \frac{1}{f_g(g(x,\theta^*_F))}\cdot\psi_F^{R_p}(V_i,x,t,\theta^*_F,h_n) - \frac{Q_{p,F}(x,t,\theta^*_F)}{f_g(g(x,\theta^*_F)}\cdot\psi_F^{f_g}(V_i,x,\theta^*_F,h_n) \\
&= \frac{1}{h_n^D}\left\{\left(\frac{S_p(Y_i,t)-\Gamma_{p,F}(x,t,\theta^*_F)}{f_g(g(x,\theta^*_F))}\right)\cdot\omega_p(g(X_i,\theta^*_F))\cdot K\left(\frac{\Delta g(X_i,x,\theta^*_F)}{h_n}\right)\right. \\
&\quad \left. - E_F\left[\left(\frac{S_p(Y_i,t)-\Gamma_{p,F}(x,t,\theta^*_F)}{f_g(g(x,\theta^*_F))}\right)\cdot\omega_p(g(X_i,\theta^*_F))\cdot K\left(\frac{\Delta g(X_i,x,\theta^*_F)}{h_n}\right)\right]\right\} \\
&\quad + \left(\frac{\Xi_{R_p}(x,t,\theta^*_F) - Q_{p,F}(x,t,\theta^*_F)\cdot\Xi_{f_g}(x,\theta^*_F)}{f_g(g(x,\theta^*_F))}\right)\psi_F^{\theta}(Z_i).
\end{aligned}$$

Defining

$$\underbrace{\Xi_{Q_p}(x,t,\theta^*_F)}_{1\times k} \equiv \frac{\Xi_{R_p}(x,t,\theta^*_F) - Q_{p,F}(x,t,\theta^*_F)\cdot\Xi_{f_g}(x,\theta^*_F)}{f_g(g(x,\theta^*_F))},$$

we can re-write $\psi_F^{Q_p}(V_i,x,t,\theta_F^*,h_n)$ as,

$$\begin{aligned}\psi_F^{Q_p}(V_i,x,t,\theta_F^*,h_n) \equiv \frac{1}{h_n^D}\Bigg\{&\left(\frac{S_p(Y_i,t)-\Gamma_{p,F}(x,t,\theta_F^*)}{f_g(g(x,\theta_F^*))}\right)\cdot\omega_p(g(X_i,\theta_F^*))\cdot K\left(\frac{\Delta g(X_i,x,\theta_F^*)}{h_n}\right)\\ &-E_F\left[\left(\frac{S_p(Y_i,t)-\Gamma_{p,F}(x,t,\theta_F^*)}{f_g(g(x,\theta_F^*))}\right)\cdot\omega_p(g(X_i,\theta_F^*))\cdot K\left(\frac{\Delta g(X_i,x,\theta_F^*)}{h_n}\right)\right]\Bigg\}\\ &+\Xi_{Q_p}(x,t,\theta_F^*)\psi_F^{\theta}(Z_i),\end{aligned} \tag{S3.4.4}$$

From the uniform linear representation in (S3.1.55) and (S3.2.38), equation (S3.4.1) becomes

$$\begin{aligned}\widehat{Q}_p(x,t,\widehat{\theta}) &= Q_{p,F}(x,t,\theta_F^*)+\frac{1}{n}\sum_{i=1}^n\psi_F^{Q_p}(V_i,x,t,\theta_F^*,h_n)+\zeta_n^{Q_p}(x,t), \quad \text{where}\\ \zeta_n^{Q_p}(x,t) &\equiv \frac{1}{f_g(g(x,\theta_F^*))}\cdot\zeta_n^{R_p}(x,t)-\frac{Q_{p,F}(x,t,\theta_F^*)}{f_g(g(x,\theta_F^*))}\cdot\zeta_n^{f_g}(x)\\ &\quad-\frac{\left(\widehat{R}_p(x,t,\widehat{\theta})-R_{p,F}(x,t,\theta_F^*)\right)\cdot\left(\widehat{f}_g(g(x,\widehat{\theta}))-f_g(x,\theta_F^*)\right)}{\widetilde{f}_g(x)^2}\\ &\quad+\frac{\widetilde{R}_p(x,t)\cdot\left(\widehat{f}_g(g(x,\widehat{\theta}))-f_g(x,\theta_F^*)\right)^2}{\widetilde{f}_g(x)^3}\end{aligned} \tag{S3.4.5}$$

Let $\epsilon>0$ be the constant in Assumption 4. From (S3.1.56) and (S3.2.39), $\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\zeta_n^{R_p}(x,t)\right| = o_p\left(\frac{1}{n^{1/2+\epsilon}}\right)$, uniformly over $\mathcal{F}$. And from (S3.1.58) and (S3.2.41),

$$\sup_{x\in\mathcal{X}}\left|\widehat{f}_g(x,\widehat{\theta})-f_g(x,\theta_F^*)\right| = O_p\left(\frac{1}{n^{1/2}\cdot h_n^D}\right) \quad \text{uniformly over } \mathcal{F}.$$
$$\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\widehat{R}_p(x,t,\widehat{\theta})-R_{p,F}(x,t,\theta_F^*)\right| = O_p\left(\frac{1}{n^{1/2}\cdot h_n^D}\right) \quad \text{uniformly over } \mathcal{F}.$$

In Assumption 4 we stated that the constant $\epsilon > 0$ satisfies $n^{1/2-\epsilon} \cdot h_n^{2D} \longrightarrow \infty$. These results, combined with (S3.4.2) and (S3.4.3) yield,

$$\begin{aligned}
\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left|\zeta_n^{Q_p}(x,t)\right| &\leq \frac{1}{\underline{f}_g} \sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left|\zeta_n^{R_p}(x,t)\right| + \frac{\overline{Q}}{\underline{f}_g} \sup_{x\in\mathcal{X}} \left|\zeta_n^{f_g}(x)\right| \\
&+ \sup_{x\in\mathcal{X}} \left|\frac{1}{\widetilde{f}_g(x)^2}\right| \sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left|\widehat{R}_p(x,t,\widehat{\theta}) - R_{p,F}(x,t,\theta_F^*)\right| \times \sup_{x\in\mathcal{X}} \left|\widehat{f}_g(x,\widehat{\theta}) - f_g(x,\theta_F^*)\right| \\
&+ \sup_{x\in\mathcal{X}} \left|\frac{1}{\widetilde{f}_g(x)^3}\right| \left(\sup_{x\in\mathcal{X}} \left|\widehat{f}_g(x,\widehat{\theta}) - f_g(x,\theta_F^*)\right|\right)^2 \\
&= o_p\left(\frac{1}{n^{1/2+\epsilon}}\right) + O_p\left(\frac{1}{n \cdot h_n^{2D}}\right) \quad \text{uniformly over } \mathcal{F} \\
&= o_p\left(\frac{1}{n^{1/2+\epsilon}}\right) \quad \text{uniformly over } \mathcal{F}.
\end{aligned} \tag{S3.4.6}$$

From (S3.1.57), (S3.2.40) and the conditions in Assumption 2 which assert that, for each $F \in \mathcal{F}$,

$$\inf_{x\in\mathcal{X}} f_g(g(x,\theta_F^*)) \geq \underline{f}_g, \quad \sup_{x\in\mathcal{X}} f_g(g(x,\theta_F^*)) \leq \overline{f}_g, \text{ and } \sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left|Q_{p,F}(x,t,\theta_F^*)\right| \leq \overline{Q}, \quad p = 1,\ldots,P.$$

we have that

$$\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left|\frac{1}{n}\sum_{i=1}^{n} \psi_F^{Q_p}(V_i,x,t,\theta_F^*,h_n)\right| = O_p\left(\frac{1}{n^{1/2} \cdot h_n^D}\right) \quad \text{uniformly over } \mathcal{F}. \tag{S3.4.7}$$

and therefore, from (S3.4.5) and (S3.4.6),

$$\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left|\widehat{Q}_p(x,t,\widehat{\theta}) - Q_{p,F}(x,t,\theta_F^*)\right| = O_p\left(\frac{1}{n^{1/2} \cdot h_n^D}\right) + o_p\left(\frac{1}{n^{1/2+\epsilon}}\right) = O_p\left(\frac{1}{n^{1/2} \cdot h_n^D}\right) \quad \text{uniformly over } \mathcal{F}.$$

In particular, since $n^{1/2-\epsilon} \cdot h_n^{2D} \longrightarrow \infty$ by Assumption 4, the previous result implies that

$$\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left|\widehat{Q}_p(x,t,\widehat{\theta}) - Q_{p,F}(x,t,\theta_F^*)\right| = o_p\left(\frac{1}{n^{1/4+\epsilon/2}}\right) \quad \text{uniformly over } \mathcal{F}.$$

Let $\psi_F^Q(V_i,x,t,\theta_F^*,h_n) \equiv \left(\psi_F^{Q_1}(V_i,x,t,\theta_F^*,h_n),\ldots,\psi_F^{Q_P}(V_i,x,t,\theta_F^*,h_n)\right)'$, $\zeta_n^Q(x,t) \equiv \left(\zeta_n^{Q_1}(x,t),\ldots,\zeta_n^{Q_P}(x,t)\right)'$. From (S3.4.5) and (S3.4.6), we have

$$\begin{aligned}
&\widehat{Q}(x,t,\widehat{\theta}) = Q_F(x,t,\theta_F^*) + \frac{1}{n}\sum_{i=1}^{n} \psi_F^Q(V_i,x,t,\theta_F^*,h_n) + \zeta_n^Q(x,t), \quad \text{where} \\
&\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left\|\zeta_n^Q(x,t)\right\| = o_p\left(\frac{1}{n^{1/2+\epsilon}}\right) \quad \text{uniformly over } \mathcal{F}.
\end{aligned}$$

where $\epsilon > 0$ is the constant described in Assumption 4. And we have,

$$\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left\|\widehat{Q}(x,t,\widehat{\theta}) - Q_F(x,t,\theta_F^*)\right\| = o_p\left(\frac{1}{n^{1/4+\epsilon/2}}\right) \quad \text{uniformly over } \mathcal{F}.$$

The previous two results prove part (ii) of Proposition S1. ■

S3.5 Asymptotic properties of $\mathcal{B}(\widehat{Q}(x,t,\widehat{\theta})) - \mathcal{B}(Q_F(x,t,\theta_F^*))$

From Assumption 5, the following second-order approximation is valid,

$$\begin{aligned}\mathcal{B}(\widehat{Q}(x,t,\widehat{\theta})) &= \mathcal{B}(Q_F(x,t,\theta_F^*)) + \nabla_Q\mathcal{B}(Q_F(x,t,\theta_F^*))\left(\widehat{Q}(x,t,\widehat{\theta}) - Q_F(x,t,\theta_F^*)\right) \\ &\quad + \frac{1}{2}\left(\widehat{Q}(x,t,\widehat{\theta}) - Q_F(x,t,\theta_F^*)\right)'\nabla_{QQ'}\mathcal{B}(\widetilde{Q}(x,t))\left(\widehat{Q}(x,t,\widehat{\theta}) - Q_F(x,t,\theta_F^*)\right),\end{aligned}$$

where $\widetilde{Q}(x,t)$ belongs in the line segment connecting $\widehat{Q}(x,t,\widehat{\theta})$ and $Q_F(x,t,\theta_F^*)$, and thus

$$\left\|\widetilde{Q}(x,t) - Q_F(x,t,\theta_F^*)\right\| \leq \left\|\widehat{Q}(x,t,\widehat{\theta}) - Q_F(x,t,\theta_F^*)\right\|.$$

From here, the results in Proposition S1 yield

$$\begin{aligned}\mathcal{B}(\widehat{Q}(x,t,\widehat{\theta})) &= \mathcal{B}(Q_F(x,t,\theta_F^*)) + \nabla_Q\mathcal{B}(Q_F(x,t,\theta_F^*))\frac{1}{n}\sum_{i=1}^n \psi_F^Q(V_i,x,t,\theta_F^*,h_n) \\ &\quad + \nabla_Q\mathcal{B}(Q_F(x,t,\theta_F^*))\zeta_n^Q(x,t) \\ &\quad + \frac{1}{2}\left(\widehat{Q}(x,t,\widehat{\theta}) - Q_F(x,t,\theta_F^*)\right)'\nabla_{QQ'}\mathcal{B}(\widetilde{Q}(x,t))\left(\widehat{Q}(x,t,\widehat{\theta}) - Q_F(x,t,\theta_F^*)\right) \\ &\equiv \mathcal{B}(Q_F(x,t,\theta_F^*)) + \frac{1}{n}\sum_{i=1}^n \psi_F^{\mathcal{B}}(V_i,x,t,\theta_F^*,h_n) + \zeta_n^{\mathcal{B}}(x,t),\end{aligned} \tag{S3.5.1}$$

where,

$$\begin{aligned}\psi_F^{\mathcal{B}}(V_i,x,t,\theta_F^*,h_n) &\equiv \nabla_Q\mathcal{B}(Q_F(x,t,\theta_F^*))\psi_F^Q(V_i,x,t,\theta_F^*,h_n) \\ &= \sum_{p=1}^P \frac{\partial\mathcal{B}\left(Q_F(x,t,\theta_F^*)\right)}{\partial Q_p}\psi_F^{Q_p}(V_i,x,t,\theta_F^*,h_n), \quad \text{and} \\ \zeta_n^{\mathcal{B}}(x,t) &\equiv \nabla_Q\mathcal{B}(Q_F(x,t,\theta_F^*))\zeta_n^Q(x,t) \\ &\quad + \frac{1}{2}\left(\widehat{Q}(x,t,\widehat{\theta}) - Q_F(x,t,\theta_F^*)\right)'\nabla_{QQ'}\mathcal{B}(\widetilde{Q}(x,t))\left(\widehat{Q}(x,t,\widehat{\theta}) - Q_F(x,t,\theta_F^*)\right).\end{aligned}$$

We proceed by noting that, under the conditions of Assumption 5 and the results in Proposition S1, we have

$$\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left\|\nabla_{QQ'}\mathcal{B}(\widetilde{Q}(x,t))\right\| = O_p(1) \quad \text{uniformly over } \mathcal{F}.$$

to see this, recall that $\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left\|Q_F(x,t,\theta_F^*)\right\| \le \overline{\Gamma}\cdot\overline{\omega} \equiv \overline{Q}$, where $\overline{\Gamma}$ is as described in Assumption 2 and recall that $\left\|\widetilde{Q}(x,t) - Q_F(x,t,\theta_F^*)\right\| \le \left\|\widehat{Q}(x,t,\widehat{\theta}) - Q_F(x,t,\theta_F^*)\right\|$. Therefore,

$$\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left\|\widetilde{Q}(x,t)\right\| \le \sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left\|\widetilde{Q}(x,t) - Q_F(x,t,\theta_F^*)\right\| + \overline{Q} \le \sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left\|\widehat{Q}(x,t,\widehat{\theta}) - Q_F(x,t,\theta_F^*)\right\| + \overline{Q}$$

