Econometric Supplement for "Estimation and inference in discrete games with uncertain behavior"

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Abstract
This document includes step-by-step derivations leading to the main econometric results in the paper. Every section in this document has the format **SX.X** and every equation has the format **(SX.X.X)**. Any section, assumption or equation that we reference here which does not have this format refers to the main paper, or to Appendix A (if it has the format AX.X.X).

S1 Asymptotic properties of $\widehat{\Delta}$ and proof of Proposition 1

As defined in equation (5), let,

$$\begin{aligned} m_S^C(X,\theta) &\equiv F_{1,2}(X_1^{ns\prime}\beta_1 - X_1^{s\prime}\Delta_1 - X_2^{s\prime}\Delta_2\ ,\ X_2^{ns\prime}\beta_2 - X_1^{s\prime}\Delta_1 - X_2^{s\prime}\Delta_2|\rho), \\ m_S^{NC}(X,\theta) &\equiv F_{1,2}(X_1^{ns\prime}\beta_1 - X_1^{s\prime}\Delta_1\ ,\ X_2^{ns\prime}\beta_2 - X_2^{s\prime}\Delta_2|\rho), \\ \Xi_S(X,\theta) &\equiv m_S^C(X,\theta) - m_S^{NC}(X,\theta). \end{aligned}$$

And recall that, from (3), our expression for $E[S|U]$ simplifies to,

$$E[S|U] = m_S^{NC}(X,\theta_0) + \pi(Z)\cdot\Xi_S(X,\theta_0).$$

We begin by describing the asymptotic properties of the semiparametric convolution weights $\widehat{\pi}(z,\theta)$ introduced in equation (22) of the paper.

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S1.1 Asymptotic properties of $\widehat{\pi}(z,\theta)$

Take the functionals we defined in Assumption E2.

$$
\begin{aligned}
\mu_I^{\Xi_S}(z,\theta) &\equiv E[(S - m_S^{NC}(X,\theta)) \cdot \Xi_S(X,\theta) \,|Z = z] \\
\mu_{II,\kappa}^{\Xi_S}(z,\theta) &\equiv E[\Xi_S(X,\theta)^{\kappa} \,|Z = z] \text{ (for } \kappa = 1,2) \\
\mu_{III,\theta_\ell}^{\Xi_S}(z,\theta) &\equiv E\left[\frac{\partial \Xi_S(X,\theta)}{\partial \theta_\ell} \cdot (S - m_S^{NC}(X,\theta)) \cdot \,\Big| Z = z\right] \text{ (for } \ell = 1,\ldots,d_\theta) \\
\mu_{IV,\theta_\ell}^{\Xi_S}(z,\theta) &\equiv E\left[\frac{\partial m_S^{NC}(X,\theta)}{\partial \theta_\ell} \cdot \Xi_S(X,\theta) \,\Big| Z = z\right] \text{ (for } \ell = 1,\ldots,d_\theta) \\
\mu_{V,\theta_\ell}^{\Xi_S}(z,\theta) &\equiv E\left[\frac{\partial \Xi_S(X,\theta)}{\partial \theta_\ell} \cdot \Xi_S(X,\theta) \,\Big| Z = z\right] \text{ (for } \ell = 1,\ldots,d_\theta) \\
\mu_{VI,\theta_\ell}^{\Xi_S}(z,\theta) &\equiv E\left[\frac{\partial \Xi_S(X,\theta)}{\partial \theta_\ell} \,\Big| Z = z\right] \text{ (for } \ell = 1,\ldots,d_\theta)
\end{aligned}
\tag{S1.1}
$$

From here, for any $z \in \text{Supp}(Z)$ (i.e, any z such that $f_Z(z) > 0$) and any $\theta \in \Theta$ we will denote

$$
R_a^{\pi}(z,\theta) \equiv \mu_I^{\Xi_S}(z,\theta) \cdot f_Z(z) \quad \text{and} \quad R_b^{\pi}(z,\theta) \equiv \mu_{II,2}^{\Xi_S}(z,\theta) \cdot f_Z(z).
$$

For any $z \in \text{Supp}(Z)$, we can write

$$
\pi(z,\theta) = \frac{R_a^{\pi}(z,\theta)}{R_b^{\pi}(z,\theta)}. \tag{S1.2}
$$

We can describe our estimator $\widehat{\pi}(z,\theta)$ using sample analogs of $R_a^{\pi}(z,\theta)$ and $R_b^{\pi}(z,\theta)$. Let

$$
\begin{aligned}
\widehat{R}_a^{\pi}(z,\theta) &\equiv \frac{1}{n \cdot h_n^{d_Z}} \sum_{i=1}^{n} \Big(S_i - m_S^{NC}(X_i,\theta)\Big) \cdot \Xi_S(X_i,\theta) \cdot K\left(\frac{Z_i - z}{h_n}\right), \\
\widehat{R}_b^{\pi}(z,\theta) &\equiv \frac{1}{n \cdot h_n^{d_Z}} \sum_{i=1}^{n} \Xi_S(X_i,\theta)^2 \cdot K\left(\frac{Z_i - z}{h_n}\right).
\end{aligned}
$$

Take a given (z,θ). As described in equation (22) of the paper, let

$$
\widehat{\pi}(z,\theta) \equiv \frac{\widehat{R}_a^{\pi}(z,\theta)}{\widehat{R}_b^{\pi}(z,\theta)}. \tag{S1.3}
$$

Next, for a given $h > 0$ let,

$$
\begin{aligned}
\nu_{a,n}^{\pi}(z,\theta;h) &\equiv \frac{1}{n} \sum_{i=1}^{n} \left(\Big(S_i - m_S^{NC}(X_i,\theta)\Big) \cdot \Xi_S(X_i,\theta) \cdot K\left(\frac{Z_i - z}{h}\right) - E\left[\Big(S - m_S^{NC}(X,\theta)\Big) \cdot \Xi_S(X,\theta) \cdot K\left(\frac{Z - z}{h}\right)\right]\right), \\
\nu_{b,n}^{\pi}(z,\theta;h) &\equiv \frac{1}{n} \sum_{i=1}^{n} \left(\Xi_S(X_i,\theta)^2 \cdot K\left(\frac{Z_i - z}{h}\right) - E\left[\Xi_S(X,\theta)^2 \cdot K\left(\frac{Z - z}{h}\right)\right]\right)
\end{aligned}
$$

By the bounded-variation properties of our kernel function described in Assumption E1 and the Lipschitz-restrictions in Assumption E3, Nolan and Pollard (1987, Lemma 22), Pakes and Pollard

(1989, Example 2.10, Lemma 2.13, Lemma 2.14) imply that the class of functions involved is *Euclidean* (see Sherman (1994, Definition 3)),

$$\mathscr{G}_a \equiv \left\{ g(s,x,z) = \left(s - m_S^{NC}(x,\theta)\right) \cdot \Xi_S(x,\theta) \cdot K\left(\frac{z-\widetilde{z}}{h}\right) \text{ for some } \theta \in \Theta, \widetilde{z} \in \mathbb{R}^{d_Z}, h > 0 \right\},$$
$$\mathscr{G}_b \equiv \left\{ g(x,z) = \Xi_S(x,\theta)^2 \cdot K\left(\frac{z-\widetilde{z}}{h}\right) \text{ for some } \theta \in \Theta, \widetilde{z} \in \mathbb{R}^{d_Z}, h > 0 \right\}.$$

Since $|\Xi_S(x,\theta)| \le 1$ and $|m_S^{NC}(x,\theta)| \le 1$ for all (x,θ), $0 \le S \le 2$, and $|K(\psi)| \le \overline{K}$ for all ψ, the corresponding envelope for the class of functions is constant, and thus trivially has finite $2+\delta$ moments. From here, Sherman (1994, Corollary 4) yields,

$$\sup_{\substack{\theta\in\Theta \\ z\in\mathbb{R}^{d_Z} \\ h>0}} \left|\nu_{a,n}^{\pi}(z,\theta;h)\right| = O_p\left(n^{-1/2}\right), \quad \sup_{\substack{\theta\in\Theta \\ z\in\mathbb{R}^{d_Z} \\ h>0}} \left|\nu_{b,n}^{\pi}(z,\theta;h)\right| = O_p\left(n^{-1/2}\right) \tag{S1.4}$$

Next, note that we can express,

$$\begin{aligned} \widehat{R}_a^{\pi}(z,\theta) &= R_a^{\pi}(z,\theta) + B_{a,n}^{\pi}(z,\theta) + \frac{1}{h_n^{d_Z}} \cdot \nu_{a,n}^{\pi}(z,\theta;h_n), \\ \widehat{R}_b^{\pi}(z,\theta) &= R_b^{\pi}(z,\theta) + B_{b,n}^{\pi}(z,\theta) + \frac{1}{h_n^{d_Z}} \cdot \nu_{b,n}^{\pi}(z,\theta;h_n), \end{aligned} \tag{S1.5}$$

where $B_{a,n}^{\pi}(z,\theta)$ and $B_{b,n}^{\pi}(z,\theta)$ represent the bias terms,

$$B_{a,n}^{\pi}(z,\theta) \equiv E\left[\frac{1}{h_n^{d_Z}} \cdot \left(S - m_S^{NC}(X,\theta)\right) \cdot \Xi_S(X,\theta) \cdot K\left(\frac{Z-z}{h_n}\right)\right] - R_a^{\pi}(z,\theta),$$
$$B_{b,n}^{\pi}(z,\theta) \equiv E\left[\frac{1}{h_n^{d_Z}} \cdot \Xi_S(X,\theta)^2 \cdot K\left(\frac{Z-z}{h_n}\right)\right] - R_b^{\pi}(z,\theta).$$

An M^{th}–order approximation combined with the smoothness restrictions in Assumption E2 and the higher-order properties of the kernel described in Assumption E1 yield,

$$\sup_{\substack{\theta\in\Theta \\ z\in\mathcal{Z}}} \left|B_{a,n}^{\pi}(z,\theta)\right| = O\left(h_n^M\right), \quad \sup_{\substack{\theta\in\Theta \\ z\in\mathcal{Z}}} \left|B_{b,n}^{\pi}(z,\theta)\right| = O\left(h_n^M\right)$$

Combined with our bandwidth convergence restrictions in Assumption E1, these results imply

$$\begin{aligned} \sup_{\substack{\theta\in\Theta \\ z\in\mathcal{Z}}} \left|\widehat{R}_a^{\pi}(z,\theta) - R_a^{\pi}(z,\theta)\right| &= O\left(h_n^M\right) + O_p\left(\frac{1}{h_n^{d_Z} \cdot n^{1/2}}\right) = o_p\left(n^{-1/4}\right), \\ \sup_{\substack{\theta\in\Theta \\ z\in\mathcal{Z}}} \left|\widehat{R}_b^{\pi}(z,\theta) - R_b^{\pi}(z,\theta)\right| &= O\left(h_n^M\right) + O_p\left(\frac{1}{h_n^{d_Z} \cdot n^{1/2}}\right) = o_p\left(n^{-1/4}\right), \end{aligned} \tag{S1.6}$$

Recall that we denote $U \equiv X \cup Z$ and $V \equiv (S, U)$. For a given (z, θ), define

$$\psi^{\pi}_{a,n}(V_i; z, \theta) \equiv \left(S_i - m^{NC}_S(X_i, \theta)\right) \cdot \Xi_S(X_i, \theta) \cdot K\left(\frac{Z_i - z}{h_n}\right) - E\left[\left(S - m^{NC}_S(X, \theta)\right) \cdot \Xi_S(X, \theta) \cdot K\left(\frac{Z - z}{h_n}\right)\right],$$
$$\psi^{\pi}_{b,n}(U_i; z, \theta) \equiv \Xi_S(X_i, \theta)^2 \cdot K\left(\frac{Z_i - z}{h_n}\right) - E\left[\Xi_S(X, \theta)^2 \cdot K\left(\frac{Z - z}{h_n}\right)\right] \tag{S1.7}$$

Note that $E[\psi^{\pi}_{a,n}(V_i; z, \theta)] = 0$ and $E[\psi^{\pi}_{b,n}(U_i; z, \theta)] = 0 \; \forall \, (z, \theta)$. The expressions in (S1.5) can be expressed as linear representation results for $\widehat{R}^{\pi}_a(z, \theta)$ and $\widehat{R}^{\pi}_b(z, \theta)$, given by

$$\widehat{R}^{\pi}_a(z, \theta) = R^{\pi}_a(z, \theta) + \frac{1}{n \cdot h_n^{d_Z}} \cdot \sum_{i=1}^{n} \psi^{\pi}_{a,n}(V_i; z, \theta) + B^{\pi}_{a,n}(z, \theta), \quad \text{where} \quad \sup_{\substack{\theta \in \Theta \\ z \in \mathcal{Z}}} \left|B^{\pi}_{a,n}(z, \theta)\right| = O\left(h_n^M\right),$$
$$\widehat{R}^{\pi}_b(z, \theta) = R^{\pi}_b(z, \theta) + \frac{1}{n \cdot h_n^{d_Z}} \cdot \sum_{i=1}^{n} \psi^{\pi}_{b,n}(U_i; z, \theta) + B^{\pi}_{b,n}(z, \theta), \quad \text{where} \quad \sup_{\substack{\theta \in \Theta \\ z \in \mathcal{Z}}} \left|B^{\pi}_{b,n}(z, \theta)\right| = O\left(h_n^M\right) \tag{S1.8}$$

Note from our bandwidth convergence restrictions in Assumption E1 that,

$$\sup_{\substack{\theta \in \Theta \\ z \in \mathcal{Z}}} \left|B^{\pi}_{a,n}(z, \theta)\right| = O\left(h_n^M\right) = o\left(n^{-1/2}\right), \quad \sup_{\substack{\theta \in \Theta \\ z \in \mathcal{Z}}} \left|B^{\pi}_{b,n}(z, \theta)\right| = O\left(h_n^M\right) = o\left(n^{-1/2}\right). \tag{S1.9}$$