Thus, from the conditions in Assumption 5 and the results in Proposition S1,

$$\begin{aligned}\sup_{F\in\mathcal{F}} P_F\left(\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left\|\nabla_{QQ'}\mathcal{B}(\widetilde{\Gamma}(x,t))\right\| > \overline{H}_Q\right) &\le \sup_{F\in\mathcal{F}} P_F\left(\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left\|\widetilde{Q}(x,t)\right\| > \overline{Q} + C_Q\right)\\ &\le \sup_{F\in\mathcal{F}} P_F\left(\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left\|\widehat{Q}(x,t,\widehat{\theta}) - Q_F(x,t,\theta_F^*)\right\| > C_Q\right) \longrightarrow 0,\end{aligned}$$

with the last result following from Proposition S1. Thus, $\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left\|\nabla_{QQ'}\mathcal{B}(\widetilde{Q}(x,t))\right\| = O_p(1)$ uniformly over $\mathcal{F}$, as claimed. From here and the results in Proposition S1, we have

$$\begin{aligned}&\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left|\left(\widehat{Q}(x,t,\widehat{\theta}) - Q_F(x,t,\theta_F^*)\right)' \nabla_{QQ'}\mathcal{B}(\widetilde{Q}(x,t))\left(\widehat{Q}(x,t,\widehat{\theta}) - Q_F(x,t,\theta_F^*)\right)\right|\\ &\le \sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left\|\nabla_{QQ'}\mathcal{B}(\widetilde{Q}(x,t))\right\| \times \left(\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left\|\widehat{Q}(x,t,\widehat{\theta}) - Q_F(x,t,\theta_F^*)\right\|\right)^2\\ &= O_p(1)\times\left(o_p\left(\frac{1}{n^{1/4+\epsilon/2}}\right)\right)^2 = o_p\left(\frac{1}{n^{1/2+\epsilon}}\right) \quad \text{uniformly over } \mathcal{F},\end{aligned}$$

where $\epsilon > 0$ is the constant in Assumption 4. By Assumption 5 and the results in Proposition S1,

$$\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left|\nabla_Q\mathcal{B}(Q_F(x,t,\theta_F^*))\zeta_n^Q(x,t)\right| \le \overline{H}_Q \cdot \sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left\|\zeta_n^Q(x,t)\right\| = o_p\left(\frac{1}{n^{1/2+\epsilon}}\right) \quad \text{uniformly over } \mathcal{F}.$$

Therefore, from the conditions described in Assumption 5 and the results in Proposition S1,

$$\begin{aligned}\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left|\zeta_n^{\mathcal{B}}(x,t)\right| &\le \sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left|\nabla_Q\mathcal{B}(Q_F(x,t,\theta_F^*))\zeta_n^Q(x,t)\right|\\ &\quad + \sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left|\left(\widehat{Q}(x,t,\widehat{\theta}) - Q_F(x,t,\theta_F^*)\right)' \nabla_{QQ'}\mathcal{B}(\widetilde{Q}(x,t))\left(\widehat{Q}(x,t,\widehat{\theta}) - Q_F(x,t,\theta_F^*)\right)\right|\\ &= o_p\left(\frac{1}{n^{1/2+\epsilon}}\right) \quad \text{uniformly over } \mathcal{F}.\end{aligned} \tag{S3.5.2}$$

Together, (S3.5.1) and (S3.5.2) yield

$$\mathcal{B}(\widehat{Q}(x,t,\widehat{\theta})) = \mathcal{B}(Q_F(x,t,\theta_F^*)) + \frac{1}{n}\sum_{i=1}^{n} \psi_F^{\mathcal{B}}(V_i,x,t,\theta_F^*,h_n) + \zeta_n^{\mathcal{B}}(x,t), \quad \text{where}$$
$$\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left\|\zeta_n^{\mathcal{B}}(x,t)\right\| = o_p\left(\frac{1}{n^{1/2+\epsilon}}\right) \quad \text{uniformly over } \mathcal{F},$$

where $\epsilon > 0$ is the constant described in Assumption 4. Finally, recall from (S3.4.7) that

$$\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left\|\frac{1}{n}\sum_{i=1}^{n} \psi_F^{Q}(V_i,x,t,\theta_F^*,h_n)\right\| = O_p\left(\frac{1}{n^{1/2}\cdot h_n^D}\right) \quad \text{uniformly over } \mathcal{F}.$$

Therefore, from here and Assumption 5,

$$\begin{aligned}\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left|\frac{1}{n}\sum_{i=1}^{n} \psi_F^{\mathcal{B}}(V_i,x,t,\theta_F^*,h_n)\right| &\leq \sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left\|\nabla_Q\mathcal{B}(Q_F(x,t,\theta_F^*))\right\| \cdot \sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left\|\frac{1}{n}\sum_{i=1}^{n} \psi_F^{Q}(V_i,x,t,\theta_F^*,h_n)\right\| \\ &\leq \overline{H}_Q \cdot O_p\left(\frac{1}{n^{1/2}\cdot h_n^D}\right) \quad \text{uniformly over } \mathcal{F}.\end{aligned}$$

Since $n^{1/2-\epsilon}\cdot h_n^{2D} \longrightarrow \infty$ by Assumption 4, the above result combined with (S3.5.1) and (S3.5.2) implies,

$$\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left|\mathcal{B}(\widehat{Q}(x,t,\widehat{\theta})) - \mathcal{B}(Q_F(x,t,\theta_F^*))\right| = o_p\left(\frac{1}{n^{1/4+\epsilon/2}}\right) \quad \text{uniformly over } \mathcal{F}. \tag{S3.5.3}$$

Next, note from Assumption 5 that, for any $s > 0$ we have

$$\begin{aligned}&\sup_{F\in\mathcal{F}} P_F\left(\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left|\mathcal{B}(\widehat{Q}(x,t,\widehat{\theta})) - \mathcal{B}(Q_F(x,t,\theta_F^*))\right| \geq s\right) \\ &\qquad \leq \sup_{F\in\mathcal{F}} P_F\left(\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left\|\widehat{Q}(x,t,\widehat{\theta}) - Q_F(x,t,\theta_F^*)\right\| \geq \left(\frac{s}{M_1}\right)\wedge M_2\right)\end{aligned}$$

In particular, from Proposition S1, if we take any positive sequence $s_n > 0$ such that $s_n \longrightarrow 0$, with $\frac{h_n^M}{s_n} \longrightarrow 0$ and $s_n \cdot n \cdot h_n^{D+1} \longrightarrow \infty$, we have

$$\begin{aligned}&\sup_{F\in\mathcal{F}} P_F\left(\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left|\mathcal{B}(\widehat{Q}(x,t,\widehat{\theta})) - \mathcal{B}(Q_F(x,t,\theta_F^*))\right| \geq s_n\right) \\ &\qquad = O\left(\frac{1}{\left((n^{1/2}\wedge r_n)\cdot h_n^{D+1}\cdot\left(\left(\frac{A_1}{M_1}\right)\cdot s_n - C_1\cdot h_n^M - \frac{C_2}{n\cdot h_n^{D+1}} - \frac{C_3}{n\cdot h_n^D}\right)\right)^q}\right)\end{aligned}$$

In particular, take the sequence b_n used in the construction of $\widehat{T}_2$. By the bandwidth convergence restrictions described in Assumption 4, we have $\frac{h_n^M}{b_n} \longrightarrow 0$ and $b_n \cdot n \cdot h_n^{D+1} \longrightarrow \infty$, and therefore,

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left|\mathcal{B}(\widehat{Q}(x,t,\widehat{\theta})) - \mathcal{B}(Q_F(x,t,\theta_F^*))\right| \geq b_n\right) \longrightarrow 0. \tag{S3.5.4}$$

S3.6 Proof of Proposition 1

S3.6.1 Asymptotic properties of $\widehat{T}_2$

Recall that $T_{2,F} \equiv \int_t T_{0,F}(t)d\mathcal{W}(t)$, where $T_{0,F}(t) \equiv E_F\left[\left(\mathcal{B}\left(Q_F(X,t,\theta_F^*)\right)\right)_+ \phi(X,t)\right]$. Our corresponding estimators are $\widehat{T}_2 \equiv \int_t \widehat{T}_0(t)d\mathcal{W}(t)$, with $\widehat{T}_0(t) \equiv \frac{1}{n}\sum_{i=1}^n \mathcal{B}\left(\widehat{Q}(X_i,t,\widehat{\theta})\right)\mathbb{1}\left\{\mathcal{B}\left(\widehat{Q}(X_i,t,\widehat{\theta})\right) \geq -b_n\right\}\phi(X_i)$. Let $\widetilde{T}_{0,F}(t) \equiv \frac{1}{n}\sum_{i=1}^n \mathcal{B}\left(\widehat{Q}(X_i,t,\widehat{\theta})\right)\mathbb{1}\left\{\mathcal{B}\left(Q_F(X_i,t,\theta_F^*)\right) \geq 0\right\}\phi(X_i)$. Note that $\widetilde{T}_{0,F}(t)$ takes $\widehat{T}_0(t)$ and replaces $\mathbb{1}\left\{\mathcal{B}\left(\widehat{Q}(X_i,t,\widehat{\theta})\right) \geq -b_n\right\}$ with $\mathbb{1}\left\{\mathcal{B}\left(Q_F(X_i,t,\theta_F^*)\right) \geq 0\right\}$. Let $\xi^a_{T_0,n}(t) \equiv \widehat{T}_0(t) - T_{0,F}(t)$, and note that $\left|\xi^a_{T_2,n}(t)\right| \leq \sum_{i=1}^n \left|\mathcal{B}\left(\widehat{Q}(X_i,t,\widehat{\theta})\right)\right|\phi(X_i,t)\left|\mathbb{1}\left\{\mathcal{B}\left(\widehat{Q}(X_i,t,\widehat{\theta})\right) \geq -b_n\right\} - \mathbb{1}\left\{\mathcal{B}\left(Q_F(X_i,t,\theta_F^*)\right) \geq 0\right\}\right|$, and

$$\begin{aligned}
&\left|\mathbb{1}\left\{\mathcal{B}\left(\widehat{Q}(X_i,t,\widehat{\theta})\right) \geq -b_n\right\} - \mathbb{1}\left\{\mathcal{B}\left(Q_F(X_i,t,\theta_F^*)\right) \geq 0\right\}\right| \\
=&\mathbb{1}\left\{\mathcal{B}\left(\widehat{Q}(X_i,t,\widehat{\theta})\right) \geq -b_n\ ,\ -2b_n \leq \mathcal{B}\left(Q_F(X_i,t,\theta_F^*)\right) < 0\right\} + \mathbb{1}\left\{\mathcal{B}\left(\widehat{Q}(X_i,t,\widehat{\theta})\right) \geq -b_n\ ,\ \mathcal{B}\left(Q_F(X_i,t,\theta_F^*)\right) < -2b_n\right\} \\
&+\mathbb{1}\left\{\mathcal{B}\left(\widehat{Q}(X_i,t,\widehat{\theta})\right) < -b_n\ ,\ \mathcal{B}\left(Q_F(X_i,t,\theta_F^*)\right) \geq 0\right\} \\
\leq&\mathbb{1}\left\{-2b_n \leq \mathcal{B}\left(Q_F(X_i,t,\theta_F^*)\right) < 0\right\} + \mathbb{1}\left\{\left|\mathcal{B}\left(\widehat{Q}(X_i,t,\widehat{\theta})\right) - \mathcal{B}\left(Q_F(X_i,t,\theta_F^*)\right)\right| \geq b_n\right\}.
\end{aligned}$$

From here, we have

$$\begin{aligned}
&\left|\xi^a_{T_2,n}(t)\right| \\
\leq&\frac{1}{n}\sum_{i=1}^n \left|\mathcal{B}\left(\widehat{Q}(X_i,t,\widehat{\theta})\right)\right|\phi(X_i,t)\mathbb{1}\left\{-2b_n \leq \mathcal{B}\left(Q_F(X_i,t,\theta_F^*)\right) < 0\right\} \\
&+\frac{1}{n}\sum_{i=1}^n \left|\mathcal{B}\left(\widehat{Q}(X_i,t,\widehat{\theta})\right)\right|\phi(X_i)\mathbb{1}\left\{\left|\mathcal{B}\left(\widehat{Q}(X_i,t,\widehat{\theta})\right) - \mathcal{B}\left(Q_F(X_i,t,\theta_F^*)\right)\right| \geq b_n\right\} \\
\leq&\frac{1}{n}\sum_{i=1}^n \left(\left|\mathcal{B}\left(Q_F(X_i,t,\theta_F^*)\right)\right| + \left|\mathcal{B}\left(\widehat{Q}(X_i,t,\widehat{\theta})\right) - \mathcal{B}\left(Q_F(X_i,t,\theta_F^*)\right)\right|\right)\phi(X_i)\mathbb{1}\left\{-2b_n \leq \mathcal{B}\left(Q_F(X_i,t,\theta_F^*)\right) < 0\right\} \\
&+\frac{1}{n}\sum_{i=1}^n \left|\mathcal{B}\left(\widehat{Q}(X_i,t,\widehat{\theta})\right)\right|\phi(X_i,t)\mathbb{1}\left\{\left|\mathcal{B}\left(\widehat{Q}(X_i,t,\widehat{\theta})\right) - \mathcal{B}\left(Q_F(X_i,t,\theta_F^*)\right)\right| \geq b_n\right\} \\
\leq&\left(2b_n + \sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\mathcal{B}\left(\widehat{Q}(x,t,\widehat{\theta})\right) - \mathcal{B}\left(Q_F(x,t,\theta_F^*)\right)\right|\right) \times \frac{1}{n}\sum_{i=1}^n \phi(X_i)\mathbb{1}\left\{-2b_n \leq \mathcal{B}\left(Q_F(X_i,t,\theta_F^*)\right) < 0\right\} \\
&+\frac{1}{n}\sum_{i=1}^n \left|\mathcal{B}\left(\widehat{Q}(X_i,t,\widehat{\theta})\right)\right|\phi(X_i)\mathbb{1}\left\{\left|\mathcal{B}\left(\widehat{Q}(X_i,t,\widehat{\theta})\right) - \mathcal{B}\left(Q_F(X_i,t,\theta_F^*)\right)\right| \geq b_n\right\}
\end{aligned}$$

Therefore,

$$
\begin{aligned}
&\left|\xi^{a}_{T_0,n}(t)\right| \\
&\leq \left(2b_n + \sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left|\mathcal{B}\left(\widehat{Q}(x,t,\widehat{\theta})\right) - \mathcal{B}\left(Q_F(x,t,\theta^*_F)\right)\right|\right) \times \frac{1}{n}\sum_{i=1}^{n}\phi(X_i)\mathbb{1}\left\{-2b_n \leq \mathcal{B}\left(Q_F(X_i,t,\theta^*_F)\right) < 0\right\} \\
&+\frac{1}{n}\sum_{i=1}^{n}\left|\mathcal{B}\left(\widehat{Q}(X_i,t,\widehat{\theta})\right)\right|\phi(X_i)\mathbb{1}\left\{\left|\mathcal{B}\left(\widehat{Q}(X_i,t,\widehat{\theta})\right) - \mathcal{B}\left(Q_F(X_i,t,\theta^*_F)\right)\right| \geq b_n\right\}
\end{aligned}
$$

From (S3.5.3), we have $\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left|\mathcal{B}\left(\widehat{Q}(x,t,\widehat{\theta})\right) - \mathcal{B}\left(Q(x,t,\theta^*_F)\right)\right| = o_p\left(\frac{1}{n^{1/4+\epsilon/2}}\right)$ uniformly over $\mathcal{F}$, where $\epsilon > 0$ is the constant described in Assumption 4. Therefore, uniformly over $\mathcal{F}$ we have

$$
\begin{aligned}
\left|\xi^{a}_{T_0,n}(t)\right| &\leq \left(2b_n + o_p\left(\frac{1}{n^{1/4+\epsilon/2}}\right)\right) \times \frac{1}{n}\sum_{i=1}^{n}\phi(X_i)\mathbb{1}\left\{-2b_n \leq \mathcal{B}\left(Q(X_i,t,\theta^*_F)\right) < 0\right\} \\
&\quad + \frac{1}{n}\sum_{i=1}^{n}\left|\mathcal{B}\left(\widehat{Q}(X_i,t,\widehat{\theta})\right)\right|\phi(X_i)\mathbb{1}\left\{\left|\mathcal{B}\left(\widehat{Q}(X_i,t,\widehat{\theta})\right) - \mathcal{B}\left(Q(X_i,t,\theta^*_F)\right)\right| \geq b_n\right\}.
\end{aligned}
\tag{S3.6.1}
$$

We will analyze each of the two summands in (S3.6.1). For a given $b > 0$ and $t \in \mathcal{T}$, let

$$
m^{a}_{T_0,n}(b,t) \equiv \frac{1}{n}\sum_{i=1}^{n}\left(\phi(X_i)\mathbb{1}\left\{-b \leq \mathcal{B}\left(Q_F(X_i,t,\theta^*_F)\right) < 0\right\} - E_F\left[\phi(X)\mathbb{1}\left\{-b \leq \mathcal{B}\left(Q_F(X,t,\theta^*_F)\right) < 0\right\}\right]\right)
$$

From Assumption 6, there exist constants $(\overline{A},\overline{V})$ such that, for each $F \in \mathcal{F}$, the following class of indicator functions is Euclidean $(\overline{A},\overline{V})$ for the constant envelope 1,

$$
\left\{m : \mathcal{X} \longrightarrow \mathbb{R} : m(x) = \mathbb{1}\left\{-b \leq \mathcal{B}\left(Q_F(x,t,\theta^*_F)\right) < 0\right\} \text{ for some } 0 < b \leq b_0 \text{ and } t \in \mathcal{T}\right\}.
$$

From here, Result S1 yields, $\sup_{\substack{0<b<b_0 \\ t\in\mathcal{T}}} \left|m^{a}_{T_0,n}(b,t)\right| = O_p\left(\frac{1}{n^{1/2}}\right)$ uniformly over $\mathcal{F}$. For n large enough $0 < 2b_n \leq b_0$. Therefore,

$$
\left|m^{a}_{T_0,n}(2b_n,t)\right| \leq \sup_{\substack{0<b<b_0 \\ t\in\mathcal{T}}} \left|m^{a}_{T_0,n}(b,t)\right| = O_p\left(\frac{1}{n^{1/2}}\right) \quad \text{uniformly over } \mathcal{F}. \tag{S3.6.2}
$$

We have $\frac{1}{n}\sum_{i=1}^{n}\phi(X_i)\mathbb{1}\left\{-2b_n \leq \mathcal{B}\left(Q_F(X_i,t,\theta^*_F)\right) < 0\right\} = m^{a}_{T_0,n}(2b_n,t) + E_F\left[\phi(X)\mathbb{1}\left\{-2b_n \leq \mathcal{B}\left(Q_F(X,t,\theta^*_F)\right) < 0\right\}\right]$. From Assumption 7, $\exists\ \underline{b}_2 > 0$ and $\overline{C}_{\mathcal{B},2} > 0$ such that, for all $0 < b \leq \underline{b}_2$,

$$
\sup_{t\in\mathcal{T}} E_F\left[\phi(X)\mathbb{1}\left\{-b \leq \mathcal{B}\left(Q_F(X,t,\theta^*_F)\right) < 0\right\}\right] \leq \overline{\phi}\cdot\overline{C}_{\mathcal{B},2}\cdot b \quad \forall\ F \in \mathcal{F},
$$

For n large enough, we have $0 < 2b_n \leq \underline{b}_2 \wedge b_0$, and from Assumption 4, we have $n^{1/2} \cdot b_n \longrightarrow \infty$. This, combined with equation (S3.6.2) and Assumption 7 yields,

$$\begin{aligned}
\sup_{t\in\mathcal{T}} \frac{1}{n}\sum_{i=1}^{n} \phi(X_i)\mathbb{1}\Big\{-2b_n \leq \mathcal{B}\left(Q_F(X_i,t,\theta_F^*)\right) < 0\Big\} &\leq O_p\left(\frac{1}{n^{1/2}}\right) + O(b_n) \\
&= b_n \cdot \left(O_p\left(\frac{1}{b_n \cdot n^{1/2}}\right) + O(1)\right) \qquad \text{(S3.6.3)} \\
&= b_n \cdot \left(o_p(1) + O(1)\right) \\
&= O_p(b_n) \quad \text{uniformly over } \mathcal{F}.
\end{aligned}$$

From Assumption 5 and (S3.5.3), $\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left|\mathcal{B}\left(\widehat{Q}(x,t,\widehat{\theta})\right)\right| = O_p(1)$ uniformly over $\mathcal{F}$. Therefore,

$$\begin{aligned}
&\sup_{t\in\mathcal{T}} \frac{1}{n}\sum_{i=1}^{n} \left|\mathcal{B}\left(\widehat{Q}(X_i,t,\widehat{\theta})\right)\right| \phi(X_i)\mathbb{1}\left\{\left|\mathcal{B}\left(\widehat{Q}(X_i,t,\widehat{\theta})\right) - \mathcal{B}\left(Q(X_i,t,\theta_F^*)\right)\right| \geq b_n\right\} \\
\leq &\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left|\mathcal{B}\left(\widehat{Q}(x,t,\widehat{\theta})\right)\right| \times \sup_{t\in\mathcal{T}} \frac{1}{n}\sum_{i=1}^{n} \phi(X_i)\mathbb{1}\left\{\left|\mathcal{B}\left(\widehat{Q}(X_i,t,\widehat{\theta})\right) - \mathcal{B}\left(Q(X_i,t,\theta_F^*)\right)\right| \geq b_n\right\} \\
= &O_p(1) \times \left(\sup_{t\in\mathcal{T}} \frac{1}{n}\sum_{i=1}^{n} \phi(X_i)\mathbb{1}\left\{\left|\mathcal{B}\left(\widehat{Q}(X_i,t,\widehat{\theta})\right) - \mathcal{B}\left(Q(X_i,t,\theta_F^*)\right)\right| \geq b_n\right\}\right), \quad \text{uniformly over } \mathcal{F}.
\end{aligned}$$

Next, note that

$$\begin{aligned}
&\sup_{F\in\mathcal{F}} P_F\left(\sup_{t\in\mathcal{T}} \frac{1}{n}\sum_{i=1}^{n} \phi(X_i)\mathbb{1}\left\{\left|\mathcal{B}\left(\widehat{Q}(X_i,t,\widehat{\theta})\right) - \mathcal{B}\left(Q(X_i,t,\theta_F^*)\right)\right| \geq b_n\right\} \neq 0\right) \\
\leq &\sup_{F\in\mathcal{F}} P_F\left(\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}} \left|\mathcal{B}(\widehat{Q}(x,t,\widehat{\theta})) - \mathcal{B}(Q_F(x,t,\theta_F^*))\right| \geq b_n\right) \longrightarrow 0
\end{aligned}$$

where the last equality follows from (S3.5.4). In particular, for any $\delta > 0$ and $\Delta > 0$,

$$\sup_{F\in\mathcal{F}} P_F\left(\sup_{t\in\mathcal{T}} \frac{1}{n}\sum_{i=1}^{n} \phi(X_i)\mathbb{1}\left\{\left|\mathcal{B}\left(\widehat{Q}(X_i,t,\widehat{\theta})\right) - \mathcal{B}\left(Q(X_i,t,\theta_F^*)\right)\right| \geq b_n\right\} \geq \frac{\delta}{n^{1/2+\Delta}}\right) \longrightarrow 0.$$

That is,

$$\sup_{t\in\mathcal{T}} \frac{1}{n}\sum_{i=1}^{n} \phi(X_i)\mathbb{1}\left\{\left|\mathcal{B}\left(\widehat{Q}(X_i,t,\widehat{\theta})\right) - \mathcal{B}\left(Q(X_i,t,\theta_F^*)\right)\right| \geq b_n\right\} = o_p\left(\frac{1}{n^{1/2+\Delta}}\right) \quad \forall\, \Delta > 0, \;\text{uniformly over } \mathcal{F}.$$

Therefore,

$$\sup_{t\in\mathcal{T}} \frac{1}{n}\sum_{i=1}^{n}\left|\mathcal{B}\left(\widehat{Q}(X_i,t,\widehat{\theta})\right)\right|\phi(X_i)\mathbb{1}\left\{\left|\mathcal{B}\left(\widehat{Q}(X_i,t,\widehat{\theta})\right)-\mathcal{B}\left(Q(X_i,t,\theta_F^*)\right)\right|\geq b_n\right\}$$
$$= o_p\left(\frac{1}{n^{1/2+\Delta}}\right) \quad \forall\ \Delta>0,\ \text{ uniformly over } \mathcal{F}.$$

Plugging the results in (S3.6.3) and in the previous expression into (S3.6.1), for any $\Delta > 0$ we have

$$\sup_{t\in\mathcal{T}} \left|\xi^a_{T_0,n}(t)\right| \leq \left(2b_n + o_p\left(\frac{1}{n^{1/4+\epsilon/2}}\right)\right)\times O_p(b_n) + o_p\left(\frac{1}{n^{1/2+\Delta}}\right)$$
$$= O_p\left(b_n^2\right) + o_p\left(\frac{b_n}{n^{1/4+\epsilon/2}}\right) + o_p\left(\frac{1}{n^{1/2+\Delta}}\right) \quad \text{uniformly over } \mathcal{F}.$$

Take any $\Delta > 0$ and note that $\left(\frac{b_n}{n^{1/4+\epsilon/2}}\right)\cdot n^{1/2+\Delta} = \left(n^{1/2+2\Delta-\epsilon}\cdot b_n^2\right)^{1/2}$. In Assumption 4 we stated that $\exists\ \delta_0 > 0$ s.t $n^{1/2+\delta_0}\cdot b_n^2 \longrightarrow 0$. Therefore, $\frac{b_n}{n^{1/4+\epsilon/2}} = o\left(\frac{1}{n^{1/2+\Delta}}\right) \quad \forall\ 0<\Delta\leq \frac{\delta_0}{2}$. From here, we obtain

$$\sup_{t\in\mathcal{T}} \left|\xi^a_{T_0,n}(t)\right| = o_p\left(\frac{1}{n^{1/2+\delta_0/2}}\right) \quad \text{uniformly over } \mathcal{F}. \tag{S3.6.4}$$

Therefore, using the linear representation result in (S3.5.1),

$$\widehat{T_0}(t) = \widetilde{T}_{0,F}(t) + \xi^a_{T_0,n}(t)$$
$$= \frac{1}{n}\sum_{i=1}^{n}\mathcal{B}\left(\widehat{Q}(X_i,t,\widehat{\theta})\right)\mathbb{1}\left\{\mathcal{B}\left(Q_F(X_i,t,\theta_F^*)\right)\geq 0\right\}\phi(X_i) + \xi^a_{T_0,n}(t)$$
$$= \frac{1}{n}\sum_{i=1}^{n}\left(\mathcal{B}(Q_F(X_i,t,\theta_F^*)) + \frac{1}{n}\sum_{j=1}^{n}\psi_F^{\mathcal{B}}(V_j,X_i,t,\theta_F^*,h_n) + \zeta_n^{\mathcal{B}}(X_i,t)\right)\mathbb{1}\left\{\mathcal{B}\left(Q_F(X_i,t,\theta_F^*)\right)\geq 0\right\}\phi(X_i) + \xi^a_{T_0,n}(t)$$
$$= \frac{1}{n}\sum_{i=1}^{n}\left(\mathcal{B}\left(Q_F(X_i,t,\theta_F^*)\right)\right)_+\phi(X_i) + \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\psi_F^{\mathcal{B}}(V_j,X_i,t,\theta_F^*,h_n)\mathbb{1}\left\{\mathcal{B}\left(Q_F(X_i,t,\theta_F^*)\right)\geq 0\right\}\phi(X_i,t)$$
$$+ \frac{1}{n}\sum_{i=1}^{n}\zeta_n^{\mathcal{B}}(X_i,t)\mathbb{1}\left\{\mathcal{B}\left(Q_F(X_i,t,\theta_F^*)\right)\geq 0\right\}\phi(X_i,t) + \xi^a_{T_0,n}(t).$$

Recall that $T_{0,F}(t) \equiv E_F\left[\left(\mathcal{B}\left(Q_F(X,t,\theta_F^*)\right)\right)_+ \phi(X)\right]$. Thus, from the above expression,

$$\begin{aligned}\widehat{T_0}(t) =&T_{0,F}(t) + \frac{1}{n}\sum_{i=1}^{n}\Big((\mathcal{B}(Q_F(X_i,t,\theta_F^*)))_+ \phi(X_i) - T_{0,F}(t)\Big)\\ &+ \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\psi_F^{\mathcal{B}}(V_j,X_i,t,\theta_F^*,h_n)\mathbb{1}\{\mathcal{B}(Q_F(X_i,t,\theta_F^*))\geq 0\}\phi(X_i)\\ &\underbrace{+\frac{1}{n}\sum_{i=1}^{n}\zeta_n^{\mathcal{B}}(X_i,t)\mathbb{1}\{\mathcal{B}(Q_F(X_i,t,\theta_F^*))\geq 0\}\phi(X_i)}_{\equiv \xi^b_{T_0,n}(t)}+\xi^a_{T_0,n}(t).\end{aligned} \tag{S3.6.5}$$