Recall from Assumption E2 that there exists a $C > 0$ such that $R^{\pi}_b(z, \theta) \geq C^2$ for all $(z, \theta) \in \mathcal{Z} \times \Theta$. Using this restriction, equations (S1.2) and (S1.3) and a second-order approximation yields

$$\begin{aligned}\widehat{\pi}(z, \theta) = \pi(z, \theta) &+ \frac{1}{R^{\pi}_b(z, \theta)} \cdot \left(\widehat{R}^{\pi}_a(z, \theta) - R^{\pi}_a(z, \theta)\right) - \frac{\pi(z, \theta)}{R^{\pi}_b(z, \theta)} \cdot \left(\widehat{R}^{\pi}_b(z, \theta) - R^{\pi}_b(z, \theta)\right) \\ &+ O_p\left(\left\|\left(\widehat{R}^{\pi}_a(z, \theta) - R^{\pi}_a(z, \theta) \, , \, \widehat{R}^{\pi}_b(z, \theta) - R^{\pi}_b(z, \theta)\right)\right\|^2\right)\end{aligned} \tag{S1.10}$$

From (S1.6), we have

$$\sup_{\substack{\theta \in \Theta \\ z \in \mathcal{Z}}} \left\|\left(\widehat{R}^{\pi}_a(z, \theta) - R^{\pi}_a(z, \theta) \, , \, \widehat{R}^{\pi}_b(z, \theta) - R^{\pi}_b(z, \theta)\right)\right\|^2 = O_p\left(\frac{1}{\left(h_n^{d_Z} \cdot n^{1/2}\right)^2}\right).$$

Using this result and plugging in the linear representations in (S1.8), the expression in (S1.10) yields,

$$\widehat{\pi}(z, \theta) = \pi(z, \theta) + \frac{1}{n \cdot h_n^{d_z}} \sum_{i=1}^{n} \psi^{\pi}_n(V_i; z, \theta) + \vartheta^{\pi}_n(z, \theta), \quad \text{where}$$
$$\sup_{\substack{\theta \in \Theta \\ z \in \mathcal{Z}}} \left|\vartheta^{\pi}_n(z, \theta)\right| = O\left(h_n^M\right) + O_p\left(\frac{1}{\left(h_n^{d_Z} \cdot n^{1/2}\right)^2}\right) = o_p\left(n^{-1/2}\right), \quad \text{and} \tag{S1.11}$$
$$\psi^{\pi}_n(V_i; z, \theta) \equiv \frac{1}{R^{\pi}_b(z, \theta)} \cdot \psi^{\pi}_{a,n}(V_i; z, \theta) - \frac{\pi(z, \theta)}{R^{\pi}_b(z, \theta)} \cdot \psi^{\pi}_{b,n}(U_i; z, \theta)$$

where $\psi^{\pi}_{a,n}(V_i; z, \theta)$ and $\psi^{\pi}_{b,n}(U_i; z, \theta)$ are the influence functions described in (S1.7). The result

$O\left(h_n^M\right) + O_p\left(\frac{1}{\left(h_n^{d_Z}\cdot n^{1/2}\right)^2}\right) = o_p\left(n^{-1/2}\right)$ follows from the bandwidth convergence restrictions in Assumption E1. Equation (S1.11) is a linear representation result for $\widehat{\pi}(z,\theta)$ and will be invoked repeatedly below. Note that, since $E[\psi_{a,n}^{\pi}(V_i;z,\theta)] = 0$ and $E[\psi_{b,n}^{\pi}(U_i;z,\theta)] = 0$ $\forall$ (z,θ), we have $E[\psi_n^{\pi}(V_i;z,\theta)] = 0$ for all (z,θ) such that $R_b^{\pi}(z,\theta) \neq 0$. In particular, we have $E[\psi_n^{\pi}(V_i;z,\theta)] = 0$ $\forall$ $(z,\theta) \in \mathcal{Z}\times\Theta$. From (S1.11) it also follows that,

$$\sup_{\substack{\theta\in\Theta\\ z\in\mathcal{Z}}} |\widehat{\pi}(z,\theta) - \pi(z,\theta)| = O_p\left(\frac{1}{h_n^{d_Z}\cdot n^{1/2}}\right) + o_p\left(n^{-1/2}\right) = o_p\left(n^{-1/4}\right). \tag{S1.12}$$

where the last result follows from the bandwidth convergence restrictions in Assumption E1. Next, we study the properties of $\frac{\partial\widehat{\pi}(z,\theta)}{\partial\theta_\ell} - \frac{\partial\pi(z,\theta)}{\partial\theta_\ell}$ for $\ell = 1,\ldots,d_\Theta$. From the expressions in (S1.2) and (S1.3), we have

$$\begin{aligned}\frac{\partial\pi(z,\theta)}{\partial\theta_\ell} &= \frac{1}{R_b^{\pi}(z,\theta)}\cdot\left(\frac{\partial R_a^{\pi}(z,\theta)}{\partial\theta_\ell} - \pi(z,\theta)\cdot\frac{\partial R_b^{\pi}(z,\theta)}{\partial\theta_\ell}\right)\\ \frac{\partial\widehat{\pi}(z,\theta)}{\partial\theta_\ell} &= \frac{1}{\widehat{R}_b^{\pi}(z,\theta)}\cdot\left(\frac{\partial\widehat{R}_a^{\pi}(z,\theta)}{\partial\theta_\ell} - \widehat{\pi}(z,\theta)\cdot\frac{\partial\widehat{R}_b^{\pi}(z,\theta)}{\partial\theta_\ell}\right)\end{aligned} \tag{S1.13}$$

Note that, using the functional definitions in (S1.1),

$$\begin{aligned}\frac{\partial R_a^{\pi}(z,\theta)}{\partial\theta_\ell} &= E\left[\frac{\partial\Xi_S(X,\theta)}{\partial\theta_\ell}\cdot(S - m_S^{NC}(X,\theta))\,\Big|Z=z\right]\cdot f_Z(z) - E\left[\frac{\partial m_S^{NC}(X,\theta)}{\partial\theta_\ell}\cdot\Xi_S(X,\theta)\,\Big|Z=z\right]\cdot f_Z(z)\\ &\equiv \left(\mu_{III,\theta_\ell}^{\Xi_S}(z,\theta) - \mu_{IV,\theta_\ell}^{\Xi_S}(z,\theta)\right)\cdot f_Z(z)\\ \frac{\partial R_b^{\pi}(z,\theta)}{\partial\theta_\ell} &= 2\cdot E\left[\frac{\partial\Xi_S(X,\theta)}{\partial\theta_\ell}\cdot\Xi_S(X,\theta)\,\Big|Z=z\right]\cdot f_Z(z)\\ &\equiv 2\cdot\mu_{V,\theta_\ell}^{\Xi_S}(z,\theta)\cdot f_Z(z).\end{aligned}$$

We have already characterized the asymptotic properties of $\widehat{R}_b^{\pi}(z,\theta)$ and $\widehat{\pi}(z,\theta)$. We now proceed to study the asymptotic properties of $\frac{\partial\widehat{R}_a^{\pi}(z,\theta)}{\partial\theta_\ell}$ and $\frac{\partial\widehat{R}_b^{\pi}(z,\theta)}{\partial\theta_\ell}$. We have

$$\begin{aligned}\frac{\partial\widehat{R}_a^{\pi}(z,\theta)}{\partial\theta_\ell} &= \frac{1}{n\cdot h_n^{d_Z}}\sum_{i=1}^{n}\left(\frac{\partial\Xi_S(X_i,\theta)}{\partial\theta_\ell}\cdot\left(S_i - m_S^{NC}(X_i,\theta)\right) - \frac{\partial m_S^{NC}(X_i,\theta)}{\partial\theta_\ell}\cdot\Xi_S(X_i,\theta)\right)\cdot K\left(\frac{Z_i - z}{h_n}\right)\\ \frac{\partial\widehat{R}_b^{\pi}(z,\theta)}{\partial\theta_\ell} &= \frac{2}{n\cdot h_n^{d_Z}}\sum_{i=1}^{n}\frac{\partial\Xi_S(X_i,\theta)}{\partial\theta_\ell}\cdot\Xi_S(X_i,\theta)\cdot K\left(\frac{Z_i - z}{h_n}\right)\end{aligned}$$

Let

$$\psi_{a,n}^{\nabla\theta_\ell\pi}(V_i;z,\theta) \equiv \left(\frac{\partial \Xi_S(X_i,\theta)}{\partial\theta_\ell}\cdot\left(S_i - m_S^{NC}(X_i,\theta)\right) - \frac{\partial m_S^{NC}(X_i,\theta)}{\partial\theta_\ell}\cdot\Xi_S(X_i,\theta)\right)\cdot K\left(\frac{Z_i-z}{h_n}\right)$$
$$- E\left[\left(\frac{\partial \Xi_S(X,\theta)}{\partial\theta_\ell}\cdot\left(S - m_S^{NC}(X,\theta)\right) - \frac{\partial m_S^{NC}(X,\theta)}{\partial\theta_\ell}\cdot\Xi_S(X,\theta)\right)\cdot K\left(\frac{Z-z}{h_n}\right)\right]$$
$$\psi_{b,n}^{\nabla\theta_\ell\pi}(U_i;z,\theta) \equiv 2\cdot\left(\frac{\partial \Xi_S(X_i,\theta)}{\partial\theta_\ell}\cdot\Xi_S(X_i,\theta)\cdot K\left(\frac{Z_i-z}{h_n}\right) - E\left[\frac{\partial \Xi_S(X,\theta)}{\partial\theta_\ell}\cdot\Xi_S(X,\theta)\cdot K\left(\frac{Z-z}{h_n}\right)\right]\right), \tag{S1.14}$$

Note that $E[\psi_{a,n}^{\nabla\theta_\ell\pi}(V;z,\theta)] = 0$ and $E[\psi_{b,n}^{\nabla\theta_\ell\pi}(U;z,\theta)] = 0$ for any (z,θ). Next, define the bias terms,

$$B_{a,n}^{\nabla\theta_\ell\pi}(z,\theta) \equiv$$
$$E\left[\frac{1}{h_n^{d_Z}}\cdot\left(\frac{\partial \Xi_S(X,\theta)}{\partial\theta_\ell}\cdot\left(S - m_S^{NC}(X,\theta)\right) - \frac{\partial m_S^{NC}(X,\theta)}{\partial\theta_\ell}\cdot\Xi_S(X,\theta)\right)\cdot K\left(\frac{Z-z}{h_n}\right)\right] - \left(\mu_{III,\theta_\ell}^{\Xi_S}(z,\theta) - \mu_{IV,\theta_\ell}^{\Xi_S}(z,\theta)\right)\cdot f_Z(z)$$
$$= E\left[\frac{1}{h_n^{d_Z}}\cdot\left(\frac{\partial \Xi_S(X,\theta)}{\partial\theta_\ell}\cdot\left(S - m_S^{NC}(X,\theta)\right) - \frac{\partial m_S^{NC}(X,\theta)}{\partial\theta_\ell}\cdot\Xi_S(X,\theta)\right)\cdot K\left(\frac{Z-z}{h_n}\right)\right] - \frac{\partial R_a^\pi(z,\theta)}{\partial\theta_\ell},$$

$$B_{b,n}^{\nabla\theta_\ell\pi}(z,\theta) \equiv$$
$$2\cdot E\left[\frac{1}{h_n^{d_Z}}\cdot\frac{\partial \Xi_S(X,\theta)}{\partial\theta_\ell}\cdot\Xi_S(X,\theta)\cdot K\left(\frac{Z-z}{h_n}\right)\right] - 2\cdot\mu_{V,\theta_\ell}^{\Xi_S}(z,\theta)\cdot f_Z(z)$$
$$= 2\cdot E\left[\frac{1}{h_n^{d_Z}}\cdot\frac{\partial \Xi_S(X,\theta)}{\partial\theta_\ell}\cdot\Xi_S(X,\theta)\cdot K\left(\frac{Z-z}{h_n}\right)\right] - \frac{\partial R_b^\pi(z,\theta)}{\partial\theta_\ell}$$

An M^{th}–order approximation combined with the smoothness restrictions in Assumption E2 and the higher-order properties of the kernel described in Assumption E1 yield,

$$\sup_{\substack{\theta\in\Theta\\ z\in\mathcal{Z}}}\left|B_{a,n}^{\nabla\theta_\ell\pi}(z,\theta)\right| = O\left(h_n^M\right) = o\left(n^{-1/2}\right), \quad \sup_{\substack{\theta\in\Theta\\ z\in\mathcal{Z}}}\left|B_{b,n}^{\nabla\theta_\ell\pi}(z,\theta)\right| = O\left(h_n^M\right) = o\left(n^{-1/2}\right), \tag{S1.15}$$

where (as before), the result $O\left(h_n^M\right) = o\left(n^{-1/2}\right)$ follows from the bandwidth convergence restrictions in Assumption E1. From here, repeating the arguments that led to the linear representations in (S1.8), our restrictions now yield the following linear representations for $\frac{\partial\widehat{R}_a^\pi(z,\theta)}{\partial\theta_\ell}$ and $\frac{\partial\widehat{R}_b^\pi(z,\theta)}{\partial\theta_\ell}$,

$$\frac{\partial\widehat{R}_a^\pi(z,\theta)}{\partial\theta_\ell} = \frac{\partial R_a^\pi(z,\theta)}{\partial\theta_\ell} + \frac{1}{n\cdot h_n^{d_Z}}\cdot\sum_{i=1}^n\psi_{a,n}^{\nabla\theta_\ell\pi}(V_i;z,\theta) + B_{a,n}^{\nabla\theta_\ell\pi}(z,\theta),\ \text{ where } \sup_{\substack{\theta\in\Theta\\ z\in\mathcal{Z}}}\left|B_{a,n}^{\nabla\theta_\ell\pi}(z,\theta)\right| = o_p\left(n^{-1/2}\right),$$
$$\frac{\partial\widehat{R}_b^\pi(z,\theta)}{\partial\theta_\ell} = \frac{\partial R_b^\pi(z,\theta)}{\partial\theta_\ell} + \frac{1}{n\cdot h_n^{d_Z}}\cdot\sum_{i=1}^n\psi_{b,n}^{\nabla\theta_\ell\pi}(U_i;z,\theta) + B_{b,n}^{\nabla\theta_\ell\pi}(z,\theta),\ \text{ where } \sup_{\substack{\theta\in\Theta\\ z\in\mathcal{Z}}}\left|B_{b,n}^{\nabla\theta_\ell\pi}(z,\theta)\right| = o_p\left(n^{-1/2}\right) \tag{S1.16}$$

From the above result, we also have

$$\begin{aligned}
\sup_{\substack{\theta\in\Theta \\ z\in\mathcal{Z}}} \left| \frac{\partial \widehat{R}_a^{\pi}(z,\theta)}{\partial\theta_\ell} - \frac{\partial R_a^{\pi}(z,\theta)}{\partial\theta_\ell} \right| &= O_p\left(\frac{1}{h_n^{d_Z}\cdot n^{1/2}}\right) + o_p\left(n^{-1/2}\right) = o_p\left(n^{-1/4}\right), \\
\sup_{\substack{\theta\in\Theta \\ z\in\mathcal{Z}}} \left| \frac{\partial \widehat{R}_b^{\pi}(z,\theta)}{\partial\theta_\ell} - \frac{\partial R_b^{\pi}(z,\theta)}{\partial\theta_\ell} \right| &= O_p\left(\frac{1}{h_n^{d_Z}\cdot n^{1/2}}\right) + o_p\left(n^{-1/2}\right) = o_p\left(n^{-1/4}\right)
\end{aligned} \tag{S1.17}$$

We are now ready to describe a linear representation result for $\frac{\partial\widehat{\pi}(z,\theta)}{\partial\theta_\ell} - \frac{\partial\pi(z,\theta)}{\partial\theta_\ell}$. To this end, we go back (S1.13). Note first that,

$$\begin{aligned}
&\frac{\partial \widehat{R}_a^{\pi}(z,\theta)}{\partial\theta_\ell} - \widehat{\pi}(z,\theta)\cdot\frac{\partial \widehat{R}_b^{\pi}(z,\theta)}{\partial\theta_\ell} = \frac{R_a^{\pi}(z,\theta)}{\partial\theta_\ell} - \pi(z,\theta)\cdot\frac{\partial R_b^{\pi}(z,\theta)}{\partial\theta_\ell} + \left(\frac{\partial \widehat{R}_a^{\pi}(z,\theta)}{\partial\theta_\ell} - \frac{\partial R_a^{\pi}(z,\theta)}{\partial\theta_\ell}\right) \\
&-\frac{\partial R_b^{\pi}(z,\theta)}{\partial\theta_\ell}\cdot\left(\widehat{\pi}(z,\theta)-\pi(z,\theta)\right) - \pi(z,\theta)\cdot\left(\frac{\widehat{R}_b^{\pi}(z,\theta)}{\partial\theta_\ell} - \frac{\partial R_b^{\pi}(z,\theta)}{\partial\theta_\ell}\right) - \xi_{I,n}^{\nabla\theta_\ell\pi}(z,\theta), \\
&\text{where} \quad \xi_{I,n}^{\nabla\theta_\ell\pi}(z,\theta) \equiv -\left(\widehat{\pi}(z,\theta)-\pi(z,\theta)\right)\cdot\left(\frac{\partial \widehat{R}_b^{\pi}(z,\theta)}{\partial\theta_\ell} - \frac{\partial R_b^{\pi}(z,\theta)}{\partial\theta_\ell}\right)
\end{aligned} \tag{S1.18}$$

From (S1.12) and (S1.17),

$$\sup_{\substack{\theta\in\Theta \\ z\in\mathcal{Z}}} \left|\xi_{I,n}^{\nabla\theta_\ell\pi}(z,\theta)\right| = o_p\left(n^{-1/2}\right). \tag{S1.19}$$

Next, we have

$$\begin{aligned}
\frac{1}{\widehat{R}_b^{\pi}(z,\theta)} &= \frac{1}{R_b^{\pi}(z,\theta)} - \frac{1}{R_b^{\pi}(z,\theta)^2}\cdot\left(\widehat{R}_b^{\pi}(z,\theta) - R_b^{\pi}(z,\theta)\right) + \xi_{II,n}^{\nabla\theta_\ell\pi}(z,\theta), \\
&\text{where} \quad \xi_{II,n}^{\nabla\theta_\ell\pi}(z,\theta) = O_p\left(\left|\widehat{R}_b^{\pi}(z,\theta) - R_b^{\pi}(z,\theta)\right|^2\right) \;\forall\; (z,\theta)\in\mathcal{Z}\times\Theta.
\end{aligned} \tag{S1.20}$$

From (S1.6),

$$\sup_{\substack{\theta\in\Theta \\ z\in\mathcal{Z}}} \left|\xi_{II,n}^{\nabla\theta_\ell\pi}(z,\theta)\right| = o_p\left(n^{-1/2}\right). \tag{S1.21}$$

From here, going back to (S1.13), we obtain,

$$\frac{\partial\widehat{\pi}(z,\theta)}{\partial\theta_\ell} = \frac{\partial\pi(z,\theta)}{\partial\theta_\ell} + \frac{1}{R_b^\pi(z,\theta)} \times \Bigg\{\left(\frac{\partial\widehat{R}_a^\pi(z,\theta)}{\partial\theta_\ell} - \frac{\partial R_a^\pi(z,\theta)}{\partial\theta_\ell}\right) - \frac{\partial R_b^\pi(z,\theta)}{\partial\theta_\ell}\cdot(\widehat{\pi}(z,\theta)-\pi(z,\theta))$$
$$-\pi(z,\theta)\cdot\left(\frac{\partial\widehat{R}_b^\pi(z,\theta)}{\partial\theta_\ell} - \frac{\partial R_b^\pi(z,\theta)}{\partial\theta_\ell}\right) - \frac{1}{R_b^\pi(z,\theta)}\cdot\left(\frac{\partial R_a^\pi(z,\theta)}{\partial\theta_\ell} - \pi(z,\theta)\cdot\frac{\partial R_b^\pi(z,\theta)}{\partial\theta_\ell}\right)\cdot\left(\widehat{R}_b^\pi(z,\theta) - R_b^\pi(z,\theta)\right)\Bigg\}$$
$$+\xi_{III,n}^{\nabla\theta_\ell\pi}(z,\theta), \quad \text{where}$$
$$\xi_{III,n}^{\nabla\theta_\ell\pi}(z,\theta) \equiv \left\{-\frac{1}{R_b^\pi(z,\theta)^2}\cdot\left(\widehat{R}_b^\pi(z,\theta)-R_b^\pi(z,\theta)\right) + \xi_{II,n}^{\nabla\theta_\ell\pi}(z,\theta)\right\}$$
$$\times\Bigg\{\left(\frac{\partial\widehat{R}_a^\pi(z,\theta)}{\partial\theta_\ell} - \frac{\partial R_a^\pi(z,\theta)}{\partial\theta_\ell}\right) - \frac{\partial R_b^\pi(z,\theta)}{\partial\theta_\ell}\cdot(\widehat{\pi}(z,\theta)-\pi(z,\theta))$$
$$-\pi(z,\theta)\cdot\left(\frac{\partial\widehat{R}_b^\pi(z,\theta)}{\partial\theta_\ell} - \frac{\partial R_b^\pi(z,\theta)}{\partial\theta_\ell}\right) - \frac{1}{R_b^\pi(z,\theta)}\cdot\left(\frac{\partial R_a^\pi(z,\theta)}{\partial\theta_\ell} - \pi(z,\theta)\cdot\frac{\partial R_b^\pi(z,\theta)}{\partial\theta_\ell}\right)\cdot\left(\widehat{R}_b^\pi(z,\theta) - R_b^\pi(z,\theta)\right)\Bigg\}$$
$$+\xi_{II,n}^{\nabla\theta_\ell\pi}(z,\theta)\cdot\left(\frac{\partial R_a^\pi(z,\theta)}{\partial\theta_\ell} - \pi(z,\theta)\cdot\frac{\partial R_b^\pi(z,\theta)}{\partial\theta_\ell}\right) \tag{S1.22}$$

Using the results in equations (S1.6), (S1.12), (S1.17), (S1.19) and (S1.21), the boundedness properties of the various functionals of Z described in Assumption E2 yield,

$$\sup_{\substack{\theta\in\Theta\\ z\in\mathcal{Z}}}\left|\xi_{III,n}^{\nabla\theta_\ell\pi}(z,\theta)\right| = o_p\left(n^{-1/2}\right). \tag{S1.23}$$

The last step is to plug in the linear representation results in (S1.8), (S1.11) and (S1.16) into (S1.22). Let

$$\xi_{IV,n}^{\nabla\theta_\ell\pi}(z,\theta) \equiv \frac{1}{R_b^\pi(z,\theta)}\times\Bigg\{B_{a,n}^{\nabla\theta_\ell\pi}(z,\theta) - \frac{\partial R_b^\pi(z,\theta)}{\partial\theta_\ell}\cdot\vartheta_n^\pi(z,\theta) - \pi(z,\theta)\cdot B_{b,n}^{\nabla\theta_\ell\pi}(z,\theta)$$
$$-\frac{1}{R_b^\pi(z,\theta)}\cdot\left(\frac{\partial R_a^\pi(z,\theta)}{\partial\theta_\ell} - \pi(z,\theta)\cdot\frac{\partial R_b^\pi(z,\theta)}{\partial\theta_\ell}\right)\cdot B_{b.n}^\pi(z,\theta)\Bigg\}$$

where $B_{b,n}^\pi(z,\theta)$ has the properties stated in (S1.9), $B_{a,n}^{\nabla\theta_\ell\pi}(z,\theta)$ and $B_{b,n}^{\nabla\theta_\ell\pi}(z,\theta)$ have the properties stated in (S1.15) and $\vartheta_n^\pi(z,\theta)$ is as described in (S1.11). From those equations and the boundedness properties of the functionals described in Assumption E2, we obtain,

$$\sup_{\substack{\theta\in\Theta\\ z\in\mathcal{Z}}}\left|\xi_{IV,n}^{\nabla\theta_\ell\pi}(z,\theta)\right| = o_p\left(n^{-1/2}\right). \tag{S1.24}$$

Let

$$\psi_n^{\nabla\theta_\ell\pi}(V_i;z,\theta) \equiv \frac{1}{R_b^\pi(z,\theta)} \times \Bigg\{ \psi_{a,n}^{\nabla\theta_\ell\pi}(V_i;z,\theta) - \frac{\partial R_b^\pi(z,\theta)}{\partial\theta_\ell} \cdot \psi_n^\pi(V_i;z,\theta) \\ - \pi(z,\theta) \cdot \psi_{b,n}^{\nabla\theta_\ell\pi}(V_i;z,\theta) - \frac{1}{R_b^\pi(z,\theta)} \cdot \left(\frac{\partial R_a^\pi(z,\theta)}{\partial\theta_\ell} - \pi(z,\theta) \cdot \frac{\partial R_b^\pi(z,\theta)}{\partial\theta_\ell} \right) \cdot \psi_{b,n}^\pi(U_i;z,\theta) \Bigg\}, \tag{S1.25}$$

where $\psi_{b,n}^\pi(U_i;z,\theta)$ is as defined in (S1.7), $\psi_n^\pi(V_i;z,\theta)$ is as defined in (S1.11), and $\psi_{a,n}^{\nabla\theta_\ell\pi}(V_i;z,\theta)$ and $\psi_{a,n}^{\nabla\theta_\ell\pi}(V_i;z,\theta)$ are as defined in (S1.14). Note from the properties of those functionals that $E[\psi_n^{\nabla\theta_\ell\pi}(V_i;z,\theta)] = 0 \;\forall\; (z,\theta) \in \mathcal{Z}\times\Theta$. Finally, let $\vartheta_n^{\nabla\theta_\ell\pi}(z,\theta) \equiv \xi_{III,n}^{\nabla\theta_\ell\pi}(z,\theta) + \xi_{IV,n}^{\nabla\theta_\ell\pi}(z,\theta)$ and note from (S1.23) and (S1.24) that,

$$\sup_{\substack{\theta\in\Theta \\ z\in\mathcal{Z}}} \left| \vartheta_n^{\nabla\theta_\ell\pi}(z,\theta) \right| = o_p\left(n^{-1/2}\right). \tag{S1.26}$$

From (S1.22) and our previous linear representation results, we have,

$$\frac{\partial\widehat{\pi}(z,\theta)}{\partial\theta_\ell} = \frac{\partial\pi(z,\theta)}{\partial\theta_\ell} + \frac{1}{n \cdot h_n^{d_Z}} \sum_{i=1}^n \psi_n^{\nabla\theta_\ell\pi}(V_i;z,\theta) + \vartheta_n^{\nabla\theta_\ell\pi}(z,\theta), \\ \text{where} \quad \sup_{\substack{\theta\in\Theta \\ z\in\mathcal{Z}}} \left| \vartheta_n^{\nabla\theta_\ell\pi}(z,\theta) \right| = o_p\left(n^{-1/2}\right). \tag{S1.27}$$

From (S1.27) it also follows that,

$$\sup_{\substack{\theta\in\Theta \\ z\in\mathcal{Z}}} \left| \frac{\partial\widehat{\pi}(z,\theta)}{\partial\theta_\ell} - \frac{\partial\pi(z,\theta)}{\partial\theta_\ell} \right| = O_p\left(\frac{1}{h_n^{d_Z} \cdot n^{1/2}} \right) + o_p\left(n^{-1/2}\right) = o_p\left(n^{-1/4}\right). \tag{S1.28}$$

Next let us study $\frac{\partial\widehat{\pi}(z,\theta)}{\partial\theta_j\partial\theta_\ell}$. To establish the asymptotic properties of our estimator, we only need to focus on the probability-limit of $\frac{\partial\widehat{\pi}(z,\theta)}{\partial\theta_j\partial\theta_\ell} - \frac{\partial\pi(z,\theta)}{\partial\theta_j\partial\theta_\ell}$ and not its asymptotic distribution. Take any $\ell, j = 1,\ldots,d_\theta$. We have,

$$\frac{\partial^2\pi(z,\theta)}{\partial\theta_j\partial\theta_\ell} = -\frac{1}{R_b^\pi(z,\theta)^2} \cdot \frac{\partial R_b^\pi(z,\theta)}{\partial\theta_j} \cdot \left(\frac{\partial R_a^\pi(z,\theta)}{\partial\theta_\ell} - \pi(z,\theta) \cdot \frac{\partial R_b^\pi(z,\theta)}{\partial\theta_\ell} \right) \\ + \frac{1}{R_b^\pi(z,\theta)} \cdot \left(\frac{\partial^2 R_a^\pi(z,\theta)}{\partial\theta_j\partial\theta_\ell} - \frac{\partial\pi(z,\theta)}{\partial\theta_j} \cdot \frac{R_b^\pi(z,\theta)}{\partial\theta_\ell} - \pi(z,\theta) \cdot \frac{\partial^2 R_b^\pi(z,\theta)}{\partial\theta_j\partial\theta_\ell} \right) \tag{S1.29}$$