Let us analyze each of the terms in (S3.6.5). From (S3.5.2) we have,

$$\begin{aligned}\sup_{t\in\mathcal{T}}\left|\xi^b_{T_0,n}(t)\right| \equiv \left|\frac{1}{n}\sum_{i=1}^{n}\zeta_n^{\mathcal{B}}(X_i,t)\mathbb{1}\{\mathcal{B}(Q_F(X_i,t,\theta_F^*))\geq 0\}\phi(X_i)\right| &\leq \overline{\phi}\cdot\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\zeta_n^{\mathcal{B}}(x,t)\right|\\ &= o_p\left(\frac{1}{n^{1/2+\epsilon}}\right)\quad \text{uniformly over } \mathcal{F}\end{aligned} \tag{S3.6.6}$$

Recall that

$$\begin{aligned}\psi_F^{\mathcal{B}}(V_j,x,t,\theta_F^*,h_n) &\equiv \nabla_Q\mathcal{B}(Q_F(x,t,\theta_F^*))\psi_F^{Q}(V_j,x,t,\theta_F^*,h_n)\\ &=\sum_{p=1}^{P}\frac{\partial\mathcal{B}\left(Q_F(x,t,\theta_F^*)\right)}{\partial Q_p}\psi_F^{Q_p}(V_j,x,t,\theta_F^*,h_n).\end{aligned}$$

For $h>0$, $x\in\mathcal{S}_X$ and $t\in\mathcal{T}$, let

$$\begin{aligned}\Lambda_F^{Q_p}(V_j,x,t,\theta_F^*,h) \equiv& \left(\frac{S_p(Y_j,t)-\Gamma_{p,F}(x,t,\theta_F^*)}{f_g(g(x,\theta_F^*))}\right)\omega_p(g(X_j,\theta_F^*))K\left(\frac{\Delta g(X_j,x,\theta_F^*)}{h}\right)\\ &- E_F\left[\left(\frac{S_p(Y_j,t)-\Gamma_{p,F}(x,t,\theta_F^*)}{f_g(g(x,\theta_F^*))}\right)\omega_p(g(X_j,\theta_F^*))K\left(\frac{\Delta g(X_j,x,\theta_F^*)}{h}\right)\right].\end{aligned}$$

From the definition in (S3.4.4), $\psi_F^{Q_p}(V_j,x,t,\theta_F^*,h_n) = \frac{1}{h_n^D}\Lambda_F^{Q_p}(V_j,x,t,\theta_F^*) + \Xi_{Q_p}(x,t,\theta_F^*)\psi_F^{\theta}(Z_j)$. For $t\in\mathcal{T}$, $h>0$ and $v_1\in\mathcal{S}_V$, $v_2\in\mathcal{S}_V$ let

$$\begin{aligned}\Lambda^a_{T_0,F}(v_1,v_2,t,h) &\equiv \sum_{p=1}^{P}\frac{\partial\mathcal{B}\left(Q_F(x_1,t,\theta_F^*)\right)}{\partial Q_p}\Lambda_F^{Q_p}(v_2,x_1,t,\theta_F^*,h)\mathbb{1}\{\mathcal{B}(Q_F(x_1,t,\theta_F^*))\geq 0\}\phi(x_1),\\ \Lambda^b_{T_0,F}(v_1,v_2,t) &\equiv \left(\sum_{p=1}^{P}\frac{\partial\mathcal{B}\left(Q_F(x_1,t,\theta_F^*)\right)}{\partial Q_p}\mathbb{1}\{\mathcal{B}(Q_F(x_1,t,\theta_F^*))\geq 0\}\phi(x_1)\Xi_{Q_p}(x_1,t,\theta_F^*)\right)\psi_F^{\theta}(z_2)\end{aligned}$$

and define

$$U^a_{T_0,n}(t,h) \equiv \frac{1}{n\cdot(n-1)}\sum_{i=1}^{n}\sum_{j\neq i}\Lambda^a_{T_0,F}(V_i,V_j,t,h), \quad \nu^a_{T_0,n}(t,h) \equiv \frac{1}{n}\sum_{i=1}^{n}\left(\Lambda^a_{T_0,F}(V_i,V_i,t,h) - E_F\left[\Lambda^a_{T_0,F}(V,V,t,h)\right]\right),$$

$$U^b_{T_0,n}(t) \equiv \frac{1}{n\cdot(n-1)}\sum_{i=1}^{n}\sum_{j\neq i}\Lambda^b_{T_0,F}(V_i,V_j,t), \qquad \nu^b_{T_0,n}(t) \equiv \frac{1}{n}\sum_{i=1}^{n}\left(\Lambda^b_{T_0,F}(V_i,V_i,t) - E_F\left[\Lambda^b_{T_0,F}(V,V,t)\right]\right). \tag{S3.6.7}$$

We have,

$$\begin{aligned}
&\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\psi^{\mathcal{B}}_F(V_j,X_i,t,\theta^*_F,h_n)\mathbb{1}\{\mathcal{B}(Q_F(X_i,t,\theta^*_F))\geq 0\}\phi(X_i)\\
=&\left(\frac{n-1}{n}\right)\cdot\frac{1}{h_n^D}\cdot U^a_{T_0,n}(t,h_n) + \left(\frac{n-1}{n}\right)\cdot U^b_{T_0,n}(t) + \frac{1}{n\cdot h_n^D}\cdot \nu^a_{T_0,n}(t,h_n) + \frac{1}{n}\cdot\nu^b_{T_0,n}(t)\\
&+\frac{1}{n\cdot h_n^D}\cdot E_F\left[\Lambda^a_{T_0,F}(V,V,t,h_n)\right] + \frac{1}{n}\cdot E_F\left[\Lambda^b_{T_0,F}(V,V,t)\right]
\end{aligned} \tag{S3.6.8}$$

Let us analyze $U^a_{T_0,n}(t,h_n)$, beginning with $\varphi^a_{T_0,F}(V_i,t,h_n) \equiv E_F\left[\Lambda^a_{T_0,F}(V_i,V_j,t,h_n) + \Lambda^a_{T_0,F}(V_j,V_i,t,h_n)\middle|V_i\right]$. Let

$$\mu^{S_p}_{F,n}(x,t) \equiv \frac{1}{h_n^D}\cdot E_F\left[\left(S_p(Y,t) - \Gamma_{p,F}(x,t,\theta^*_F)\right)\omega_p(g(X,\theta^*_F))K\left(\frac{\Delta g(X,x,\theta^*_F)}{h_n}\right)\right].$$

Recall that in Assumption 2 we defined $\Omega_{R_p,0}(g,t) \equiv E_F\left[S_p(Y,t)\middle|g(X,\theta^*_F) = g\right]$. By iterated expectations, $\mu^{S_p}_{F,n}(x,t) = \frac{1}{h_n^D}\int_u\left(\Omega_{R_p,0}(u,t) - \Gamma_{p,F}(x,t,\theta^*_F)\right)\omega_p(u)K\left(\frac{u-g(x,\theta^*_F)}{h_n}\right)f_g(u)du$. From here, using the smoothness properties described in Assumption 2, performing an M^{th}–order approximation and noting that $\Omega_{R_p,0}(g(x,\theta^*_F),t) = \Gamma_{p,F}(x,t,\theta^*_F)$, there exists a finite constant $\overline{B}_{\mu_S} > 0$ such that,

$$\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\mu^{S_p}_{F,n}(x,t)\right| \leq \overline{B}_{\mu_S}\cdot h_n^M \quad \forall\ F\in\mathcal{F}. \tag{S3.6.9}$$

We have,

$$\frac{1}{h_n^D}\Lambda^{Q_p}_F(V_j,x,t,\theta^*_F,h_n) = \frac{1}{h_n^D}\left(\frac{S_p(Y_j,t) - \Gamma_{p,F}(x,t,\theta^*_F)}{f_g(g(x,\theta^*_F))}\right)\cdot\omega_p(g(X_j,\theta^*_F))\cdot K\left(\frac{\Delta g(X_j,x,\theta^*_F)}{h_n}\right) - \frac{\mu^{S_p}_{F,n}(x,t)}{f_g(g(x,\theta^*_F))}.$$

And, from here,

$$\frac{1}{h_n^D}\cdot E_F\left[\frac{\partial\mathcal{B}\left(Q_F(X_i,t,\theta_F^*)\right)}{\partial Q_p}\Lambda_F^{Q_p}(V_j,X_i,t,\theta_F^*,h_n)\mathbb{1}\left\{\mathcal{B}\left(Q_F(X_i,t,\theta_F^*)\right)\geq 0\right\}\phi(X_i)\middle|V_i\right]$$

$$=\frac{\partial\mathcal{B}\left(Q_F(X_i,t,\theta_F^*)\right)}{\partial Q_p}\frac{1}{f_g(g(X_i,\theta_F^*))}\mathbb{1}\left\{\mathcal{B}\left(Q_F(X_i,t,\theta_F^*)\right)\geq 0\right\}\phi(X_i)\underbrace{\left(\mu_{F,n}^{S_p}(X_i,t)-\mu_{F,n}^{S_p}(X_i,t)\right)}_{=0} \quad \text{(S3.6.10)}$$

$$=0 \quad \forall\, F\in\mathcal{F}.$$

From Assumptions 2 and 5

$$\sup_{(x,t)\in\mathcal{S}_X\times\mathcal{T}}\left|\frac{\partial\mathcal{B}\left(Q_F(x,t,\theta_F^*)\right)}{\partial Q_p}\frac{1}{f_g(g(x,\theta_F^*))}\mathbb{1}\left\{\mathcal{B}\left(Q_F(x,t,\theta_F^*)\right)\geq 0\right\}\phi(x)\right|$$

$$\leq\overline{\phi}\cdot\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\frac{\partial\mathcal{B}\left(Q_F(x,t,\theta_F^*)\right)}{\partial Q_p}\frac{1}{f_g(g(x,\theta_F^*))}\right|\leq\overline{\phi}\cdot\frac{\overline{H}_Q}{\underline{f}_g}\quad\forall\, F\in\mathcal{F}$$

Also from Assumption 2, $\exists\ \overline{C}_{\Xi_{Q_p}}>0$ s.t $\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left\|\Xi_{Q_p}(x,t,\theta_F^*)\right\|\leq\overline{C}_{\Xi_{Q_p}}\quad\forall\, F\in\mathcal{F}$. Therefore, from Assumptions 2 and 5,

$$\sup_{(x,t)\in\mathcal{S}_X\times\mathcal{T}}\left\|\frac{\partial\mathcal{B}\left(Q_F(x,t,\theta_F^*)\right)}{\partial Q_p}\mathbb{1}\left\{\mathcal{B}\left(Q_F(x,t,\theta_F^*)\right)\geq 0\right\}\phi(x)\Xi_{Q_p}(x,t,\theta_F^*)\right\|$$

$$\leq\overline{\phi}\cdot\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left\|\frac{\partial\mathcal{B}\left(Q_F(x,t,\theta_F^*)\right)}{\partial Q_p}\Xi_{Q_p}(x,t,\theta_F^*)\right\|\leq\overline{\phi}\cdot\overline{H}_Q\cdot\overline{C}_{\Xi_{Q_p}}\equiv\overline{C}_{\Xi_{T_2}}\quad\forall\, F\in\mathcal{F}. \quad \text{(S3.6.11)}$$

Thus, from (S3.6.10),

$$\frac{1}{h_n^D}\cdot E_F\left[\frac{\partial\mathcal{B}\left(Q_F(X_i,t,\theta_F^*)\right)}{\partial Q_p}\Lambda_F^{Q_p}(V_j,X_i,t,\theta_F^*,h_n)\mathbb{1}\left\{\mathcal{B}\left(Q_F(X_i,t,\theta_F^*)\right)\geq 0\right\}\phi(X_i)\middle|V_i\right]=0,\ \text{ and therefore,}$$

$$E_F\left[\Lambda_{T_0,F}^{a}(V_j,V_i,\theta_F^*,t,h_n)\right]=0\quad\forall\, t\in\mathcal{T},\ \forall\, F\in\mathcal{F} \quad \text{(S3.6.12)}$$

Next, let us analyze $\frac{1}{h_n^D}\cdot E_F\left[\frac{\partial\mathcal{B}\left(Q_F(X_i,t,\theta_F^*)\right)}{\partial Q_p}\Lambda_F^{Q_p}(V_j,X_i,t,\theta_F^*,h_n)\mathbb{1}\left\{\mathcal{B}\left(Q_F(X_i,t,\theta_F^*)\right)\geq 0\right\}\phi(X_i)\middle|V_j\right]$. Recall from Assumption 7 that we defined

$$\Omega_{T_0}^{p}(y,t,g)=E_F\left[\left(S_p(y,t)-\Gamma_{p,F}(X,t,\theta_F^*)\right)\frac{\partial\mathcal{B}\left(Q_F(X,t,\theta_F^*)\right)}{\partial Q_p}\phi(X)\mathbb{1}\left\{\mathcal{B}\left(Q_F(X,t,\theta_F^*)\right)\geq 0\right\}\middle|g(X,\theta_F^*)=g\right]$$

Using iterated expectations, we have

$$
\begin{aligned}
&\frac{1}{h_n^D}\cdot E_F\left[\frac{\partial\mathcal{B}\left(Q_F(X_i,t,\theta_F^*)\right)}{\partial Q_p}\Lambda_F^{Q_p}(V_j,X_i,t,\theta_F^*,h_n)\mathbb{1}\left\{\mathcal{B}\left(Q_F(X_i,t,\theta_F^*)\right)\geq 0\right\}\phi(X_i)\middle|V_j\right]\\
&=\frac{1}{h_n^D}E_F\left[\frac{\Omega_{T_0}^p(Y_j,t,g(X_i,\theta_F^*))}{f_g(g(X_i,\theta_F^*))}K\left(\frac{\Delta g(X_i,X_j,\theta_F^*)}{h_n}\right)\middle|V_j\right]\cdot\omega_p(g(X_j,\theta_F^*))\\
&-E_F\left[\frac{\mu_{F,n}^{S_p}(X_i,t)}{f_g(g(X_i,\theta_F^*))}\frac{\partial\mathcal{B}\left(Q_F(X_i,t,\theta_F^*)\right)}{\partial Q_p}\mathbb{1}\left\{\mathcal{B}\left(Q_F(X_i,t,\theta_F^*)\right)\geq 0\right\}\phi(X_i)\right]
\end{aligned}
$$