Take any $\ell, j = 1,\ldots,d_\theta$. We have,

$$\frac{\partial^2\widehat{\pi}(z,\theta)}{\partial\theta_j\partial\theta_\ell} = -\frac{1}{\widehat{R}_b^\pi(z,\theta)^2} \cdot \frac{\partial\widehat{R}_b^\pi(z,\theta)}{\partial\theta_j} \cdot \left(\frac{\partial\widehat{R}_a^\pi(z,\theta)}{\partial\theta_\ell} - \widehat{\pi}(z,\theta) \cdot \frac{\partial\widehat{R}_b^\pi(z,\theta)}{\partial\theta_\ell} \right) \\ + \frac{1}{\widehat{R}_b^\pi(z,\theta)} \cdot \left(\frac{\partial^2\widehat{R}_a^\pi(z,\theta)}{\partial\theta_j\partial\theta_\ell} - \frac{\partial\widehat{\pi}(z,\theta)}{\partial\theta_j} \cdot \frac{\widehat{R}_b^\pi(z,\theta)}{\partial\theta_\ell} - \widehat{\pi}(z,\theta) \cdot \frac{\partial^2\widehat{R}_b^\pi(z,\theta)}{\partial\theta_j\partial\theta_\ell} \right) \tag{S1.30}$$

From our previous results in equations (S1.6)-(S1.28), we have already established that, under our assumptions,

$$\sup_{\substack{\theta\in\Theta\\ z\in\mathcal{Z}}}\left|\frac{1}{\widehat{R}_b^{\pi}(z,\theta)^2}-\frac{1}{R_b^{\pi}(z,\theta)^2}\right|=o_p(1),\quad \sup_{\substack{\theta\in\Theta\\ z\in\mathcal{Z}}}\left|\frac{\partial\widehat{R}_b^{\pi}(z,\theta)}{\partial\theta_\ell}-\frac{\partial R_b^{\pi}(z,\theta)}{\partial\theta_\ell}\right|=o_p(1)\ \ (\text{for } \ell=1,\dots,d_\theta)$$
$$\sup_{\substack{\theta\in\Theta\\ z\in\mathcal{Z}}}\left|\frac{\partial\widehat{R}_a^{\pi}(z,\theta)}{\partial\theta_\ell}-\frac{\partial R_a^{\pi}(z,\theta)}{\partial\theta_\ell}\right|=o_p(1)\ \ (\text{for } \ell=1,\dots,d_\theta),\quad \sup_{\substack{\theta\in\Theta\\ z\in\mathcal{Z}}}|\widehat{\pi}(z,\theta)-\pi(z,\theta)|=o_p(1), \tag{S1.31}$$
$$\sup_{\substack{\theta\in\Theta\\ z\in\mathcal{Z}}}\left|\frac{\partial\widehat{\pi}(z,\theta)}{\partial\theta_\ell}-\frac{\partial \pi(z,\theta)}{\partial\theta_\ell}\right|=o_p(1)\ \ (\text{for } \ell=1,\dots,d_\theta).$$

Thus, comparing (S1.29) and (S1.30), in order to obtain $\sup_{\substack{\theta\in\Theta\\ z\in\mathcal{Z}}}\left|\frac{\partial^2\widehat{\pi}(z,\theta)}{\partial\theta_j\partial\theta_\ell}-\frac{\partial^2\pi(z,\theta)}{\partial\theta_j\partial\theta_\ell}\right|=o_p(1)$ for $j,\ell=1,\dots,d_\theta$, we only need to show that $\sup_{\substack{\theta\in\Theta\\ z\in\mathcal{Z}}}\left|\frac{\partial^2\widehat{R}_a^{\pi}(z,\theta)}{\partial\theta_j\partial\theta_\ell}-\frac{\partial^2 R_a^{\pi}(z,\theta)}{\partial\theta_j\partial\theta_\ell}\right|=o_p(1)$ and $\sup_{\substack{\theta\in\Theta\\ z\in\mathcal{Z}}}\left|\frac{\partial^2\widehat{R}_b^{\pi}(z,\theta)}{\partial\theta_j\partial\theta_\ell}-\frac{\partial^2 R_b^{\pi}(z,\theta)}{\partial\theta_j\partial\theta_\ell}\right|=o_p(1)$. Take the functionals we defined in Assumption E2.

$$\begin{aligned}
\delta_{I,\theta_\ell,\theta_j}(z,\theta)&\equiv E\left[\frac{\partial^2\Xi_S(X,\theta)}{\partial\theta_j\partial\theta_\ell}\cdot\left(S-m_S^{NC}(X,\theta)\right)\,\middle|Z=z\right]\ (\text{for } j,\ell=1,\dots,d_\theta)\\
\delta_{II,\theta_\ell,\theta_j}(z,\theta)&\equiv E\left[\frac{\partial\Xi_S(X,\theta)}{\partial\theta_\ell}\cdot\frac{\partial m_S^{NC}(X,\theta)}{\partial\theta_j}\,\middle|Z=z\right]\ (\text{for } j,\ell=1,\dots,d_\theta)\\
\delta_{III,\theta_\ell,\theta_j}(z,\theta)&\equiv E\left[\frac{\partial^2 m_S^{NC}(X,\theta)}{\partial\theta_j\partial\theta_\ell}\cdot\Xi_S(X,\theta)\,\middle|Z=z\right]\ (\text{for } j,\ell=1,\dots,d_\theta)\\
\delta_{IV,\theta_\ell,\theta_j}(z,\theta)&\equiv E\left[\frac{\partial^2\Xi_S(X,\theta)}{\partial\theta_j\partial\theta_\ell}\cdot\Xi_S(X,\theta)\,\middle|Z=z\right]\ (\text{for } j,\ell=1,\dots,d_\theta)\\
\delta_{V,\theta_\ell,\theta_j}(z,\theta)&\equiv E\left[\frac{\partial\Xi_S(X,\theta)}{\partial\theta_\ell}\cdot\frac{\partial\Xi_S(X,\theta)}{\partial\theta_j}\,\middle|Z=z\right]\ (\text{for } j,\ell=1,\dots,d_\theta)
\end{aligned}\tag{S1.32}$$

We have,

$$\begin{aligned}
\frac{\partial^2 R_a^{\pi}(z,\theta)}{\partial\theta_j\partial\theta_\ell}&=\left(\delta_{I,\theta_\ell,\theta_j}(z,\theta)-\delta_{II,\theta_\ell,\theta_j}(z,\theta)-\delta_{II,\theta_j,\theta_\ell}(z,\theta)-\delta_{III,\theta_\ell,\theta_j}(z,\theta)\right)\cdot f_Z(z),\\
\frac{\partial^2 R_b^{\pi}(z,\theta)}{\partial\theta_j\partial\theta_\ell}&=2\cdot\left(\delta_{IV,\theta_\ell,\theta_j}(z,\theta)+\delta_{V,\theta_\ell,\theta_j}(z,\theta)\right)\cdot f_Z(z).
\end{aligned}$$

And,

$$\begin{aligned}\frac{\partial^2 \widehat{R}_a^{\pi}(z,\theta)}{\partial\theta_j\partial\theta_\ell} =& \frac{1}{n\cdot h_n^{d_Z}}\sum_{i=1}^{n}\frac{\partial^2 \Xi_S(X_i,\theta)}{\partial\theta_j\partial\theta_\ell}\cdot\left(S_i - m_S^{NC}(X_i,\theta)\right)\cdot K\left(\frac{Z_i-z}{h_n}\right)\\ &-\frac{1}{n\cdot h_n^{d_Z}}\sum_{i=1}^{n}\frac{\partial \Xi_S(X_i,\theta)}{\partial\theta_\ell}\cdot\frac{\partial m_S^{NC}(X_i,\theta)}{\partial\theta_j}\cdot K\left(\frac{Z_i-z}{h_n}\right)\\ &-\frac{1}{n\cdot h_n^{d_Z}}\sum_{i=1}^{n}\frac{\partial^2 m_S^{NC}(X_i,\theta)}{\partial\theta_j\partial\theta_\ell}\cdot\Xi_S(X_i,\theta)\cdot K\left(\frac{Z_i-z}{h_n}\right)\\ &-\frac{1}{n\cdot h_n^{d_Z}}\sum_{i=1}^{n}\frac{\partial m_S^{NC}(X_i,\theta)}{\partial\theta_\ell}\cdot\frac{\partial \Xi_S(X_i,\theta)}{\partial\theta_j}\cdot K\left(\frac{Z_i-z}{h_n}\right),\\ \frac{\partial^2 \widehat{R}_b^{\pi}(z,\theta)}{\partial\theta_j\partial\theta_\ell} =& \frac{2}{n\cdot h_n^{d_Z}}\sum_{i=1}^{n}\frac{\partial^2 \Xi_S(X_i,\theta)}{\partial\theta_j\partial\theta_\ell}\cdot\Xi_S(X_i,\theta)\cdot K\left(\frac{Z_i-z}{h_n}\right)\\ &+\frac{2}{n\cdot h_n^{d_Z}}\sum_{i=1}^{n}\frac{\partial \Xi_S(X_i,\theta)}{\partial\theta_\ell}\cdot\frac{\partial \Xi_S(X_i,\theta)}{\partial\theta_j}\cdot K\left(\frac{Z_i-z}{h_n}\right).\end{aligned} \tag{S1.33}$$

From here, we can express

$$\begin{aligned}&\frac{\partial^2 \widehat{R}_a^{\pi}(z,\theta)}{\partial\theta_j\partial\theta_\ell} = \frac{\partial^2 R_a^{\pi}(z,\theta)}{\partial\theta_j\partial\theta_\ell}\\ &+\frac{1}{n\cdot h_n^{d_Z}}\sum_{i=1}^{n}\left(\frac{\partial^2 \Xi_S(X_i,\theta)}{\partial\theta_j\partial\theta_\ell}\cdot\left(S_i-m_S^{NC}(X_i,\theta)\right)\cdot K\left(\frac{Z_i-z}{h_n}\right)-E\left[\frac{\partial^2 \Xi_S(X,\theta)}{\partial\theta_j\partial\theta_\ell}\cdot\left(S-m_S^{NC}(X,\theta)\right)\cdot K\left(\frac{Z-z}{h_n}\right)\right]\right)\\ &-\frac{1}{n\cdot h_n^{d_Z}}\sum_{i=1}^{n}\left(\frac{\partial \Xi_S(X_i,\theta)}{\partial\theta_\ell}\cdot\frac{\partial m_S^{NC}(X_i,\theta)}{\partial\theta_j}\cdot K\left(\frac{Z_i-z}{h_n}\right)-E\left[\frac{\partial \Xi_S(X,\theta)}{\partial\theta_\ell}\cdot\frac{\partial m_S^{NC}(X,\theta)}{\partial\theta_j}\cdot K\left(\frac{Z-z}{h_n}\right)\right]\right)\\ &-\frac{1}{n\cdot h_n^{d_Z}}\sum_{i=1}^{n}\left(\frac{\partial^2 m_S^{NC}(X_i,\theta)}{\partial\theta_j\partial\theta_\ell}\cdot\Xi_S(X_i,\theta)\cdot K\left(\frac{Z_i-z}{h_n}\right)-E\left[\frac{\partial^2 m_S^{NC}(X,\theta)}{\partial\theta_j\partial\theta_\ell}\cdot\Xi_S(X,\theta)\cdot K\left(\frac{Z-z}{h_n}\right)\right]\right)\\ &-\frac{1}{n\cdot h_n^{d_Z}}\sum_{i=1}^{n}\left(\frac{\partial m_S^{NC}(X_i,\theta)}{\partial\theta_\ell}\cdot\frac{\partial \Xi_S(X_i,\theta)}{\partial\theta_j}\cdot K\left(\frac{Z_i-z}{h_n}\right)-E\left[\frac{\partial m_S^{NC}(X,\theta)}{\partial\theta_\ell}\cdot\frac{\partial \Xi_S(X,\theta)}{\partial\theta_j}\cdot K\left(\frac{Z-z}{h_n}\right)\right]\right)\\ &+\left(E\left[\frac{1}{h_n^{d_Z}}\cdot\frac{\partial^2 \Xi_S(X,\theta)}{\partial\theta_j\partial\theta_\ell}\cdot\left(S-m_S^{NC}(X,\theta)\right)\cdot K\left(\frac{Z-z}{h_n}\right)\right]-\delta_{I,\theta_\ell,\theta_j}(z,\theta)\cdot f_Z(z)\right)\\ &-\left(E\left[\frac{1}{h_n^{d_Z}}\cdot\frac{\partial \Xi_S(X,\theta)}{\partial\theta_\ell}\cdot\frac{\partial m_S^{NC}(X,\theta)}{\partial\theta_j}\cdot K\left(\frac{Z-z}{h_n}\right)\right]-\delta_{II,\theta_\ell,\theta_j}(z,\theta)\cdot f_Z(z)\right)\\ &-\left(E\left[\frac{1}{h_n^{d_Z}}\cdot\frac{\partial^2 m_S^{NC}(X,\theta)}{\partial\theta_j\partial\theta_\ell}\cdot\Xi_S(X,\theta)\cdot K\left(\frac{Z-z}{h_n}\right)\right]-\delta_{III,\theta_\ell,\theta_j}(z,\theta)\cdot f_Z(z)\right)\\ &-\left(E\left[\frac{1}{h_n^{d_Z}}\cdot\frac{\partial m_S^{NC}(X,\theta)}{\partial\theta_\ell}\cdot\frac{\partial \Xi_S(X,\theta)}{\partial\theta_j}\cdot K\left(\frac{Z-z}{h_n}\right)\right]-\delta_{II,\theta_j,\theta_\ell}(z,\theta)\cdot f_Z(z)\right)\end{aligned} \tag{S1.34}$$

and,

$$
\begin{aligned}
\frac{\partial^2 \widehat{R}_b^\pi(z,\theta)}{\partial\theta_j\partial\theta_\ell} &= \frac{\partial^2 R_b^\pi(z,\theta)}{\partial\theta_j\partial\theta_\ell} \\
&+ \frac{2}{n\cdot h_n^{d_Z}}\sum_{i=1}^n\left(\frac{\partial^2\Xi_S(X_i,\theta)}{\partial\theta_j\partial\theta_\ell}\cdot\Xi_S(X_i,\theta)\cdot K\left(\frac{Z_i-z}{h_n}\right)-E\left[\frac{\partial^2\Xi_S(X,\theta)}{\partial\theta_j\partial\theta_\ell}\cdot\Xi_S(X,\theta)\cdot K\left(\frac{Z-z}{h_n}\right)\right]\right) \\
&+ \frac{2}{n\cdot h_n^{d_Z}}\sum_{i=1}^n\left(\frac{\partial\Xi_S(X_i,\theta)}{\partial\theta_\ell}\cdot\frac{\partial\Xi_S(X_i,\theta)}{\partial\theta_j}\cdot K\left(\frac{Z_i-z}{h_n}\right)-E\left[\frac{\partial\Xi_S(X,\theta)}{\partial\theta_\ell}\cdot\frac{\partial\Xi_S(X,\theta)}{\partial\theta_j}\cdot K\left(\frac{Z-z}{h_n}\right)\right]\right) \quad \text{(S1.35)} \\
&+ 2\cdot\left(E\left[\frac{1}{h_n^{d_Z}}\cdot\frac{\partial^2\Xi_S(X,\theta)}{\partial\theta_j\partial\theta_\ell}\cdot\Xi_S(X,\theta)\cdot K\left(\frac{Z-z}{h_n}\right)\right]-\delta_{IV,\theta_\ell,\theta_j}(z,\theta)\cdot f_Z(z)\right) \\
&+ 2\cdot\left(E\left[\frac{1}{h_n^{d_Z}}\cdot\frac{\partial\Xi_S(X,\theta)}{\partial\theta_\ell}\cdot\frac{\partial\Xi_S(X,\theta)}{\partial\theta_j}\cdot K\left(\frac{Z-z}{h_n}\right)\right]-\delta_{V,\theta_\ell,\theta_j}(z,\theta)\cdot f_Z(z)\right)
\end{aligned}
$$

By the bounded-variation properties of our kernel function described in Assumption E1 and the Lipschitz-restrictions in Assumption E3, Nolan and Pollard (1987, Lemma 22), Pakes and Pollard (1989, Example 2.10, Lemma 2.13, Lemma 2.14) imply that the classes of functions that produce the empirical processes described next are Euclidean. The existence of $2+\delta$-moments in Assumptions I1 and E3, and the boundedness restrictions in part (iii) of Assumption G1 imply that the corresponding envelopes for these classes also have finite $2+\delta$-moments. From here, Sherman (1994, Corollary 4) can be used to show that,

$$
\begin{aligned}
&\sup_{\substack{\theta\in\Theta\\ z\in\mathcal{Z}}}\left|\frac{1}{n\cdot h_n^{d_Z}}\sum_{i=1}^n\left(\frac{\partial^2\Xi_S(X_i,\theta)}{\partial\theta_j\partial\theta_\ell}\cdot\left(S_i-m_S^{NC}(X_i,\theta)\right)\cdot K\left(\frac{Z_i-z}{h_n}\right)-E\left[\frac{\partial^2\Xi_S(X,\theta)}{\partial\theta_j\partial\theta_\ell}\cdot\left(S-m_S^{NC}(X,\theta)\right)\cdot K\left(\frac{Z-z}{h_n}\right)\right]\right)\right| = o_p(1), \\
&\sup_{\substack{\theta\in\Theta\\ z\in\mathcal{Z}}}\left|\frac{1}{n\cdot h_n^{d_Z}}\sum_{i=1}^n\left(\frac{\partial\Xi_S(X_i,\theta)}{\partial\theta_\ell}\cdot\frac{\partial m_S^{NC}(X_i,\theta)}{\partial\theta_j}\cdot K\left(\frac{Z_i-z}{h_n}\right)-E\left[\frac{\partial\Xi_S(X,\theta)}{\partial\theta_\ell}\cdot\frac{\partial m_S^{NC}(X,\theta)}{\partial\theta_j}\cdot K\left(\frac{Z-z}{h_n}\right)\right]\right)\right| = o_p(1), \\
&\sup_{\substack{\theta\in\Theta\\ z\in\mathcal{Z}}}\left|\frac{1}{n\cdot h_n^{d_Z}}\sum_{i=1}^n\left(\frac{\partial^2 m_S^{NC}(X_i,\theta)}{\partial\theta_j\partial\theta_\ell}\cdot\Xi_S(X_i,\theta)\cdot K\left(\frac{Z_i-z}{h_n}\right)-E\left[\frac{\partial^2 m_S^{NC}(X,\theta)}{\partial\theta_j\partial\theta_\ell}\cdot\Xi_S(X,\theta)\cdot K\left(\frac{Z-z}{h_n}\right)\right]\right)\right| = o_p(1), \\
&\sup_{\substack{\theta\in\Theta\\ z\in\mathcal{Z}}}\left|\frac{1}{n\cdot h_n^{d_Z}}\sum_{i=1}^n\left(\frac{\partial m_S^{NC}(X_i,\theta)}{\partial\theta_\ell}\cdot\frac{\partial\Xi_S(X_i,\theta)}{\partial\theta_j}\cdot K\left(\frac{Z_i-z}{h_n}\right)-E\left[\frac{\partial m_S^{NC}(X,\theta)}{\partial\theta_\ell}\cdot\frac{\partial\Xi_S(X,\theta)}{\partial\theta_j}\cdot K\left(\frac{Z-z}{h_n}\right)\right]\right)\right| = o_p(1), \\
&\sup_{\substack{\theta\in\Theta\\ z\in\mathcal{Z}}}\left|\frac{1}{n\cdot h_n^{d_Z}}\sum_{i=1}^n\left(\frac{\partial^2\Xi_S(X_i,\theta)}{\partial\theta_j\partial\theta_\ell}\cdot\Xi_S(X_i,\theta)\cdot K\left(\frac{Z_i-z}{h_n}\right)-E\left[\frac{\partial^2\Xi_S(X,\theta)}{\partial\theta_j\partial\theta_\ell}\cdot\Xi_S(X,\theta)\cdot K\left(\frac{Z-z}{h_n}\right)\right]\right)\right| = o_p(1), \\
&\sup_{\substack{\theta\in\Theta\\ z\in\mathcal{Z}}}\left|\frac{1}{n\cdot h_n^{d_Z}}\sum_{i=1}^n\left(\frac{\partial\Xi_S(X_i,\theta)}{\partial\theta_\ell}\cdot\frac{\partial\Xi_S(X_i,\theta)}{\partial\theta_j}\cdot K\left(\frac{Z_i-z}{h_n}\right)-E\left[\frac{\partial\Xi_S(X,\theta)}{\partial\theta_\ell}\cdot\frac{\partial\Xi_S(X,\theta)}{\partial\theta_j}\cdot K\left(\frac{Z-z}{h_n}\right)\right]\right)\right| = o_p(1)
\end{aligned}
$$

And for the bias terms, the smoothness restrictions in Assumption E2 yield,

$$\sup_{\substack{\theta\in\Theta\\ z\in\mathcal{Z}}}\left|E\left[\frac{1}{h_n^{d_Z}}\cdot\frac{\partial^2\Xi_S(X,\theta)}{\partial\theta_j\partial\theta_\ell}\cdot\left(S-m_S^{NC}(X,\theta)\right)\cdot K\left(\frac{Z-z}{h_n}\right)\right]-\delta_{I,\theta_\ell,\theta_j}(z,\theta)\cdot f_Z(z)\right|=o(1),$$

$$\sup_{\substack{\theta\in\Theta\\ z\in\mathcal{Z}}}\left|E\left[\frac{1}{h_n^{d_Z}}\cdot\frac{\partial\Xi_S(X,\theta)}{\partial\theta_\ell}\cdot\frac{\partial m_S^{NC}(X,\theta)}{\partial\theta_j}\cdot K\left(\frac{Z-z}{h_n}\right)\right]-\delta_{II,\theta_\ell,\theta_j}(z,\theta)\cdot f_Z(z)\right|=o(1),$$

$$\sup_{\substack{\theta\in\Theta\\ z\in\mathcal{Z}}}\left|E\left[\frac{1}{h_n^{d_Z}}\cdot\frac{\partial^2 m_S^{NC}(X,\theta)}{\partial\theta_j\partial\theta_\ell}\cdot\Xi_S(X,\theta)\cdot K\left(\frac{Z-z}{h_n}\right)\right]-\delta_{III,\theta_\ell,\theta_j}(z,\theta)\cdot f_Z(z)\right|=o(1),$$

$$\sup_{\substack{\theta\in\Theta\\ z\in\mathcal{Z}}}\left|E\left[\frac{1}{h_n^{d_Z}}\cdot\frac{\partial m_S^{NC}(X,\theta)}{\partial\theta_\ell}\cdot\frac{\partial\Xi_S(X,\theta)}{\partial\theta_j}\cdot K\left(\frac{Z-z}{h_n}\right)\right]-\delta_{II,\theta_j,\theta_\ell}(z,\theta)\cdot f_Z(z)\right|=o(1),$$

$$\sup_{\substack{\theta\in\Theta\\ z\in\mathcal{Z}}}\left|E\left[\frac{1}{h_n^{d_Z}}\cdot\frac{\partial^2\Xi_S(X,\theta)}{\partial\theta_j\partial\theta_\ell}\cdot\Xi_S(X,\theta)\cdot K\left(\frac{Z-z}{h_n}\right)\right]-\delta_{IV,\theta_\ell,\theta_j}(z,\theta)\cdot f_Z(z)\right|=o(1),$$

$$\sup_{\substack{\theta\in\Theta\\ z\in\mathcal{Z}}}\left|E\left[\frac{1}{h_n^{d_Z}}\cdot\frac{\partial\Xi_S(X,\theta)}{\partial\theta_\ell}\cdot\frac{\partial\Xi_S(X,\theta)}{\partial\theta_j}\cdot K\left(\frac{Z-z}{h_n}\right)\right]-\delta_{V,\theta_\ell,\theta_j}(z,\theta)\cdot f_Z(z)\right|=o(1).$$

Combined with equations (S1.34)-(S1.35), these results yield,

$$\begin{aligned}&\sup_{\substack{\theta\in\Theta\\ z\in\mathcal{Z}}}\left|\frac{\partial^2\widehat{R}_a^{\pi}(z,\theta)}{\partial\theta_j\partial\theta_\ell}-\frac{\partial^2 R_a^{\pi}(z,\theta)}{\partial\theta_j\partial\theta_\ell}\right|=o_p(1),\quad \ell,j=1,\ldots,d_\theta,\\ &\sup_{\substack{\theta\in\Theta\\ z\in\mathcal{Z}}}\left|\frac{\partial^2\widehat{R}_b^{\pi}(z,\theta)}{\partial\theta_j\partial\theta_\ell}-\frac{\partial^2 R_b^{\pi}(z,\theta)}{\partial\theta_j\partial\theta_\ell}\right|=o_p(1),\quad \ell,j=1,\ldots,d_\theta.\end{aligned}\tag{S1.36}$$

From the expressions in (S1.29) and (S1.30), the results summarized in (S1.31), combined with (S1.36) yield,

$$\sup_{\substack{\theta\in\Theta\\ z\in\mathcal{Z}}}\left|\frac{\partial^2\widehat{\pi}(z,\theta)}{\partial\theta_j\partial\theta_\ell}-\frac{\partial^2\pi(z,\theta)}{\partial\theta_j\partial\theta_\ell}\right|=o_p(1)\text{ for } j,\ell=1,\ldots,d_\theta.\tag{S1.37}$$

To summarize, our restrictions yield the linear representation results,

$$\begin{aligned}\widehat{\pi}(z,\theta)&=\pi(z,\theta)+\frac{1}{n\cdot h_n^{d_z}}\sum_{i=1}^{n}\psi_n^{\pi}(V_i;z,\theta)+\vartheta_n^{\pi}(z,\theta),\\ &\quad\text{where }\sup_{\substack{\theta\in\Theta\\ z\in\mathcal{Z}}}\left|\vartheta_n^{\pi}(z,\theta)\right|=o_p\left(n^{-1/2}\right),\\ \frac{\partial\widehat{\pi}(z,\theta)}{\partial\theta_\ell}&=\frac{\partial\pi(z,\theta)}{\partial\theta_\ell}+\frac{1}{n\cdot h_n^{d_Z}}\sum_{i=1}^{n}\psi_n^{\nabla\theta_\ell\pi}(V_i;z,\theta)+\vartheta_n^{\nabla\theta_\ell\pi}(z,\theta),\\ &\quad\text{where }\sup_{\substack{\theta\in\Theta\\ z\in\mathcal{Z}}}\left|\vartheta_n^{\nabla\theta_\ell\pi}(z,\theta)\right|=o_p\left(n^{-1/2}\right),\quad \ell=1,\ldots,d_\theta,\end{aligned}\tag{S1.38}$$

where the influence function $\psi_n^{\pi}(V_i;z,\theta)$ is as described in equation (S1.11), and the influence function $\psi_n^{\nabla\theta_\ell\pi}(V_i;z,\theta)$ is as described in equation (S1.25). Finally,

$$\begin{aligned}
&\sup_{\substack{\theta\in\Theta\\ z\in\mathcal{Z}}}|\widehat{\pi}(z,\theta)-\pi(z,\theta)| = o_p\left(n^{-1/4}\right),\\
&\sup_{\substack{\theta\in\Theta\\ z\in\mathcal{Z}}}\left|\frac{\partial\widehat{\pi}(z,\theta)}{\partial\theta_\ell}-\frac{\partial\pi(z,\theta)}{\partial\theta_\ell}\right| = o_p\left(n^{-1/4}\right), \quad \ell=1,\ldots,d_\theta,\\
&\sup_{\substack{\theta\in\Theta\\ z\in\mathcal{Z}}}\left|\frac{\partial^2\widehat{\pi}(z,\theta)}{\partial\theta_j\partial\theta_\ell}-\frac{\partial^2\pi(z,\theta)}{\partial\theta_j\partial\theta_\ell}\right| = o_p(1), \quad j,\ell=1,\ldots,d_\theta.
\end{aligned} \tag{S1.39}$$