We have,

$$
\begin{aligned}
&\frac{1}{h_n^D}E_F\left[\frac{\Omega_{T_0}^p(Y_j,t,g(X_i,\theta_F^*))}{f_g(g(X_i,\theta_F^*))}K\left(\frac{\Delta g(X_i,X_j,\theta_F^*)}{h_n}\right)\middle|V_j\right]\cdot\omega_p(g(X_j,\theta_F^*))\\
=&\frac{1}{h_n^D}\int\frac{\Omega_{T_0}^p(Y_j,t,u)}{f_g(u)}K\left(\frac{u-g(X_j,\theta_F^*)}{h_n}\right)f_g(u)du\cdot\omega_p(g(X_j,\theta_F^*))
\end{aligned}
$$

Since $\omega_p(g)\neq 0 \Leftrightarrow g\in\mathcal{G}$, the above expression is nonzero only if $g(X_j,\theta_F^*)\in\mathcal{G}$, and since the kernel K has bounded support such that $K(\psi)\neq 0$ if and only if $\|\psi\|\leq S$ and since $h_n\to 0$, for large enough n, $\left\|\frac{u-g}{h_n}\right\|\leq S$ for $g\in\mathcal{G}$ implies $f_g(u)\geq \underline{f}_g>0$. Therefore, for large enough n the terms $f_g(u)$ in the numerator and in the denominator of the previous expression can cancel each other out (since they are nonzero) and we have

$$
\begin{aligned}
&\frac{1}{h_n^D}E_F\left[\frac{\Omega_{T_0}^p(Y_j,t,g(X_i,\theta_F^*))}{f_g(g(X_i,\theta_F^*))}K\left(\frac{\Delta g(X_i,X_j,\theta_F^*)}{h_n}\right)\middle|V_j\right]\cdot\omega_p(g(X_j,\theta_F^*))\\
=&\frac{1}{h_n^D}\int\Omega_{T_0}^p(Y_j,t,u)K\left(\frac{u-g(X_j,\theta_F^*)}{h_n}\right)du\cdot\omega_p(g(X_j,\theta_F^*))\quad\forall\ t\in\mathcal{T},\ \forall\ F\in\mathcal{F}
\end{aligned}
$$

From here, the smoothness conditions described in Assumption 7 and an M^{th}–order approximation imply that there exists a finite constant $\overline{B}_{\Omega_{T_0}}>0$ such that

$$
\begin{aligned}
&\frac{1}{h_n^D}E_F\left[\frac{\Omega_{T_0}^p(Y_j,t,g(X_i,\theta_F^*))}{f_g(g(X_i,\theta_F^*))}K\left(\frac{\Delta g(X_i,X_j,\theta_F^*)}{h_n}\right)\middle|V_j\right]\cdot\omega_p(g(X_j,\theta_F^*))\\
=&\Omega_{T_0}^p(Y_j,t,g(X_j,\theta_F^*))\cdot\omega_p(g(X_j,\theta_F^*))+B_{\Omega_{T_0},n}^p(Y_j,X_j,t)\cdot\omega_p(g(X_j,\theta_F^*)),\\
&\text{where}\quad\sup_{\substack{(y,x)\in\mathcal{S}_{Y,X}\\ t\in\mathcal{T}}}\left|B_{\Omega_{T_0},n}^p(y,x,t)\cdot\omega_p(g(x,\theta_F^*))\right|\leq\overline{B}_{\Omega_{T_0}}\cdot h_n^M\quad\forall\ F\in\mathcal{F}.
\end{aligned}
$$

Next, from Assumptions 2 and 5 and from the result in (S3.6.9),

$$\sup_{t\in\mathcal{T}}\left|E_F\left[\frac{\mu^{S_p}_{F,n}(X_i,t)}{f_g(g(X_i,\theta^*_F))}\frac{\partial\mathcal{B}\big(Q_F(X_i,t,\theta^*_F)\big)}{\partial Q_p}\mathbb{1}\{\mathcal{B}(Q_F(X_i,t,\theta^*_F))\geq 0\}\phi(X_i)\right]\right|\leq\overline{\phi}\cdot\frac{\overline{H}_Q}{\underline{f}_g}\cdot\overline{B}_{\mu_S}\cdot h_n^M\quad\forall\, F\in\mathcal{F}.$$

These results combined yield,

$$\begin{aligned}&\frac{1}{h_n^D}\cdot E_F\left[\frac{\partial\mathcal{B}\big(Q_F(X_i,t,\theta^*_F)\big)}{\partial Q_p}\Lambda^{Q_p}_F(V_j,X_i,t,\theta^*_F,h_n)\mathbb{1}\{\mathcal{B}(Q_F(X_i,t,\theta^*_F))\geq 0\}\phi(X_i)\middle|V_j\right]\\&=\Omega^p_{T_0}(Y_j,t,g(X_j,\theta^*_F))\cdot\omega_p(g(X_j,\theta^*_F))+B^p_{T_0,n}(Y_j,X_j,t),\\&\quad\text{where}\quad\sup_{\substack{(y,x)\in\mathcal{S}_{Y,X}\\t\in\mathcal{T}}}\left|B^p_{T_0,n}(y,x,t)\right|\leq\left(\overline{B}_{\Omega_{T_0}}+\overline{\phi}\cdot\frac{\overline{H}_Q}{\underline{f}_g}\cdot\overline{B}_{\mu_S}\right)\cdot h_n^M\equiv\overline{C}^a_{T_0}\cdot h_n^M\quad\forall\,F\in\mathcal{F}.\end{aligned}\tag{S3.6.13}$$

Denote $\sum_{p=1}^P B^p_{T_0,n}(Y_i,X_i,t)\equiv\overline{B}_{T_0,n}(Y_i,X_i,t)$. Combining (S3.6.12) and (S3.6.13), we have

$$\begin{aligned}\frac{1}{h_n^D}\cdot\varphi^a_{T_0,F}(V_i,t,h_n)&\equiv\frac{1}{h_n^D}\cdot E_F\left[\Lambda^a_{T_0,F}(V_i,V_j,t,h_n)+\Lambda^a_{T_0,F}(V_j,V_i,t,h_n)\middle|V_i\right]\\&\equiv\sum_{p=1}^P\Omega^p_{T_0}(Y_i,t,g(X_i,\theta^*_F))\cdot\omega_p(g(X_i,\theta^*_F))+\overline{B}_{T_0,n}(Y_i,X_i,t),\\&\quad\text{where}\quad\sup_{\substack{(y,x)\in\mathcal{S}_{Y,X}\\t\in\mathcal{T}}}\left|\overline{B}_{T_0,n}(y,x,t)\right|\leq P\cdot\overline{C}^a_{T_0}\cdot h_n^M\equiv\overline{C}^b_{T_0}\cdot h_n^M\quad\forall\,F\in\mathcal{F}.\end{aligned}\tag{S3.6.14}$$

Next, let $\vartheta^a_{T_0,F}(V_i,V_j,t,h)\equiv\Lambda^a_{T_0,F}(V_i,V_j,t,h)+\Lambda^a_{T_0,F}(V_j,V_i,t,h)-\varphi^a_{T_0,F}(V_i,t,h)-\varphi^a_{T_0,F}(V_j,t,h)$. Note that $\vartheta^a_{T_0,F}(V_i,V_j,t,h)=\vartheta^a_{T_0,F}(V_j,V_i,t,h)$ and $E_F\left[\vartheta^a_{T_0,F}(V_i,V_j,t,h)|V_i\right]=E_F\left[\vartheta^a_{T_0,F}(V_i,V_j,t,h)|V_j\right]=0$. Define $\widetilde{U}^a_{T_0,n}(t,h)=\frac{1}{n(n-1)}\sum_{i=1}^n\sum_{j\neq i}\vartheta^a_{T_0,F}(V_i,V_j,t,h)$. From the conditions in Assumption 6, there exist constants $(\overline{A},\overline{V})$ such that, for each $F\in\mathcal{F}$, the following class of functions is Euclidean $(\overline{A},\overline{V})$ for the constant envelope 1, $\left\{m:\mathcal{X}\longrightarrow\mathbb{R}: m(x)=\mathbb{1}\left\{\mathcal{B}\big(Q_F(x,t,\theta^*_F)\big)\geq 0\right\}\text{ for some } t\in\mathcal{T}\right\}$. Combining this with the bounded-variation properties of the kernel K and the conditions in Assumptions 2-5 and 7, applying Pakes and Pollard (1989, Example 2.10) and Nolan and Pollard (1987, Lemma 20) (or Sherman (1994, Lemma 5)), and Pakes and Pollard (1989, Lemma 2.14), the class of functions $\left\{m:\mathcal{S}^2_V\longrightarrow\mathbb{R}\text{: } m(v_1,v_2)=\vartheta^a_{T_0,F}(v_1,v_2,t,h)\text{ for some } h>0\right\}$ is Euclidean for an envelope $\overline{G}^a_{T_0}(\cdot)$ s.t $E_F\left[\overline{G}^a_{T_0}(V_1,V_2)^{4q}\right]\leq\overline{C}_4<\infty\ \forall\,F\in\mathcal{F}$ (with $(V_1,V_2)\sim F\otimes F$) and q being the integer in Assumption 1. From here, Result S1 yields $\sup_{t\in\mathcal{T},h>0}\left|\widetilde{U}^a_{T_0,n}(t,h)\right|=O_p\left(\frac{1}{n}\right)$ uniformly over $\mathcal{F}$. Therefore,

$$\sup_{t\in\mathcal{T}}\left|\widetilde{U}^a_{T_0,n}(t,h_n)\right|\leq\sup_{t\in\mathcal{T},h>0}\left|\widetilde{U}^a_{T_0,n}(t,h)\right|=O_p\left(\frac{1}{n}\right)\quad\text{uniformly over }\mathcal{F}\tag{S3.6.15}$$

The Hoeffding decomposition of $U^a_{T_0,n}(t,h_n)$ (see Serfling (1980, pages 177-178) or Sherman (1994, equations (6)-(7))) yields,

$$\begin{aligned}\frac{1}{h_n^D}\cdot U^a_{T_0,n}(t,h_n) &= \frac{1}{n}\sum_{i=1}^{n}\frac{1}{h_n^D}\cdot\varphi^a_{T_0,F}(V_i,t,h_n)+\frac{1}{2h_n^D}\cdot\widetilde{U}^a_{T_0,n}(t,h_n)\\ &=\frac{1}{n}\sum_{i=1}^{n}\left(\sum_{p=1}^{P}\Omega^p_{T_0}(Y_i,t,g(X_i,\theta^*_F))\cdot\omega_p(g(X_i,\theta^*_F))\right)+\frac{1}{n}\sum_{i=1}^{n}\overline{B}_{T_0,n}(Y_i,X_i,t)+\frac{1}{2h_n^D}\cdot\widetilde{U}^a_{T_0,n}(t,h_n),\end{aligned} \tag{S3.6.16}$$

where, from (S3.6.14) and (S3.6.15),

$$\begin{aligned}\left|\frac{1}{n}\sum_{i=1}^{n}\overline{B}_{T_0,n}(Y_i,X_i,t)+\frac{1}{2h_n^D}\cdot\widetilde{U}^a_{T_0,n}(t,h_n)\right| &\leq \sup_{\substack{(y,x)\in\mathcal{S}_{Y,X}\\ t\in\mathcal{T}}}\left|\overline{B}_{T_0,n}(y,x,t)\right|+\sup_{t\in\mathcal{T},h>0}\left|\widetilde{U}^a_{T_0,n}(t,h)\right|\\ &= O(h_n^M)+O_p\left(\frac{1}{n\cdot h_n^D}\right)=o_p\left(\frac{1}{n^{1/2+\epsilon}}\right)\quad\forall\ F\in\mathcal{F}\end{aligned} \tag{S3.6.17}$$

where (once again), $\epsilon>0$ is the constant described in Assumption 4. Next, note that

$$E_F\left[\Omega^p_{T_0}(Y,t,g(X,\theta^*_F))\cdot\omega_p(g(X,\theta^*_F)\right]=0\quad\forall\ t\in\mathcal{T},\ \forall\ F\in\mathcal{F} \tag{S3.6.18}$$

for each $p=1,\ldots,P$. Let $(V_1,V_2)\sim F\otimes F$. Then, by the definition of $\Omega^p_{T_0}$, we have

$$\begin{aligned}&\Omega^p_{T_0}(Y_1,t,g(X_1,\theta^*_F))\cdot\omega_p(g(X_1,\theta^*_F))=\\ &E_F\left[\Big(S_p(Y_1,X_2,t)-\Gamma_{p,F}(X_2,t,\theta^*_F)\Big)\frac{\partial\mathcal{B}\Big(Q_F(X_2,t,\theta^*_F)\Big)}{\partial Q_p}\phi(X_2)\mathbb{1}\left\{\mathcal{B}\left(Q_F(X_2,t,\theta^*_F)\right)\geq 0\right\}\middle|g(X_2,\theta^*_F)=g(X_1,\theta^*_F),V_1\right]\\ &\cdot\omega_p(g(X_1,\theta^*_F))\end{aligned}$$

Therefore, by iterated expectations, we have

$$\begin{aligned}&E_F\left[\Omega^p_{T_0}(Y_1,t,g(X_1,\theta^*_F))\cdot\omega_p(g(X_1,\theta^*_F))\right]=\\ &E_F\Bigg[E_F\left[\Big(S_p(Y_1,X_2,t)-\Gamma_{p,F}(X_2,t,\theta^*_F)\Big)\cdot\omega_p(g(X_1,\theta^*_F))\middle|g(X_1,\theta^*_F)=g(X_2,\theta^*_F),V_2\right]\\ &\quad\cdot\frac{\partial\mathcal{B}\Big(Q_F(X_2,t,\theta^*_F)\Big)}{\partial Q_p}\phi(X_2)\mathbb{1}\left\{\mathcal{B}\left(Q_F(X_2,t,\theta^*_F)\right)\geq 0\right\}\Bigg]\\ &=E_F\left[\underbrace{\Big(\Gamma_{p,F}(X_2,t,\theta^*_F)-\Gamma_{p,F}(X_2,t,\theta^*_F)\Big)}_{=0}\cdot\omega_p(g(X_2,\theta^*_F))\frac{\partial\mathcal{B}\Big(Q_F(X_2,t,\theta^*_F)\Big)}{\partial Q_p}\phi(X_2)\mathbb{1}\left\{\mathcal{B}\left(Q_F(X_2,t,\theta^*_F)\right)\geq 0\right\}\right]\\ &=0.\end{aligned}$$