S1.2 Asymptotic properties of $\left|\widehat{Q}_{S,\mathcal{Z}}(\theta)-Q_{S,\mathcal{Z}}(\theta)\right|$

Recall that the population objective function of our conditional GMM estimator is

$$Q_{S,\mathcal{Z}}(\theta)\equiv\frac{1}{2}\cdot E[\tau_{\mathcal{Z}}(U,\theta)^2], \quad \text{where} \quad \tau_{\mathcal{Z}}(u,\theta)\equiv E\left[\varphi_S(V,\theta)\cdot\mathbb{1}_{\mathcal{Z}}\{U\leq u\}\right],$$

and $\varphi_S(V_i,\theta)\equiv S_i-m_S^{NC}(X_i,\theta)-\pi(Z_i,\theta)\cdot\Xi_S(X_i,\theta)$. Our sample objective function is,

$$\widehat{Q}_{S,\mathcal{Z}}(\theta)\equiv\frac{1}{2}\cdot\frac{1}{n}\sum_{j=1}^{n}\widehat{\tau}_{\mathcal{Z}}(U_j,\theta)^2 \tag{S1.40}$$

where,

$$\begin{aligned}
\widehat{\tau}_{\mathcal{Z}}(u,\theta)&=\frac{1}{n-1}\sum_{i\neq j}\widehat{\varphi}_S(V_i,\theta)\cdot\mathbb{1}_{\mathcal{Z}}\{U_i\leq u\},\\
\widehat{\varphi}_S(V_i,\theta)&\equiv S_i-m_S^{NC}(X_i,\theta)-\widehat{\pi}(Z_i,\theta)\cdot\Xi_S(X_i,\theta)
\end{aligned}$$

We have,

$$\begin{aligned}
\widehat{\tau}_{\mathcal{Z}}(u,\theta)-\frac{1}{n}\sum_{i=1}^{n}\varphi_S(V_i,\theta)\cdot\mathbb{1}_{\mathcal{Z}}\{U_i\leq u\} &= \frac{1}{n}\sum_{i=1}^{n}\left(\widehat{\varphi}_S(V_i,\theta)-\varphi_S(V_i,\theta)\right)\cdot\mathbb{1}_{\mathcal{Z}}\{U_i\leq u\}\\
&= -\frac{1}{n}\sum_{i=1}^{n}\left[\widehat{\pi}(Z_i,\theta)-\pi(Z_i,\theta)\right]\cdot\Xi_S(X_i,\theta)\cdot\mathbb{1}_{\mathcal{Z}}\{U_i\leq u\}
\end{aligned}$$

Note that $|\Xi_S(x,\theta)|\leq 1\ \forall\ (x,\theta)$, and $\mathbb{1}_{\mathcal{Z}}\{U_i\leq u\}\leq\mathbb{1}\{Z_i\in\mathcal{Z}\}\ \forall\ u$. Thus,

$$\sup_{\substack{\theta\in\Theta\\ u\in\mathbb{R}^{d_U}}}\left|\widehat{\tau}_{\mathcal{Z}}(u,\theta)-\frac{1}{n}\sum_{i=1}^{n}\varphi_S(V_i,\theta)\cdot\mathbb{1}_{\mathcal{Z}}\{U_i\leq u\}\right|\leq\sup_{\substack{\theta\in\Theta\\ z\in\mathcal{Z}}}|\widehat{\pi}(z,\theta)-\pi(z,\theta)|=o_p(1) \tag{S1.41}$$

Given the Lipschitz restrictions in Assumption E3, Pakes and Pollard (1989, Lemmas 2.12, 2.13 and 2.14)) imply that the class of functions $\{\varphi_S(V,\theta)\cdot\mathbb{1}_{\mathcal{Z}}\{u\leq c\}$ for some $\theta\in\Theta$, $c\in\mathbb{R}^{d_U}\}$ is a *Euclidean* class (see Sherman (1994, Definition 3)), and given the existence of $2+\delta$-moments in Assumptions I1 and E3, and the regularity properties of the parametric distribution described in Assumption G1, the corresponding envelope for the class also has finite $2+\delta$-moments. From here, Sherman

(1994, Corollary 4) yields,

$$\sup_{\substack{\theta\in\Theta\\ u\in\mathbb{R}^{d_U}}}\left|\frac{1}{n}\sum_{i=1}^{n}(\varphi_S(V_i,\theta)\cdot\mathbb{1}_{\mathcal{Z}}\{U_i\leq u\}-\tau_{\mathcal{Z}}(u,\theta))\right|=o_p(1).$$

Combined with the result in (S1.41), this yields,

$$\sup_{\substack{\theta\in\Theta\\ u\in\mathbb{R}^{d_U}}}|\widehat{\tau}_{\mathcal{Z}}(u,\theta)-\tau_{\mathcal{Z}}(u,\theta)|=o_p(1). \tag{S1.42}$$

Plugging (S1.42) into (S1.40),

$$\widehat{Q}_{S,\mathcal{Z}}(\theta)=\frac{1}{2}\cdot\frac{1}{n}\sum_{j=1}^{n}\tau_{\mathcal{Z}}(U_j,\theta)^2+\xi_{I,n}^{\widehat{Q}}(\theta),\quad \text{Where}\quad \sup_{\theta\in\Theta}\left|\xi_{I,n}^{\widehat{Q}}(\theta)\right|=o_p(1). \tag{S1.43}$$

Recall that $Q_{S,\mathcal{Z}}(\theta)\equiv\frac{1}{2}\cdot E\left[\tau_{\mathcal{Z}}(U,\theta)^2\right]$. Given the Lipschitz restrictions in Assumption E3, Pakes and Pollard (1989, Lemmas 2.13 and 2.14)) imply that the class of functions $\{\tau(u,\theta)\ \text{ for some } \theta\in\Theta\}$ is Euclidean, and the regularity properties of the parametric distribution described in Assumption G1, the corresponding envelope for the class also has finite $2+\delta$-moments. From here, Sherman (1994, Corollary 4) yields,

$$\sup_{\theta\in\Theta}\left|\frac{1}{n}\sum_{j=1}^{n}\tau_{\mathcal{Z}}(U_j,\theta)^2-E\left[\tau_{\mathcal{Z}}(U,\theta)^2\right]\right|=o_p(1). \tag{S1.44}$$

Plugging (S1.44) into (S1.43), we obtain

$$\sup_{\theta\in\Theta}\left|\widehat{Q}_{S,\mathcal{Z}}(\theta)-Q_{S,\mathcal{Z}}(\theta)\right|=o_p(1). \tag{S1.45}$$

S1.3 Asymptotic properties of $\left|\frac{\partial^2\widehat{Q}_{S,\mathcal{Z}}(\theta)}{\partial\theta\partial\theta'}-\frac{\partial^2 Q_{S,\mathcal{Z}}(\theta)}{\partial\theta\partial\theta'}\right|$

We have

$$\begin{aligned}\frac{\partial^2 Q_{S,\mathcal{Z}}(\theta)}{\partial\theta\partial\theta'}&=E\left[\frac{\partial^2\tau_{\mathcal{Z}}(U,\theta)}{\partial\theta\partial\theta'}\cdot\tau_{\mathcal{Z}}(U,\theta)\right]+E\left[\frac{\partial\tau_{\mathcal{Z}}(U,\theta)}{\partial\theta}\cdot\frac{\partial\tau_{\mathcal{Z}}(U,\theta)}{\partial\theta}'\right],\\ \frac{\partial^2\widehat{Q}_{S,\mathcal{Z}}(\theta)}{\partial\theta\partial\theta'}&=\frac{1}{n}\sum_{j=1}^{n}\left(\frac{\partial^2\widehat{\tau}_{\mathcal{Z}}(U_j,\theta)}{\partial\theta\partial\theta'}\cdot\widehat{\tau}_{\mathcal{Z}}(U_j,\theta)+\frac{\partial\widehat{\tau}_{\mathcal{Z}}(U_j,\theta)}{\partial\theta}\cdot\frac{\partial\widehat{\tau}_{\mathcal{Z}}(U_j,\theta)}{\partial\theta}'\right).\end{aligned} \tag{S1.46}$$

Where,

$$\frac{\partial \tau_{\mathcal{Z}}(u,\theta)}{\partial \theta} = E\left[\frac{\partial \varphi_S(V,\theta)}{\partial \theta} \cdot \mathbb{1}_{\mathcal{Z}}\{U \leq u\}\right],$$
$$\frac{\partial^2 \tau_{\mathcal{Z}}(u,\theta)}{\partial \theta \partial \theta'} = E\left[\frac{\partial^2 \varphi_S(V,\theta)}{\partial \theta \partial \theta'} \cdot \mathbb{1}_{\mathcal{Z}}\{U \leq u\}\right],$$
$$\frac{\partial \widehat{\tau}_{\mathcal{Z}}(u,\theta)}{\partial \theta} = \frac{1}{n}\sum_{i=1}^{n} \frac{\partial \widehat{\varphi}_S(V_i,\theta)}{\partial \theta} \cdot \mathbb{1}_{\mathcal{Z}}\{U_i \leq u\},$$
$$\frac{\partial^2 \widehat{\tau}_{\mathcal{Z}}(u,\theta)}{\partial \theta \partial \theta'} = \frac{1}{n}\sum_{i=1}^{n} \frac{\partial^2 \widehat{\varphi}_S(V_i,\theta)}{\partial \theta \partial \theta'} \cdot \mathbb{1}_{\mathcal{Z}}\{U_i \leq u\}.$$

Recall that,

$$\widehat{\varphi}_S(V_i,\theta) \equiv S_i - m_S^{NC}(X_i,\theta) - \widehat{\pi}(Z_i,\theta) \cdot \Xi_S(X_i,\theta),$$
$$\varphi_S(V_i,\theta) \equiv S_i - m_S^{NC}(X_i,\theta) - \pi(Z_i,\theta) \cdot \Xi_S(X_i,\theta).$$

Thus,

$$\begin{aligned}\frac{\partial \widehat{\varphi}_S(V_i,\theta)}{\partial \theta} - \frac{\partial \varphi_S(V_i,\theta)}{\partial \theta} = &\left[\frac{\partial \pi(Z_i,\theta)}{\partial \theta} - \frac{\partial \widehat{\pi}(Z_i,\theta)}{\partial \theta}\right] \cdot \Xi_S(X_i,\theta) \\ &+ \frac{\partial \Xi_S(X_i,\theta)}{\partial \theta} \cdot [\pi(Z_i,\theta) - \widehat{\pi}(Z_i,\theta)],\end{aligned} \tag{S1.47}$$

and,

$$\begin{aligned}&\frac{\partial^2 \widehat{\varphi}_S(V_i,\theta)}{\partial \theta \partial \theta'} - \frac{\partial^2 \varphi_S(V_i,\theta)}{\partial \theta \partial \theta'} = \left[\frac{\partial \pi(Z_i,\theta)}{\partial \theta} - \frac{\partial \widehat{\pi}(Z_i,\theta)}{\partial \theta}\right] \cdot \frac{\partial \Xi_S(X_i,\theta)}{\partial \theta}' \\ &+ \left[\frac{\partial^2 \pi(Z_i,\theta)}{\partial \theta \partial \theta'} - \frac{\partial^2 \widehat{\pi}(Z_i,\theta)}{\partial \theta \partial \theta'}\right] \cdot \Xi_S(X_i,\theta) + \frac{\partial \Xi_S(X_i,\theta)}{\partial \theta} \cdot \left[\frac{\partial \pi(Z_i,\theta)}{\partial \theta} - \frac{\partial \widehat{\pi}(Z_i,\theta)}{\partial \theta}\right]' \\ &+ \frac{\partial^2 \Xi_S(X_i,\theta)}{\partial \theta \partial \theta'} \cdot [\pi(Z_i,\theta) - \widehat{\pi}(Z_i,\theta)].\end{aligned} \tag{S1.48}$$

Under the existence-of-moments restriction in Assumption I1, and the regularity conditions in Assumption E3, we have

$$\sup_{\theta \in \Theta}\left(\frac{1}{n}\sum_{i=1}^{n}\left\|\frac{\partial \Xi_S(X_i,\theta)}{\partial \theta}\right\|\right) = O_p(1), \quad \text{and} \quad \sup_{\theta \in \Theta}\left(\frac{1}{n}\sum_{i=1}^{n}\left\|\frac{\partial^2 \Xi_S(X_i,\theta)}{\partial \theta \partial \theta'}\right\|\right) = O_p(1).$$

Furthermore, $|\Xi_S(x,\theta)| \leq 1$ for all (x,θ). Thus, from the expressions in (S1.47)-(S1.48), the uniform convergence results in (S1.39)-(S1.39) yield

$$\begin{aligned}&\sup_{\substack{\theta \in \Theta \\ u \in \mathbb{R}^{d_U}}}\left|\frac{\partial \widehat{\tau}_{\mathcal{Z}}(u,\theta)}{\partial \theta} - \frac{1}{n-1}\sum_{i \neq j} \frac{\partial \varphi_S(V_i,\theta)}{\partial \theta} \cdot \mathbb{1}_{\mathcal{Z}}\{U_i \leq u\}\right| = o_p(1), \\ &\sup_{\substack{\theta \in \Theta \\ u \in \mathbb{R}^{d_U}}}\left|\frac{\partial^2 \widehat{\tau}_{\mathcal{Z}}(u,\theta)}{\partial \theta \partial \theta'} - \frac{1}{n-1}\sum_{i \neq j} \frac{\partial^2 \varphi_S(V_i,\theta)}{\partial \theta \partial \theta'} \cdot \mathbb{1}_{\mathcal{Z}}\{U_i \leq u\}\right| = o_p(1),\end{aligned} \tag{S1.49}$$