By Assumption 7, $\exists\ \overline{\eta}_{\Omega_{T_0}} > 0$ s.t $\sup_{t\in\mathcal{T}} E_F\left[\left|\Omega^p_{T_0}(Y,t,g(X,\theta^*_F))\cdot\omega_p(g(X,\theta^*_F)\right|^2\right] \le \overline{\eta}_{\Omega_{T_0}}\ \forall F\in\mathcal{F}$. From here, a Chebyshev inequality yields

$$\sup_{t\in\mathcal{T}}\left|\frac{1}{n}\sum_{i=1}^{n}\left(\sum_{p=1}^{P}\Omega^p_{T_0}(Y_i,t,g(X_i,\theta^*_F))\cdot\omega_p(g(X_i,\theta^*_F))\right)\right| = O_p\left(\frac{1}{n^{1/2}}\right)\quad \text{uniformly over } \mathcal{F}.$$

(S3.6.16), (S3.6.17) and the previous expression yield,

$$\sup_{t\in\mathcal{T}}\frac{1}{h_n^D}\cdot U^a_{T_0,n}(t,h_n) = O_p\left(\frac{1}{n^{1/2}}\right)\quad \text{uniformly over } \mathcal{F}. \tag{S3.6.19}$$

In (S3.6.7), we defined the U-statistic $U^b_{T_0,n}(t)\equiv\frac{1}{n\cdot(n-1)}\sum_{i=1}^n\sum_{j\neq i}\Lambda^b_{T_0,F}(V_i,V_j,t)$, where $\Lambda^b_{T_0,F}(v_1,v_2,t)\equiv\left(\sum_{p=1}^P\frac{\partial\mathcal{B}\left(Q_F(x_1,t,\theta^*_F)\right)}{\partial Q_p}\mathbb{1}\left\{\mathcal{B}\left(Q_F(x_1,t,\theta^*_F)\right)\ge 0\right\}\phi(x_1)\Xi_{Q_p}(x_1,t,\theta^*_F)\right)\psi^\theta_F(z_2)$. Define,

$$\begin{aligned}\Xi^p_{T_0,F}(t) &\equiv E_F\left[\frac{\partial\mathcal{B}\left(Q_F(X,t,\theta^*_F)\right)}{\partial Q_p}\mathbb{1}\left\{\mathcal{B}\left(Q_F(X,t,\theta^*_F)\right)\ge 0\right\}\phi(X)\Xi_{Q_p}(X,t,\theta^*_F)\right],\\ \Xi_{T_0,F}(t) &\equiv \sum_{p=1}^{P}\Xi^p_{T_0,F}(t)\end{aligned} \tag{S3.6.20}$$

Let $\varphi^b_{T_0,F}(V_i,t)\equiv E_F\left[\Lambda^b_{T_0,F}(V_i,V_j,t)+\Lambda^b_{T_0,F}(V_j,V_i,t)\middle|V_i\right]$. Then, $\varphi^b_{T_0,F}(V_i,t)=\Xi_{T_0,F}(t)\psi^\theta_F(Z_i)$. Note that $E_F\left[\varphi^b_{T_0,F}(V_i,t)\right]=0\ \forall\ t\in\mathcal{T},F\in\mathcal{F}$. Therefore, $E_F\left[\Lambda^b_{T_0,F}(V_i,V_j)\right]=0\ \ \forall\ t\in\mathcal{T}\ \forall\ t\in\mathcal{T},F\in\mathcal{F}$. Now let $\vartheta^b_{T_0,F}(V_i,V_j,t)\equiv\Lambda^b_{T_0,F}(V_i,V_j,t)+\Lambda^b_{T_0,F}(V_j,V_i,t)-\varphi^b_{T_0,F}(V_i,t)-\varphi^b_{T_0,F}(V_j,t)$, and define the degenerate U-statistic $\widetilde{U}^b_{T_0,n}(t)=\frac{1}{n(n-1)}\sum_{i=1}^n\sum_{j\neq i}\vartheta^b_{T_0,F}(V_i,V_j,t)$. From Assumption 1, $E_F\left[\left\|\psi^\theta_F(Z)\right\|^{4q}\right]\le\overline{\mu}_\psi\ \forall\ F\in\mathcal{F}$, where q is the integer described there. Also, as we stated in (S3.6.11), Assumptions 2 and 5 imply $\sup_{t\in\mathcal{T}}\left\|\Xi_{T_0,F}(t)\right\|\le P\cdot\overline{C}_{\Xi_{T_2}}\ \forall\ F\in\mathcal{F}$. Therefore, there exists an envelope $\overline{G}_{T_0}(\cdot)$ and a finite constant $\overline{\mu}_{T_0^b}>0$ such that $\sup_{t\in\mathcal{T}}\left|\vartheta^b_{T_0,F}(v_1,v_2,t)\right|\le\overline{G}_{T_0}(v_1,v_2)$ and $E_F\left[\overline{G}_{T_0}(V_1,V_2)^{4q}\right]\le\overline{\mu}_{T_2^b}$ for all $F\in\mathcal{F}$ (with $(V_1,V_2)\sim F\otimes F$). From here, applying Result S1 we obtain

$$\sup_{t\in\mathcal{T}}\left|\widetilde{U}^b_{T_0,n}(t)\right| = O_p\left(\frac{1}{n}\right)\quad \text{uniformly over } \mathcal{F}. \tag{S3.6.21}$$

The Hoeffding decomposition of $U^b_{T_0,n}(t)$ is,

$$\begin{aligned}U^b_{T_0,n}(t) &= \frac{1}{n}\sum_{i=1}^{n}\varphi^b_{T_0}(V_i,t)+\frac{1}{2}\widetilde{U}^b_{T_0,n}(t)\\ &= \Xi_{T_0,F}(t)\frac{1}{n}\sum_{i=1}^{n}\psi^\theta_F(Z_i)+O_p\left(\frac{1}{n}\right)\quad \text{uniformly over } \mathcal{F}.\end{aligned} \tag{S3.6.22}$$

From Assumption 1, $\left\|\frac{1}{n}\sum_{i=1}^{n}\psi_F^{\theta}(Z_i)\right\| = O_p\left(\frac{1}{n^{1/2}}\right)$ uniformly over $\mathcal{F}$. Combined with (S3.6.11),

$$\sup_{t\in\mathcal{T}}\left\|\Xi_{T_0,F}(t)\frac{1}{n}\sum_{i=1}^{n}\psi_F^{\theta}(Z_i)\right\| \leq \overline{C}_{\Xi_{T_0}}\cdot\left\|\frac{1}{n}\sum_{i=1}^{n}\psi_F^{\theta}(Z_i)\right\| = O_p\left(\frac{1}{n^{1/2}}\right) \quad \text{uniformly over } \mathcal{F}. \tag{S3.6.23}$$

(S3.6.22) and (S3.6.23) yield

$$\sup_{t\in\mathcal{T}}\left|U_{T_0,n}^{b}(t)\right| = O_p\left(\frac{1}{n^{1/2}}\right) \quad \text{uniformly over } \mathcal{F}. \tag{S3.6.24}$$

Let $\nu_{T_0,n}^{a}(t,h)$ be as in (S3.6.7). The same arguments that led to (S3.6.15) yield

$$\sup_{t\in\mathcal{T}}\left|\nu_{T_0,n}^{a}(t,h_n)\right| \leq \sup_{t\in\mathcal{T},h>0}\left|\nu_{T_0,n}^{a}(t,h)\right| = O_p\left(\frac{1}{n^{1/2}}\right) \quad \text{uniformly over } \mathcal{F}. \tag{S3.6.25}$$

And from (S3.6.11), we have

$$\sup_{t\in\mathcal{T}}\left|E_F\left[\Lambda_{T_0,F}^{a}(V,V,t,h_n)\right]\right| \leq \sup_{t\in\mathcal{T},h>0}\left|E_F\left[\Lambda_{T_0,F}^{a}(V,V,t,h)\right]\right| \leq P\cdot\overline{C}_{\Xi_{T_0}} \quad \forall\ F\in\mathcal{F} \tag{S3.6.26}$$

Next, take the process $\nu_{T_0,n}^{b}(t)$ defined in (S3.6.7). The same arguments that led to (S3.6.21) yield

$$\sup_{t\in\mathcal{T}}\left|\nu_{T_0,n}^{b}(t)\right| = O_p\left(\frac{1}{n^{1/2}}\right) \quad \text{uniformly over } \mathcal{F}. \tag{S3.6.27}$$

By Assumption 1, there exists a finite constant $\overline{\eta}_{\psi} > 0$ such that $E_F\left[\left\|\psi_F^{\theta}(Z)\right\|\right] \leq \overline{\eta}_{\psi}$ for all $F\in\mathcal{F}$. Combined with the result in (S3.6.11), this yields

$$\sup_{t\in\mathcal{T}}\left|E_F\left[\Lambda_{T_2,F}^{b}(V,V,t)\right]\right| \leq P\cdot\overline{C}_{\Xi_{T_0}}\cdot\overline{\eta}_{\psi} \quad \forall\ F\in\mathcal{F}. \tag{S3.6.28}$$

S3.6.2 Proof of Proposition 1

From (S3.6.8), we have

$$\begin{aligned}
&\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\psi_F^{\mathcal{B}}(V_j,X_i,t,\theta_F^*,h_n)\mathbb{1}\left\{\mathcal{B}\left(Q_F(X_i,t,\theta_F^*)\right)\geq 0\right\}\phi(X_i)\\
=&\frac{1}{h_n^D}\cdot U_{T_0,n}^{a}(t,h_n) + U_{T_0,n}^{b}(t) - \frac{1}{n}\cdot\frac{1}{h_n^D}\cdot U_{T_0,n}^{a}(t,h_n) - \frac{1}{n}\cdot U_{T_0,n}^{b}(t) + \frac{1}{n\cdot h_n^D}\cdot\nu_{T_0,n}^{a}(t,h_n) + \frac{1}{n}\cdot\nu_{T_0,n}^{b}(t)\\
&+\frac{1}{n\cdot h_n^D}\cdot E_F\left[\Lambda_{T_0,F}^{a}(V,V,t,h_n)\right] + \frac{1}{n}\cdot E_F\left[\Lambda_{T_0,F}^{b}(V,V,t)\right]
\end{aligned}$$

Using (S3.6.16) and (S3.6.22), the above expression becomes

$$\begin{aligned}&\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\psi_F^{\mathcal{B}}(V_j,X_i,t,\theta_F^*,h_n)\mathbb{1}\{\mathcal{B}(Q_F(X_i,t,\theta_F^*))\geq 0\}\phi(X_i)\\ =&\frac{1}{n}\sum_{i=1}^{n}\left(\left(\sum_{p=1}^{P}\Omega_{T_0}^{p}(Y_i,t,g(X_i,\theta_F^*))\cdot\omega_p(g(X_i,\theta_F^*))\right)+\Xi_{T_0,F}(t)\psi_F^{\theta}(Z_i)\right)+\xi_{T_0,n}^{c}(t),\end{aligned} \tag{S3.6.29}$$

where

$$\begin{aligned}\xi_{T_0,n}^{c}(t)\equiv\frac{1}{n}\sum_{i=1}^{n}\overline{B}_{T_0,n}(Y_i,X_i,t)+\frac{1}{2h_n^D}\cdot\widetilde{U}_{T_0,n}^{a}(t,h_n)+\frac{1}{2}\cdot\widetilde{U}_{T_0,n}^{b}(t)-\frac{1}{n}\cdot\frac{1}{h_n^D}\cdot U_{T_0,n}^{a}(t,h_n)-\frac{1}{n}\cdot U_{T_0,n}^{b}(t)\\ +\frac{1}{n\cdot h_n^D}\cdot\nu_{T_0,n}^{a}(t,h_n)+\frac{1}{n}\cdot\nu_{T_0,n}^{b}(t)+\frac{1}{n\cdot h_n^D}\cdot E_F\left[\Lambda_{T_0,F}^{a}(V,V,t,h_n)\right]+\frac{1}{n}\cdot E_F\left[\Lambda_{T_0,F}^{b}(V,V,t)\right]\end{aligned}$$

Combining (S3.6.17), (S3.6.19), (S3.6.21), (S3.6.24), (S3.6.25), (S3.6.26), (S3.6.27), and (S3.6.28),

$$\begin{aligned}\sup_{t\in\mathcal{T}}\left|\xi_{T_0,n}^{c}(t)\right|&\leq o_p\left(\frac{1}{n^{1/2+\epsilon}}\right)+O_p\left(\frac{1}{n\cdot h_n^D}\right)+O_p\left(\frac{1}{n}\right)+O_p\left(\frac{1}{n^{3/2}\cdot h_n^D}\right)+O_p\left(\frac{1}{n^{3/2}}\right)\\ &+O_p\left(\frac{1}{n^{3/2}\cdot h_n^D}\right)+O_p\left(\frac{1}{n^{3/2}}\right)+O\left(\frac{1}{n\cdot h_n^D}\right)+O\left(\frac{1}{n}\right)\\ &=o_p\left(\frac{1}{n^{1/2+\epsilon}}\right)\quad\text{uniformly over }\mathcal{F}.\end{aligned} \tag{S3.6.30}$$

where $\epsilon>0$ is as described in Assumption 4. Let

$$\psi_F^{T_0}(V_i,t)\equiv\left((\mathcal{B}(Q_F(X_i,t,\theta_F^*)))_+\phi(X_i)-T_{0,F}(t)\right)+\sum_{p=1}^{P}\Omega_{T_0}^{p}(Y_i,t,g(X_i,\theta_F^*))\cdot\omega_p(g(X_i,\theta_F^*))+\Xi_{T_0,F}(t)\psi_F^{\theta}(Z_i). \tag{S3.6.31}$$

Let $\Delta\equiv\epsilon\wedge(\delta_0/2)$. Plugging (S3.6.29), (S3.6.30), (S3.6.6) and (S3.6.4) into (S3.6.5), we have

$$\widehat{T_0}(t)=T_{0,F}(t)+\frac{1}{n}\sum_{i=1}^{n}\psi_F^{T_0}(V_i,t)+\varepsilon_n^{T_0}(t),\quad\text{where}\quad\sup_{t\in\mathcal{T}}\left|\varepsilon_n^{T_0}(t)\right|=o_p\left(\frac{1}{n^{1/2+\Delta}}\right)\quad\text{uniformly over }\mathcal{F}. \tag{S3.6.32}$$