Next, given the Lipschitz restrictions in Assumption E3, Pakes and Pollard (1989, Lemmas 2.12, 2.13 and 2.14)) imply that the class of functions

$$\left\{\frac{\partial \varphi_S(V,\theta)}{\partial \theta}\cdot \mathbb{1}_{\mathcal{Z}}\{u\le c\}\ \text{ for some } \theta\in\Theta,\, c\in\mathbb{R}^{d_U}\right\},$$
$$\left\{\frac{\partial^2 \varphi_S(V,\theta)}{\partial \theta\partial\theta'}\cdot \mathbb{1}_{\mathcal{Z}}\{u\le c\}\ \text{ for some } \theta\in\Theta,\, c\in\mathbb{R}^{d_U}\right\}$$

are Euclidean. The regularity properties of the parametric distribution described in Assumption G1, and the existence of $2+\delta$-moment restrictions described in Assumption I1 imply that the corresponding envelope for the class also has finite $2+\delta$-moments. From here, Sherman (1994, Corollary 4) yields,

$$\sup_{\substack{\theta\in\Theta\\ u\in\mathbb{R}^{d_U}}}\left|\frac{1}{n}\sum_{i=1}^{n}\left(\frac{\partial\varphi_S(V_i,\theta)}{\partial\theta}\cdot\mathbb{1}_{\mathcal{Z}}\{U_i\le u\}-\frac{\partial\tau_{\mathcal{Z}}(u,\theta)}{\partial\theta}\right)\right|=o_p(1),$$
$$\sup_{\substack{\theta\in\Theta\\ u\in\mathbb{R}^{d_U}}}\left|\frac{1}{n}\sum_{i=1}^{n}\left(\frac{\partial^2\varphi_S(V_i,\theta)}{\partial\theta\partial\theta'}\cdot\mathbb{1}_{\mathcal{Z}}\{U_i\le u\}-\frac{\partial^2\tau_{\mathcal{Z}}(u,\theta)}{\partial\theta\partial\theta'}\right)\right|=o_p(1).$$

Combined with the results in (S1.49), this yields,

$$\begin{aligned}&\sup_{\substack{\theta\in\Theta\\ u\in\mathbb{R}^{d_U}}}\left|\frac{\partial\widehat{\tau}_{\mathcal{Z}}(u,\theta)}{\partial\theta}-\frac{\partial\tau_{\mathcal{Z}}(u,\theta)}{\partial\theta}\right|=o_p(1),\\ &\sup_{\substack{\theta\in\Theta\\ u\in\mathbb{R}^{d_U}}}\left|\frac{\partial^2\widehat{\tau}_{\mathcal{Z}}(u,\theta)}{\partial\theta\partial\theta'}-\frac{\partial^2\tau_{\mathcal{Z}}(u,\theta)}{\partial\theta\partial\theta'}\right|=o_p(1)\end{aligned}\tag{S1.50}$$

Combining this result with (S1.42) and plugging back into (S1.46),

$$\begin{aligned}\frac{\partial^2\widehat{Q}_{S,\mathcal{Z}}(\theta)}{\partial\theta\partial\theta'}&=\frac{1}{n}\sum_{j=1}^{n}\left(\frac{\partial^2\tau_{\mathcal{Z}}(U_j,\theta)}{\partial\theta\partial\theta'}\cdot\tau_{\mathcal{Z}}(U_j,\theta)+\frac{\partial\tau_{\mathcal{Z}}(U_j,\theta)}{\partial\theta}\cdot\frac{\partial\tau_{\mathcal{Z}}(U_j,\theta)}{\partial\theta}'\right)+\xi_{I,n}^{\nabla_{\theta\theta'}\widehat{Q}}(\theta),\\ &\text{where}\quad\sup_{\theta\in\Theta}\left|\xi_{I,n}^{\nabla_{\theta\theta'}\widehat{Q}}(\theta)\right|=o_p(1).\end{aligned}\tag{S1.51}$$

Finally, the Lipschitz restrictions in Assumption E3, Pakes and Pollard (1989, Lemmas 2.12, 2.13 and 2.14)) imply that the following class of functions is Euclidean (element-wise),

$$\left\{\frac{\partial^2\tau(u,\theta)}{\partial\theta\partial\theta'}\cdot\tau_{\mathcal{Z}}(u,\theta)+\frac{\partial\tau_{\mathcal{Z}}(u,\theta)}{\partial\theta}\cdot\frac{\partial\tau_{\mathcal{Z}}(u,\theta)}{\partial\theta}'\ \text{ for some }\theta\in\Theta\right\}$$

The regularity properties of the parametric distribution described in Assumption G1, and the existence of $2+\delta$-moment restrictions described in Assumption I1 imply that the corresponding envelope for the class also has finite $2+\delta$-moments. From here, Sherman (1994, Corollary 4) and the definition of $\frac{\partial^2 Q_{S,\mathcal{Z}}(\theta)}{\partial\theta\partial\theta'}$ in (S1.46) yield,

$$\sup_{\theta\in\Theta}\left|\frac{1}{n}\sum_{j=1}^{n}\left[\left(\frac{\partial^2\tau_{\mathcal{Z}}(U_j,\theta)}{\partial\theta\partial\theta'}\cdot\tau_{\mathcal{Z}}(U_j,\theta)+\frac{\partial\tau_{\mathcal{Z}}(U_j,\theta)}{\partial\theta}\cdot\frac{\partial\tau_{\mathcal{Z}}(U_j,\theta)}{\partial\theta}'\right)-\frac{\partial^2 Q_{S,\mathcal{Z}}(\theta)}{\partial\theta\partial\theta'}\right]\right|=o_p(1) \qquad \text{(S1.52)}$$

Plugging this back into (S1.51), we finally obtain,

$$\sup_{\theta\in\Theta}\left|\frac{\partial^2\widehat{Q}_{S,\mathcal{Z}}(\theta)}{\partial\theta\partial\theta'}-\frac{\partial^2 Q_{S,\mathcal{Z}}(\theta)}{\partial\theta\partial\theta'}\right|=o_p(1). \qquad \text{(S1.53)}$$

S1.4 Consistency

Under the restrictions of Assumptions G1, G2 and I1, our estimator first-step MLE estimator $\widehat{\gamma}$ satisfies $\widehat{\gamma}\xrightarrow{p}\gamma_0$. Next, given the restrictions in Assumption I4, θ_0 is the unique minimizer of $Q_{S,\mathcal{Z}}(\theta)$ over Θ. Furthermore, $Q_{S,\mathcal{Z}}(\theta)$ is continuous by the restrictions in Assumption I3. Also, as we showed in (S1.45), under our restrictions we have $\sup_{\theta\in\Theta}\left|\widehat{Q}_{S,\mathcal{Z}}(\theta)-Q_{S,\mathcal{Z}}(\theta)\right|=o_p(1)$. Thus, since $\widehat{\gamma}\xrightarrow{p}\gamma_0$, we have, $\sup_{\Delta\in\Theta}\left|\widehat{Q}_{S,\mathcal{Z}}(\widehat{\gamma},\Delta)-Q_{S,\mathcal{Z}}(\gamma_0,\Delta)\right|=o_p(1)$. From here, Newey and McFadden (1994, Theorem 2.1) yields $\widehat{\Delta}\xrightarrow{p}\Delta_0$. Thus, grouping $\widehat{\theta}\equiv\left(\widehat{\gamma}',\widehat{\Delta}'\right)'$, we have $\widehat{\theta}\xrightarrow{p}\theta_0$.

S1.5 Asymptotic distribution

Since $\theta_0\in\text{int}(\Theta)$ by assumption, smoothness of our sample objective function $\widehat{Q}_{S,\mathcal{Z}}(\theta)$ implies that, with probability approaching one (w.p.a.1), $\widehat{\theta}$ satisfies the first-order conditions $\frac{\partial\widehat{Q}_{S,\mathcal{Z}}(\widehat{\theta})}{\partial\Delta}=0$. From here, a linear approximation yields,

$$0=\frac{\partial\widehat{Q}_{S,\mathcal{Z}}(\widehat{\theta})}{\partial\Delta}=\frac{\partial\widehat{Q}_{S,\mathcal{Z}}(\theta_0)}{\partial\Delta}+\frac{\partial^2\widehat{Q}_{S,\mathcal{Z}}(\widetilde{\theta})}{\partial\Delta\partial\gamma'}\cdot(\widehat{\gamma}-\gamma_0)+\frac{\partial^2\widehat{Q}_{S,\mathcal{Z}}(\widetilde{\theta})}{\partial\Delta\partial\Delta'}\cdot\left(\widehat{\Delta}-\Delta_0\right)$$

As usual, $\widetilde{\theta}$ belongs in the line segment connecting $\widehat{\theta}$ and θ_0 and, since $\widehat{\theta}\xrightarrow{p}\theta_0$, it satisfies $\widetilde{\theta}\xrightarrow{p}\theta_0$. Given the uniform convergence result $\sup_{\theta\in\Theta}\left|\frac{\partial^2\widehat{Q}_{S,\mathcal{Z}}(\theta)}{\partial\theta\partial\theta'}-\frac{\partial^2 Q_{S,\mathcal{Z}}(\theta)}{\partial\theta\partial\theta'}\right|=o_p(1)$ in equation (S1.53) and the assumption that $\frac{\partial^2 Q_{S,\mathcal{Z}}(\theta)}{\partial\Delta\partial\Delta'}^{-1}$ exists and is a continuous function of θ in an open neighborhood $\mathcal{N}$ of θ_0 (see Assumption I3), it follows that with probability approaching one, we can express,

$$\widehat{\Delta}-\Delta_0=-\left(\frac{\partial^2 Q_{S,\mathcal{Z}}(\theta_0)}{\partial\Delta\partial\Delta'}^{-1}+o_p(1)\right)\times\left[\frac{\partial\widehat{Q}_{S,\mathcal{Z}}(\theta_0)}{\partial\Delta}+\left(\frac{\partial^2 Q_{S,\mathcal{Z}}(\theta_0)}{\partial\Delta\partial\gamma'}+o_p(1)\right)\cdot(\widehat{\gamma}-\gamma_0)\right]$$

Plugging in the linear representation result of our first-stage estimator $\widehat{\gamma}$, we obtain,

$$\widehat{\Delta}-\Delta_0=-\left(\frac{\partial^2 Q_{S,\mathcal{Z}}(\theta_0)}{\partial\Delta\partial\Delta'}^{-1}+o_p(1)\right)\times\left[\frac{\partial\widehat{Q}_{S,\mathcal{Z}}(\theta_0)}{\partial\Delta}+\left(\frac{\partial^2 Q_{S,\mathcal{Z}}(\theta_0)}{\partial\Delta\partial\gamma'}+o_p(1)\right)\cdot\left(\frac{1}{n}\sum_{i=1}^{n}\psi(V_i;\gamma_0)+o_p(n^{-1/2})\right)\right] \qquad \text{(S1.54)}$$

Our next step is to show that, under our restrictions, we can obtain a linear representation result for $\frac{\partial \widehat{Q}_{S,\mathcal{Z}}(\theta_0)}{\partial \Delta}$.

S1.5.1 A linear representation for $\frac{\partial \widehat{Q}_{S,\mathcal{Z}}(\theta_0)}{\partial \Delta}$

We have,

$$
\begin{aligned}
\frac{\partial \widehat{Q}_{S,\mathcal{Z}}(\theta)}{\partial \Delta} &= \frac{1}{n}\sum_{j=1}^{n}\left[\left(\frac{1}{n}\sum_{i=1}^{n}\widehat{\varphi}_S(V_i,\theta)\cdot \mathbb{1}_{\mathcal{Z}}\{U_i \le U_j\}\right)\times\left(\frac{1}{n}\sum_{\ell=1}^{n}\frac{\partial \widehat{\varphi}_S(V_\ell,\theta)}{\partial \Delta}\cdot \mathbb{1}_{\mathcal{Z}}\{U_\ell \le U_j\}\right)\right] \\
&= \frac{1}{n^3}\sum_{j=1}^{n}\sum_{i\neq j}\sum_{\ell\neq i,j}\widehat{\varphi}_S(V_i,\theta)\cdot\frac{\partial \widehat{\varphi}_S(V_\ell,\theta)}{\partial \Delta}\cdot \mathbb{1}_{\mathcal{Z}}\{U_i \le U_j\}\cdot \mathbb{1}_{\mathcal{Z}}\{U_\ell \le U_j\} \\
&+ \frac{1}{n}\times\left[\frac{1}{n^2}\sum_{j=1}^{n}\sum_{i\neq j}\left(\widehat{\varphi}_S(V_i,\theta)\cdot\frac{\partial \widehat{\varphi}_S(V_i,\theta)}{\partial \Delta}\cdot \mathbb{1}_{\mathcal{Z}}\{U_i \le U_j\} + \widehat{\varphi}_S(V_j,\theta)\cdot\frac{\partial \widehat{\varphi}_S(V_i,\theta)}{\partial \Delta}\cdot \mathbb{1}_{\mathcal{Z}}\{U_i \le U_j\}\right.\right. \\
&\qquad\qquad \left.\left. + \widehat{\varphi}_S(V_i,\theta)\cdot\frac{\partial \widehat{\varphi}_S(V_j,\theta)}{\partial \Delta}\cdot \mathbb{1}_{\mathcal{Z}}\{U_i \le U_j\}\right)\right] \\
&+ \frac{1}{n^2}\times\left[\frac{1}{n}\sum_{j=1}^{n}\widehat{\varphi}_S(V_j,\theta)\cdot\frac{\partial \widehat{\varphi}_S(V_j,\theta)}{\partial \Delta}\right]
\end{aligned}
$$