The influence function $\psi_F^{T_0}(V,t)$ has two key features,

$$\begin{aligned}&(i)\quad E_F\left[\psi_F^{T_0}(V,t)\right]=0\quad\forall\ t\in\mathcal{T},\ \forall\ F\in\mathcal{F},\\ &(ii)\quad P_F\left(\mathcal{B}(\Gamma_F(X,t,\theta_F^*))<0\ \middle|\ X\in\mathcal{X}_F^*\right)=1\implies P_F\left(\psi_F^{T_0}(V,t)=0\right)=1.\end{aligned} \tag{S3.6.33}$$

Part (i) of (S3.6.33) follows from (S3.6.18), $E_F\left[\psi_F^{\theta}(Z)\right]=0$, and $E_F\left[\left(\mathcal{B}\left(Q_F(X_i,t,\theta_F^*)\right)\right)_+\phi(X_i)-T_{0,F}(t)\right]=0\ \forall\ t\in\mathcal{T}$. For part (ii), note that, from (8), having $P_F\left(\mathcal{B}\left(\Gamma_F(X,t,\theta_F^*)\right)<0\middle|X\in\mathcal{X}_F^*\right)=1$ implies that

$P_F\left(\left(\mathcal{B}\left(Q_F(X,t,\theta_F^*)\right)\right)_+\phi(X)=0\right)=P_F\left(\left(\mathcal{B}\left(\Gamma_F(X,t,\theta_F^*)\right)\right)_+\cdot\mathcal{H}\left(\omega(g(X,\theta_F^*))\right)\phi(X)=0\right)=1$, and $T_{0,F}(t)=0$. Also, if $P_F\left(\mathcal{B}\left(\Gamma_F(X,t,\theta_F^*)\right)<0\middle|X\in\mathcal{X}_F^*\right)=1$, then

$$
\begin{aligned}
&P_F\left(\phi(X)\mathbb{1}\left\{\mathcal{B}(Q_F(X,t,\theta_F^*))\geq 0\right\}=0\;\middle|\;g(X,\theta_F^*)\in\mathcal{G}\right)\\
=&P_F\left(\phi(X)\mathbb{1}\left\{\mathcal{B}(\Gamma_F(X,t,\theta_F^*))\cdot\mathcal{H}(\omega(g(X,\theta_F^*)))\geq 0\right\}=0\;\middle|\;g(X,\theta_F^*)\in\mathcal{G}\right)=1.
\end{aligned}
$$

Thus, for any (y,t,g), the above result implies

$$
\begin{aligned}
&\Omega_{T_0}^{p}(y,t,g)\cdot\omega_p(g)\\
=&E_F\left[\left(S_p(y,t)-\Gamma_{p,F}(X,t,\theta_F^*)\right)\frac{\partial\mathcal{B}\left(Q_F(X,t,\theta_F^*)\right)}{\partial Q_p}\underbrace{\phi(X)\mathbb{1}\left\{\mathcal{B}(Q_F(X,t,\theta_F^*))\geq 0\right\}}_{\substack{=0\ F\text{-a.s if } g(X,\theta_F^*)=g\in\mathcal{G}\\ (\text{i.e, if } \omega_p(g)\neq 0)}}\;\middle|\;g(X,\theta_F^*)=g\right]\cdot\underbrace{\omega_p(g)}_{\substack{=0\text{ if}\\ g\notin\mathcal{G}}}\\
=&0.
\end{aligned}
$$

Therefore, $P_F\left(\mathcal{B}\left(\Gamma_F(X,t,\theta_F^*)\right)<0\middle|X\in\mathcal{X}_F^*\right)=1\implies P_F\left(\Omega_{T_0}^{p}(Y,t,g(X,\theta_F^*))\cdot\omega_p(g(X,\theta_F^*))=0\right)=1$. Finally, recall from (A4) that $\Xi_{Q_p}(x,t,\theta_F^*)=0\;\;\forall\, x: g(x,\theta_F^*)\notin\mathcal{G}$. Thus, for each $t\in\mathcal{T}$, we have $\phi(x)\Xi_{Q_p}(x,t,\theta_F^*)=0\quad\forall\, x\notin\mathcal{X}_F^*$. And from our definition of $\Xi_{T_0,F}^{p}(t)$ and $\Xi_{T_0,F}(t)$ in (S3.6.20), if $P_F\left(\mathcal{B}\left(\Gamma_F(X,t,\theta_F^*)\right)<0\middle|X\in\mathcal{X}_F^*\right)=1$, then

$$
\begin{aligned}
\Xi_{T_0,F}^{p}(t)&\equiv E_F\left[\frac{\partial\mathcal{B}\left(Q_F(X,t,\theta_F^*)\right)}{\partial Q_p}\mathbb{1}\left\{\mathcal{B}(Q_F(X,t,\theta_F^*))\geq 0\right\}\phi(X)\Xi_{Q_p}(X,t,\theta_F^*)\right]\\
&=E_F\left[\frac{\partial\mathcal{B}\left(Q_F(X,t,\theta_F^*)\right)}{\partial Q_p}\underbrace{\mathbb{1}\left\{\mathcal{B}(\Gamma_F(X,t,\theta_F^*))\cdot\mathcal{H}(\omega(g(X,\theta_F^*)))\geq 0\right\}}_{=0\text{ if } X\in\mathcal{X}_F^*}\underbrace{\phi(X)\Xi_{Q_p}(X,t,\theta_F^*)}_{\substack{=0\\ \text{if } X\notin\mathcal{X}_F^*}}\right]\\
&=0\quad\forall\, p=1,\ldots,P.\\
\implies\quad \Xi_{T_0,F}(t)&\equiv\sum_{p=1}^{P}\Xi_{T_0,F}^{p}(t)=0.
\end{aligned}
$$

Thus, $\Xi_{T_0,F}(t)\psi_F^{\theta}(Z)=0$. Combined, these results yield that, if $P_F\left(\mathcal{B}\left(\Gamma_F(X,t,\theta_F^*)\right)<0\;\middle|\;X\in\mathcal{X}_F^*\right)=1$, then we must have $P_F\left(\psi_F^{T_0}(V,t)=0\right)=1$, which establishes part (ii) of (S3.6.33). Equipped with (S3.6.32), we can characterize a linear representation for $\widehat{T}_2$ by recalling that, $\widehat{T}_2\equiv\int_t\widehat{T}_0(t)d\mathcal{W}(t)$, and $T_{2,F}\equiv\int_t T_{0,F}(t)d\mathcal{W}(t)$, with $\int_t d\mathcal{W}(t)=1$. Let

$$
\psi_F^{T_2}(V_i)\equiv\int_t\psi_F^{T_0}(V_i,t)d\mathcal{W}(t),\quad \varepsilon_n^{T_2}\equiv\int_t\varepsilon_n^{T_0}(t)d\mathcal{W}(t).\tag{S3.6.34}
$$

From (S3.6.32), we have

$$\widehat{T}_2 = T_{2,F} + \frac{1}{n}\sum_{i=1}^{n} \psi_F^{T_2}(V_i) + \varepsilon_n^{T_2},$$
$$\text{where} \quad \left|\varepsilon_n^{T_2}\right| \leq \sup_{t\in\mathcal{T}} \left|\varepsilon_n^{T_0}(t)\right| \cdot \underbrace{\int_t d\mathcal{W}(t)}_{=1} = o_p\left(\frac{1}{n^{1/2+\Delta}}\right) \quad \text{uniformly over } \mathcal{F}. \tag{S3.6.35}$$

Equations (S3.6.31)**,** (S3.6.32)**,** (S3.6.34) **and** (S3.6.35) **prove the linear representation result in Proposition 1.** ■

S3.6.3 Properties of the influence function $\psi_F^{T_2}(V)$

From (S3.6.33), we have the following properties for the influence function $\psi_F^{T_2}(V)$,

$$(i) \quad E_F\left[\psi_F^{T_2}(V)\right] = 0 \quad \forall\, F \in \mathcal{F},$$
$$(ii) \quad P_F\left(\mathcal{B}(\Gamma_F(X,t,\theta_F^*)) < 0 \,\middle|\, X \in \mathcal{X}_F^*\right) = 1 \ \text{ for } \mathcal{W}\text{−a.e } t \in \mathcal{T} \implies P_F\left(\psi_F^{T_2}(V) = 0\right) = 1. \tag{S3.6.36}$$

Part (i) of (S3.6.36) follows directly from part (i) of (S3.6.33) since,

$$E_F\left[\psi_F^{T_2}(V)\right] = E_F\left[\int_t \psi_F^{T_0}(V,t)d\mathcal{W}(t)\right] = \int_t \underbrace{\left(E_F\left[\psi_F^{T_0}(V,t)\right]\right)}_{=0\ \forall\ t\in\mathcal{T}} d\mathcal{W}(t) = 0.$$

Similarly, part (ii) of (S3.6.36) follows directly from part (ii) of (S3.6.33) since,

$$P_F\left(\mathcal{B}(\Gamma_F(X,t,\theta_F^*)) < 0 \,\middle|\, X \in \mathcal{X}_F^*\right) = 1 \ \text{ for } \mathcal{W}\text{−a.e } t \in \mathcal{T} \implies P_F\left(\psi_F^{T_0}(V,t) = 0 \ \text{ for } \mathcal{W}\text{−a.e } t \in \mathcal{T}\ \right) = 1.$$

And,

$$P_F\left(\psi_F^{T_0}(V,t) = 0 \ \text{ for } \mathcal{W}\text{−a.e } t \in \mathcal{T}\ \right) = 1 \implies P_F\left(\psi_F^{T_2}(V) = 0\right) = P_F\left(\int_t \psi_F^{T_0}(V,t)d\mathcal{W}(t) = 0\right) = 1.$$

S4 An estimator for the influence function $\psi_F^{T_2}(V)$

Here we describe how we obtained the influence function estimators discussed in Section 4.5.2 of the paper and in Appendix A, which are used in the construction of and $\widehat{\sigma}_2^2$.

S4.1 Construction of our estimators

Fix g, x and t. We will first decompose the functionals defined in Assumption 2 as follows,

$$\begin{aligned}
&(1)\quad \Omega^{d,\ell}_{f_g}(g) \equiv E_F\left[\frac{\partial g_d(X,\theta^*_F)}{\partial\theta_\ell}\bigg| g(X,\theta^*_F)=g\right] = \frac{A^{d,\ell}_{f_g}(g)}{f_g(g)},\\
&\qquad \text{where}\quad A^{d,\ell}_{f_g}(g) \equiv \Omega^{d,\ell}_{f_g}(g)\cdot f_g(g),\\
&(2)\quad \Omega^{d,\ell}_{R_p,1}(g,t) \equiv E_F\left[S_p(Y,t)\omega_p(g(X,\theta^*_F))\frac{\partial g_d(X,\theta^*_F)}{\partial\theta_\ell}\bigg| g(X,\theta^*_F)=g\right] = \frac{A^{d,\ell}_{R_p,1}(g,t)}{f_g(g)},\\
&\qquad \text{where}\quad A^{d,\ell}_{R_p,1}(g,t) \equiv \Omega^{d,\ell}_{R_p,1}(g,t)\cdot f_g(g),\\
&(3)\quad \Omega_{R_p,2}(g,t) \equiv E_F\left[S_p(Y,t)\omega_p(g(X,\theta^*_F))\Big| g(X,\theta^*_F)=g\right] = \frac{A_{R_p,2}(g,t)}{f_g(g)},\\
&\qquad \text{where}\quad A_{R_p,2}(g,t) \equiv \Omega_{R_p,2}(g,t)\cdot f_g(g),\\
&(4)\quad \Omega^{\ell}_{R_p,3}(g,t) \equiv E_F\left[S_p(Y,t)\frac{\partial\omega_p(g(X,\theta^*_F))}{\partial\theta_\ell}\bigg| g(X,\theta^*_F)=g\right] = \frac{A^{\ell}_{R_p,3}(g,t)}{f_g(g)},\\
&\qquad \text{where}\quad A^{\ell}_{R_p,3}(g,t) \equiv \Omega^{\ell}_{R_p,3}(g,t)\cdot f_g(g).
\end{aligned} \tag{S4.1.1}$$

In addition to the above functionals, the following derivatives will be relevant,

$$\frac{\partial\Omega^{d,\ell}_{f_g}(g)}{\partial g_d},\quad \frac{\partial\Omega^{d,\ell}_{R_p,1}(g,t)}{\partial g_d},\quad \text{and}\quad \frac{\partial\Omega_{R_p,2}(g,t)}{\partial g_d}.$$

From the expressions in (S4.1.1), we have

$$\begin{aligned}
\frac{\partial\Omega^{d,\ell}_{f_g}(g)}{\partial g_d} &= \frac{\partial A^{d,\ell}_{f_g}(g)}{\partial g_d}\cdot\frac{1}{f_g(g)} - \frac{A_{f_g}(g)}{f_g(g)^2}\cdot\frac{\partial f_g(g)}{\partial g_d},\\
\frac{\partial\Omega^{d,\ell}_{R_p,1}(g,t)}{\partial g_d} &= \frac{\partial A^{d,\ell}_{R_p,1}(g,t)}{\partial g_d}\cdot\frac{1}{f_g(g)} - \frac{A^{d,\ell}_{R_p,1}(g,t)}{f_g(g)^2}\cdot\frac{\partial f_g(g)}{\partial g_d},\\
\frac{\partial\Omega_{R_p,2}(g,t)}{\partial g_d} &= \frac{\partial A_{R_p,2}(g,t)}{\partial g_d}\cdot\frac{1}{f_g(g)} - \frac{A_{R_p,2}(g,t)}{f_g(g)^2}\cdot\frac{\partial f_g(g)}{\partial g_d},
\end{aligned}$$

We will estimate the above functionals and the derivatives described using the same type of kernel estimators we employed to construct $\widehat{T}_2$. Our estimator of $f_g(g)$ is

$$\widehat{f_g}(g) = \frac{1}{n\cdot h_n^D}\sum_{i=1}^{n} K\left(\frac{g(X_i,\widehat{\theta})-g}{h_n}\right),$$

From here, we estimate

$$(1)\quad \widehat{\Omega}^{d,\ell}_{f_g}(g)=\frac{\widehat{A}^{d,\ell}_{f_g}(g)}{\widehat{f}_g(g)},\quad \widehat{A}^{d,\ell}_{f_g}(g)\equiv\frac{1}{nh_n^D}\sum_{i=1}^{n}\frac{\partial g_d(X_i,\widehat{\theta})}{\partial\theta_\ell}K\left(\frac{g(X_i,\widehat{\theta})-g}{h_n}\right),$$

$$(2)\quad \widehat{\Omega}^{d,\ell}_{R_p,1}(g,t)=\frac{\widehat{A}^{d,\ell}_{R_p,1}(g,t)}{\widehat{f}_g(g)},\quad \widehat{A}^{d,\ell}_{R_p,1}(g,t)\equiv\frac{1}{n\cdot h_n^D}\sum_{i=1}^{n}S_p(Y_i,t)\omega_p(g(X_i,\widehat{\theta}))\frac{\partial g_d(X_i,\widehat{\theta})}{\partial\theta_\ell}K\left(\frac{g(X_i,\widehat{\theta})-g}{h_n}\right),$$

$$(3)\quad \widehat{\Omega}_{R_p,2}(g,t)=\frac{\widehat{A}_{R_p,2}(g,t)}{\widehat{f}_g(g)},\quad \widehat{A}_{R_p,2}(g,t)\equiv\frac{1}{n\cdot h_n^D}\sum_{i=1}^{n}S_p(Y_i,t)\omega_p(g(X_i,\widehat{\theta}))K\left(\frac{g(X_i,\widehat{\theta})-g}{h_n}\right),$$

$$(4)\quad \widehat{\Omega}^{\ell}_{R_p,3}(g,t)=\frac{\widehat{A}^{\ell}_{R_p,3}(g,t)}{\widehat{f}_g(g)},\quad \widehat{A}^{\ell}_{R_p,3}(g,t)\equiv\frac{1}{n\cdot h_n^D}\sum_{i=1}^{n}S_p(Y_i,t)\frac{\partial\omega_p(g(X_i,\widehat{\theta}))}{\partial\theta_\ell}K\left(\frac{g(X_i,\widehat{\theta})-g}{h_n}\right),$$

From here, we have

$$\frac{\partial\widehat{\Omega}^{d,\ell}_{f_g}(g)}{\partial g_d}=\frac{\partial\widehat{A}^{d,\ell}_{f_g}(g)}{\partial g_d}\cdot\frac{1}{\widehat{f}_g(g)}-\frac{\widehat{A}_{f_g}(g)}{\widehat{f}_g(g)^2}\cdot\frac{\partial\widehat{f}_g(g)}{\partial g_d},$$

$$\frac{\partial\widehat{\Omega}^{d,\ell}_{R_p,1}(g,t)}{\partial g_d}=\frac{\partial\widehat{A}^{d,\ell}_{R_p,1}(g,t)}{\partial g_d}\cdot\frac{1}{\widehat{f}_g(g)}-\frac{\widehat{A}^{d,\ell}_{R_p,1}(g,t)}{\widehat{f}_g(g)^2}\cdot\frac{\partial\widehat{f}_g(g)}{\partial g_d},$$

$$\frac{\partial\widehat{\Omega}_{R_p,2}(g,t)}{\partial g_d}=\frac{\partial\widehat{A}_{R_p,2}(g,t)}{\partial g_d}\cdot\frac{1}{\widehat{f}_g(g)}-\frac{\widehat{A}_{R_p,2}(g,t)}{\widehat{f}_g(g)^2}\cdot\frac{\partial\widehat{f}_g(g)}{\partial g_d},$$

Under Assumptions 1-4, for each d, ℓ and p we have that, uniformly over $\mathcal{F}$,

$$\begin{aligned}\sup_{x\in\mathcal{X}}\left|\widehat{\Omega}^{d,\ell}_{f_g}(g(x,\widehat{\theta}))-\Omega^{d,\ell}_{f_g}(g(x,\theta^*_F))\right|&=o_p(1),\\ \sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\widehat{\Omega}^{d,\ell}_{R_p,1}(g(x,\widehat{\theta}),t)-\Omega^{d,\ell}_{R_p,1}(g(x,\theta^*_F),t)\right|&=o_p(1),\\ \sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\widehat{\Omega}_{R_p,2}(g(x,\widehat{\theta}),t)-\Omega_{R_p,2}(g(x,\theta^*_F),t)\right|&=o_p(1),\\ \sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\widehat{\Omega}^{\ell}_{R_p,3}(g(x,\widehat{\theta}),t)-\Omega^{\ell}_{R_p,3}(g(x,\theta^*_F),t)\right|&=o_p(1),\end{aligned}\tag{S4.1.2}$$

and,

$$\sup_{x\in\mathcal{X}}\left|\frac{\partial\widehat{\Omega}^{d,\ell}_{f_g}(g(x,\widehat{\theta}))}{\partial g_d}-\frac{\partial\Omega^{d,\ell}_{f_g}(g(x,\theta^*_F))}{\partial g_d}\right|=o_p(1),$$

$$\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\frac{\partial\widehat{\Omega}^{d,\ell}_{R_p,1}(g(x,\widehat{\theta}),t)}{\partial g_d}-\frac{\partial\Omega^{d,\ell}_{R_p,1}(g(x,\theta^*_F),t)}{\partial g_d}\right|=o_p(1), \tag{S4.1.3}$$

$$\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left|\frac{\partial\widehat{\Omega}_{R_p,2}(g(x,\widehat{\theta}),t)}{\partial g_d}-\frac{\partial\Omega_{R_p,2}(g(x,\theta^*_F),t)}{\partial g_d}\right|=o_p(1).$$

Next, we estimate $\Xi_{f_g}(x,\theta^*_F)$ and $\Xi_{R_p}(x,t,\theta^*_F)$, where these functionals are described in (A1) and $\Xi_{Q_p}(x,t,\theta^*_F)$, where this functional is described in equation (A3). Our estimators are,

$$\begin{aligned}
\widehat{\Xi}_{\ell,f_g}(x,\widehat{\theta}) &\equiv \sum_{d=1}^{D}\left(\frac{\partial g_d(x,\widehat{\theta})}{\partial\theta_\ell}\cdot\frac{\partial\widehat{f_g}(g(x,\widehat{\theta}))}{\partial g_d}-\frac{\partial\left[\widehat{\Omega}^{d,\ell}_{f_g}(g(x,\widehat{\theta}))\widehat{f_g}(g(x,\widehat{\theta}))\right]}{\partial g_d}\right),\\
\widehat{\Xi}_{f_g}(x,\widehat{\theta}) &\equiv \left(\widehat{\Xi}_{1,f_g}(x,\widehat{\theta}),\ldots,\widehat{\Xi}_{k,f_g}(x,\widehat{\theta})\right),\\
\widehat{\Xi}_{\ell,R_p}(x,t,\widehat{\theta}) &\equiv \sum_{d=1}^{D}\left(\frac{\partial\left[\widehat{\Omega}_{R_p,2}(g(x,\widehat{\theta}),t)\widehat{f_g}(g(x,\widehat{\theta}))\right]}{\partial g_d}\cdot\frac{\partial g_d(x,\widehat{\theta})}{\partial\theta_\ell}-\frac{\partial\left[\widehat{\Omega}^{d,\ell}_{R_p,1}(g(x,\widehat{\theta}),t)\widehat{f_g}(g(x,\widehat{\theta}))\right]}{\partial g_d}\right.\\
&\qquad\left.+\widehat{\Omega}^{\ell}_{R_p,3}(g(x,\widehat{\theta}),t)\cdot\widehat{f_g}(g(x,\widehat{\theta}))\right),\\
\widehat{\Xi}_{R_p}(x,t,\widehat{\theta}) &\equiv \left(\widehat{\Xi}_{1,R_p}(x,t,\widehat{\theta}),\ldots,\widehat{\Xi}_{k,R_p}(x,t,\widehat{\theta})\right),\\
\widehat{\Xi}_{Q_p}(x,t,\widehat{\theta}) &\equiv \frac{\widehat{\Xi}_{R_p}(x,t,\widehat{\theta})-\widehat{Q}_p(x,t,\widehat{\theta})\cdot\widehat{\Xi}_{f_g}(x,\widehat{\theta})}{\widehat{f_g}(g(x,\widehat{\theta}))},
\end{aligned}$$

where $\widehat{Q}_p(x,t,\widehat{\theta})$ is the estimator we described in (13). Under the conditions in Assumptions 1-4 (see (S4.1.2)-(S4.1.3)), we obtain that, for each p and uniformly over $\mathcal{F}$,

$$\sup_{(x,t)\in\mathcal{X}\times\mathcal{T}}\left\|\widehat{\Xi}_{Q_p}(x,t,\widehat{\theta})-\Xi_{Q_p}(x,t,\theta^*_F)\right\|=o_p(1). \tag{S4.1.4}$$

For a given t, we estimate the functional $\Xi_{T_0,F}(t)$ in Proposition 1, (eq. A15) with

$$\begin{aligned}
\widehat{\Xi}^p_{T_0}(t) &\equiv \frac{1}{n}\sum_{i=1}^{n}\frac{\partial\mathcal{B}\left(\widehat{Q}(X_i,t,\widehat{\theta})\right)}{\partial\widehat{Q}_p}\mathbb{1}\left\{\mathcal{B}\left(\widehat{Q}(X_i,t,\widehat{\theta})\right)\geq -b_n\right\}\phi(X_i,t)\widehat{\Xi}_{Q_p}(X_i,t,\widehat{\theta}),\\
\widehat{\Xi}_{T_0}(t) &\equiv \sum_{p=1}^{P}\widehat{\Xi}^p_{T_0}(t),
\end{aligned}$$

Under the conditions in Assumptions 1-4 (equations S4.1.2-S4.1.4), uniformly over $\mathcal{F}$ we have

$$\sup_{t\in\mathcal{T}}\left\|\widehat{\Xi}_{T_0}(t)-\Xi_{T_0,F}(t)\right\|=o_p(1),$$
$$\left\|\int_t\widehat{\Xi}_{T_0}(t)d\mathcal{W}(t)-\int_t\Xi_{T_0,F}(t)d\mathcal{W}(t)\right\|=o_p(1) \tag{S4.1.5}$$

Next, for a given x, t, we estimate $\widehat{\Gamma}_p(x,t,\widehat{\theta})=\frac{\frac{1}{n\cdot h_n^D}\sum_{i=1}^n S_p(Y_i,t)K\left(\frac{\Delta g(X_i,x,\widehat{\theta})}{h_n}\right)}{\widehat{f}_g(g(x,\widehat{\theta}))}$. For a given (y,g,t), we estimate the functional $\Omega^p_{T_0}(y,t,g)$ described in Assumption 7 (eq. 18) with

$$\widehat{\Omega}^p_{T_0}(y,t,g)=$$
$$\frac{\frac{1}{n\cdot h_n^D}\sum_{i=1}^n\left(S_p(y,t)-\widehat{\Gamma}_p(X_i,t,\widehat{\theta})\right)\frac{\partial\mathcal{B}(\widehat{Q}(X_i,t,\widehat{\theta}))}{\partial Q_p}\phi(X_i)\mathbb{1}\left\{\mathcal{B}\left(\widehat{Q}(X_i,t,\widehat{\theta})\right)\geq -b_n\right\}K\left(\frac{g(X_i,\widehat{\theta})-g}{h_n}\right)}{\widehat{f}_g(g)}$$

Under the conditions in Assumptions 1-6 and 7 (i.e, the conditions of Proposition 1), for each p and uniformly over $\mathcal{F}$ we have

$$\sup_{\substack{(y,x)\in\mathcal{S}_{Y,X}\\ t\in\mathcal{T}}}\left|\widehat{\Omega}^p_{T_0}\left(y,t,g(x,\widehat{\theta})\right)\cdot\omega_p(g(x,\widehat{\theta}))-\Omega^p_{T_2}(y,t,g(x,\theta^*_F))\cdot\omega_p(g(x,\theta^*_F))\right|=o_p(1),$$
$$\sup_{(y,x)\in\mathcal{S}_{Y,X}}\left|\int_t\widehat{\Omega}^p_{T_0}\left(y,t,g(x,\widehat{\theta})\right)d\mathcal{W}(t)\cdot\omega_p(g(x,\widehat{\theta}))-\int_t\Omega^p_{T_0}(y,t,g(x,\theta^*_F))d\mathcal{W}(t)\cdot\omega_p(g(x,\theta^*_F))\right|=o_p(1),$$
$$\sup_{(x,t)\in\mathcal{S}_X\times\mathcal{T}}\left|\mathcal{B}\left(\widehat{Q}(x,t,\widehat{\theta})\right)\mathbb{1}\left\{\mathcal{B}\left(\widehat{Q}(x,t,\widehat{\theta})\right)\geq -b_n\right\}\phi(x)-\left(\mathcal{B}\left(\widehat{Q}(x,t,\theta^*_F)\right)\right)_+\phi(x)\right|=o_p(1),$$
$$\sup_{x\in\mathcal{S}_X}\left|\int_t\mathcal{B}\left(\widehat{Q}(x,t,\widehat{\theta})\right)\mathbb{1}\left\{\mathcal{B}\left(\widehat{Q}(x,t,\widehat{\theta})\right)\geq -b_n\right\}d\mathcal{W}(t)\phi(x)-\int_t\left(\mathcal{B}\left(\widehat{Q}(x,t,\theta^*_F)\right)\right)_+d\mathcal{W}(t)\phi(x)\right|=o_p(1),$$
$$\left|\widehat{T}_2-T_{2,F}\right|=o_p(1) \tag{S4.1.6}$$

(the last line follows directly from equation (19) in Proposition 1). The last component is an estimator of the influence function $\psi^\theta_F(Z)$ for the estimator $\widehat{\theta}$. From Assumption 8, we have at hand an estimator $\widehat{\psi}^\theta(Z)$ that satisfies, $\frac{1}{n}\sum_{i=1}^n\left\|\widehat{\psi}^\theta(Z_i)-\psi^\theta_F(Z_i)\right\|^2=o_p(1)$ uniformly over $\mathcal{F}$. Our estimator for the influence function $\psi^{T_2}_F(V)$ of the statistic $\widehat{T}_2$ is,

$$\widehat{\psi}^{T_2}(V_i)\equiv\left(\int_t\mathcal{B}\left(\widehat{Q}(X_i,t,\widehat{\theta})\right)\mathbb{1}\left\{\mathcal{B}\left(\widehat{Q}(X_i,t,\widehat{\theta})\right)\geq -b_n\right\}d\mathcal{W}(t)\phi(X_i)-\widehat{T}_2\right)$$
$$+\sum_{p=1}^P\int_t\widehat{\Omega}^p_{T_0}(Y_i,t,g(X_i,\widehat{\theta}))d\mathcal{W}(t)\cdot\omega(g(X_i,\widehat{\theta}))+\int_t\widehat{\Xi}_{T_0}(t)d\mathcal{W}(t)\widehat{\psi}^\theta(Z_i)$$

From the results in (S4.1.5) and (S4.1.6), we have that under the conditions in Assumptions 1-8 in the paper, $\frac{1}{n}\sum_{i=1}^{n}\left|\widehat{\psi}^{T_2}(V_i) - \psi_F^{T_2}(V_i)\right|^2 = o_p(1)$ uniformly over $\mathcal{F}$. ■

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