Writing $\widehat{\varphi}_S(V_i,\theta) = \varphi(V_i,\theta) + [\widehat{\varphi}_S(V_i,\theta) - \varphi(V_i,\theta)]$ and $\frac{\partial\widehat{\varphi}_S(V_\ell,\theta)}{\partial\Delta} = \frac{\partial\varphi(V_\ell,\theta)}{\partial\Delta} + \left[\frac{\partial\widehat{\varphi}_S(V_\ell,\theta)}{\partial\Delta} - \frac{\partial\varphi(V_\ell,\theta)}{\partial\Delta}\right]$, we obtain,

$$\begin{aligned}
&\frac{\partial\widehat{Q}_{S,\mathcal{Z}}(\theta)}{\partial\Delta} = \\
&\frac{1}{n^3}\sum_{j=1}^{n}\sum_{i\neq j}\sum_{\ell\neq i,j}\varphi_S(V_i,\theta)\cdot\frac{\partial\varphi_S(V_\ell,\theta)}{\partial\Delta}\cdot\mathbb{1}_{\mathcal{Z}}\{U_i\leq U_j\}\cdot\mathbb{1}_{\mathcal{Z}}\{U_\ell\leq U_j\} \\
&+\frac{1}{n^3}\sum_{j=1}^{n}\sum_{i\neq j}\sum_{\ell\neq i,j}\varphi_S(V_i,\theta)\cdot\left[\frac{\partial\widehat{\varphi}_S(V_\ell,\theta)}{\partial\Delta}-\frac{\partial\varphi_S(V_\ell,\theta)}{\partial\Delta}\right]\cdot\mathbb{1}_{\mathcal{Z}}\{U_i\leq U_j\}\cdot\mathbb{1}_{\mathcal{Z}}\{U_\ell\leq U_j\} \\
&+\frac{1}{n^3}\sum_{j=1}^{n}\sum_{i\neq j}\sum_{\ell\neq i,j}\frac{\partial\varphi_S(V_\ell,\theta)}{\partial\Delta}\cdot[\widehat{\varphi}_S(V_i,\theta)-\varphi_S(V_i,\theta)]\cdot\mathbb{1}_{\mathcal{Z}}\{U_i\leq U_j\}\cdot\mathbb{1}_{\mathcal{Z}}\{U_\ell\leq U_j\} \\
&+\frac{1}{n^3}\sum_{j=1}^{n}\sum_{i\neq j}\sum_{\ell\neq i,j}[\widehat{\varphi}_S(V_i,\theta)-\varphi_S(V_i,\theta)]\cdot\left[\frac{\partial\widehat{\varphi}_S(V_\ell,\theta)}{\partial\Delta}-\frac{\partial\varphi_S(V_\ell,\theta)}{\partial\Delta}\right]\cdot\mathbb{1}_{\mathcal{Z}}\{U_i\leq U_j\}\cdot\mathbb{1}_{\mathcal{Z}}\{U_\ell\leq U_j\} \\
&+\frac{1}{n}\times\left[\frac{1}{n^2}\sum_{j=1}^{n}\sum_{i\neq j}\left(\widehat{\varphi}_S(V_i,\theta)\cdot\frac{\partial\widehat{\varphi}_S(V_i,\theta)}{\partial\Delta}\cdot\mathbb{1}_{\mathcal{Z}}\{U_i\leq U_j\}+\widehat{\varphi}_S(V_j,\theta)\cdot\frac{\partial\widehat{\varphi}_S(V_i,\theta)}{\partial\Delta}\cdot\mathbb{1}_{\mathcal{Z}}\{U_i\leq U_j\}\right.\right. \\
&\qquad\qquad\left.\left.+\widehat{\varphi}_S(V_i,\theta)\cdot\frac{\partial\widehat{\varphi}_S(V_j,\theta)}{\partial\Delta}\cdot\mathbb{1}_{\mathcal{Z}}\{U_i\leq U_j\}\right)\right] \\
&+\frac{1}{n^2}\times\left[\frac{1}{n}\sum_{j=1}^{n}\widehat{\varphi}_S(V_j,\theta)\cdot\frac{\partial\widehat{\varphi}_S(V_j,\theta)}{\partial\Delta}\right]
\end{aligned} \tag{S1.55}$$

Recall that, $\widehat{\varphi}_S(V_i,\theta)\equiv S_i - m_S^{NC}(X_i,\theta) - \widehat{\pi}(Z_i,\theta)\cdot\Xi_S(X_i,\theta)$, and $\varphi_S(V_i,\theta)\equiv S_i - m_S^{NC}(X_i,\theta) - \pi(Z_i,\theta)\cdot\Xi_S(X_i,\theta)$. Thus,

$$\begin{aligned}
\widehat{\varphi}_S(V_i,\theta)-\varphi_S(V_i,\theta) &= -[\widehat{\pi}(Z_i,\theta)-\pi(Z_i,\theta)]\cdot\Xi_S(X_i,\theta), \\
\frac{\partial\widehat{\varphi}_S(V_\ell,\theta)}{\partial\Delta}-\frac{\partial\varphi_S(V_\ell,\theta)}{\partial\Delta} &= -\left[\frac{\partial\widehat{\pi}(Z_\ell,\theta)}{\partial\Delta}-\frac{\partial\pi(Z_\ell,\theta)}{\partial\Delta}\right]\cdot\Xi_S(X_\ell,\theta)-[\widehat{\pi}(Z_\ell,\theta)-\pi(Z_\ell,\theta)]\cdot\frac{\partial\Xi_S(X_\ell,\theta)}{\partial\Delta}
\end{aligned} \tag{S1.56}$$

Let the influence functions $\psi_n^{\pi}(v;z,\theta)$ and $\psi_n^{\nabla\Delta\pi}(v;z,\theta)$ be as described in equations (S1.11) and (S1.25), and let $\vartheta_n^{\pi}(z,\theta)$ and $\vartheta_n^{\nabla\Delta\pi}(z,\theta)$ be the remainders of the linear representations described in (S1.11) and (S1.27). Plugging in these linear representations into (S1.56),

$$\begin{aligned}
\widehat{\varphi}_S(V_i,\theta)-\varphi_S(V_i,\theta) &= -\left(\frac{1}{n\cdot h_n^{d_Z}}\sum_{k=1}^{n}\psi_n^{\pi}(V_k;Z_i,\theta)+\vartheta_n^{\pi}(Z_i,\theta)\right)\cdot\Xi_S(X_i,\theta), \\
\frac{\partial\widehat{\varphi}_S(V_\ell,\theta)}{\partial\Delta}-\frac{\partial\varphi_S(V_\ell,\theta)}{\partial\Delta} &= -\left(\frac{1}{n\cdot h_n^{d_Z}}\sum_{k=1}^{n}\psi_n^{\nabla\Delta\pi}(V_k;Z_\ell,\theta)+\vartheta_n^{\nabla\Delta\pi}(Z_\ell,\theta)\right)\cdot\Xi_S(X_\ell,\theta) \\
&\quad-\left(\frac{1}{n\cdot h_n^{d_Z}}\sum_{k=1}^{n}\psi_n^{\pi}(V_k;Z_\ell,\theta)+\vartheta_n^{\pi}(Z_\ell,\theta)\right)\cdot\frac{\partial\Xi_S(X_\ell,\theta)}{\partial\Delta}
\end{aligned}$$

From here,

$$
\begin{aligned}
&\frac{1}{n^3}\sum_{j=1}^{n}\sum_{i\neq j}\sum_{\ell\neq i,j}\varphi_S(V_i,\theta)\cdot\left[\frac{\partial\widehat{\varphi}_S(V_\ell,\theta)}{\partial\Delta}-\frac{\partial\varphi_S(V_\ell,\theta)}{\partial\Delta}\right]\cdot\mathbb{1}_{\mathcal{Z}}\{U_i\leq U_j\}\cdot\mathbb{1}_{\mathcal{Z}}\{U_\ell\leq U_j\}\\
&=-\frac{1}{n^3}\sum_{j=1}^{n}\sum_{i\neq j}\sum_{\ell\neq i,j}\varphi_S(V_i,\theta)\cdot\left(\frac{1}{n\cdot h_n^{d_Z}}\sum_{k=1}^{n}\psi_n^{\nabla\Delta\pi}(V_k;Z_\ell,\theta)+\vartheta_n^{\nabla\Delta\pi}(Z_\ell,\theta)\right)\cdot\Xi_S(X_\ell,\theta)\cdot\mathbb{1}_{\mathcal{Z}}\{U_i\leq U_j\}\cdot\mathbb{1}_{\mathcal{Z}}\{U_\ell\leq U_j\}\\
&\quad-\frac{1}{n^3}\sum_{j=1}^{n}\sum_{i\neq j}\sum_{\ell\neq i,j}\varphi_S(V_i,\theta)\cdot\left(\frac{1}{n\cdot h_n^{d_Z}}\sum_{k=1}^{n}\psi_n^{\pi}(V_k;Z_\ell,\theta)+\vartheta_n^{\pi}(Z_\ell,\theta)\right)\cdot\frac{\partial\Xi_S(X_\ell,\theta)}{\partial\Delta}\cdot\mathbb{1}_{\mathcal{Z}}\{U_i\leq U_j\}\cdot\mathbb{1}_{\mathcal{Z}}\{U_\ell\leq U_j\}
\end{aligned}
$$

And,

$$
\begin{aligned}
&\frac{1}{n^3}\sum_{j=1}^{n}\sum_{i\neq j}\sum_{\ell\neq i,j}\frac{\partial\varphi_S(V_\ell,\theta)}{\partial\Delta}\cdot[\widehat{\varphi}_S(V_i,\theta)-\varphi_S(V_i,\theta)]\cdot\mathbb{1}_{\mathcal{Z}}\{U_i\leq U_j\}\cdot\mathbb{1}_{\mathcal{Z}}\{U_\ell\leq U_j\}\\
&=-\frac{1}{n^3}\sum_{j=1}^{n}\sum_{i\neq j}\sum_{\ell\neq i,j}\frac{\partial\varphi_S(V_\ell,\theta)}{\partial\Delta}\cdot\left(\frac{1}{n\cdot h_n^{d_Z}}\sum_{k=1}^{n}\psi_n^{\pi}(V_k;Z_i,\theta)+\vartheta_n^{\pi}(Z_i,\theta)\right)\cdot\Xi_S(X_i,\theta)\cdot\mathbb{1}_{\mathcal{Z}}\{U_i\leq U_j\}\cdot\mathbb{1}_{\mathcal{Z}}\{U_\ell\leq U_j\}
\end{aligned}
$$

In what follows, let $(V_1, V_2, V_3, V_4) \sim F_V \otimes F_V \otimes F_V \otimes F_V$ (four randomly drawn observations from our i.i.d sample $(V_i)_{i=1}^n$). Let,

$$
\begin{aligned}
&g^a_\Delta(V_1, V_2, V_3;\theta) \equiv \varphi_S(V_1,\theta)\cdot \frac{\partial \varphi_S(V_3,\theta)}{\partial \Delta}\cdot \mathbb{1}_{\mathcal{Z}}\{U_1 \le U_2\}\cdot \mathbb{1}_{\mathcal{Z}}\{U_3 \le U_2\},\\
&g^b_{\Delta,n}(V_1, V_2, V_3, V_4;\theta) \equiv \frac{1}{h_n^{d_Z}} \times \Bigg(\varphi_S(V_1,\theta)\cdot\Bigg(\Xi_S(X_3,\theta)\cdot \psi_n^{\nabla\Delta\pi}(V_4;Z_3,\theta) + \frac{\partial \Xi_S(X_3,\theta)}{\partial\Delta}\cdot \psi_n^{\pi}(V_4;Z_3,\theta)\Bigg)\\
&+ \frac{\partial \varphi_S(V_3,\theta)}{\partial\Delta}\cdot \Xi_S(X_1,\theta)\cdot \psi_n^{\pi}(V_4;Z_1,\theta)\Bigg)\cdot \mathbb{1}_{\mathcal{Z}}\{U_1 \le U_2\}\cdot \mathbb{1}_{\mathcal{Z}}\{U_3 \le U_2\},\\
&g^c_{\Delta,n}(V_1, V_2, V_3;\theta) \equiv \frac{1}{h_n^{d_Z}} \times \Bigg(\varphi_S(V_1,\theta)\cdot \Xi_S(X_3,\theta)\cdot \Big[\psi_n^{\nabla\Delta\pi}(V_1;Z_3,\theta) + \psi_n^{\nabla\Delta\pi}(V_2;Z_3,\theta) + \psi_n^{\nabla\Delta\pi}(V_3;Z_3,\theta)\Big]\\
&+ \varphi_S(V_1,\theta)\cdot \frac{\partial \Xi_S(X_3,\theta)}{\partial\Delta}\cdot [\psi_n^{\pi}(V_1;Z_3,\theta) + \psi_n^{\pi}(V_2;Z_3,\theta) + \psi_n^{\pi}(V_3;Z_3,\theta)]\\
&+ \frac{\partial \varphi_S(V_3,\theta)}{\partial\Delta}\cdot \Xi_S(X_1,\theta)\cdot [\psi_n^{\pi}(V_1;Z_1,\theta) + \psi_n^{\pi}(V_2;Z_1,\theta) + \psi_n^{\pi}(V_3;Z_1,\theta)]\Bigg)\cdot \mathbb{1}_{\mathcal{Z}}\{U_1 \le U_2\}\cdot \mathbb{1}_{\mathcal{Z}}\{U_3 \le U_2\},\\
&g^d_{\Delta,n}(V_1, V_2, V_3;\theta) \equiv \Bigg(\varphi_S(V_1,\theta)\cdot\Bigg(\Xi_S(X_3,\theta)\cdot \vartheta_n^{\nabla\Delta\pi}(Z_3,\theta) + \frac{\partial \Xi_S(X_3,\theta)}{\partial\Delta}\cdot \vartheta_n^{\pi}(Z_3,\theta)\Bigg)\\
&+ \frac{\partial \varphi_S(V_3,\theta)}{\partial\Delta}\cdot \Xi_S(X_1,\theta)\cdot \vartheta_n^{\pi}(Z_1,\theta)\Bigg)\cdot \mathbb{1}_{\mathcal{Z}}\{U_1 \le U_2\}\cdot \mathbb{1}_{\mathcal{Z}}\{U_3 \le U_2\},\\
&g^e_{\Delta,n}(V_1, V_2, V_3;\theta) \equiv \Bigg(\Xi_S(X_1,\theta)\cdot \Xi_S(X_3,\theta)\cdot [\widehat{\pi}(Z_1,\theta) - \pi(Z_1,\theta)]\cdot \Bigg[\frac{\partial \widehat{\pi}(Z_3,\theta)}{\partial\Delta} - \frac{\partial \pi(Z_3,\theta)}{\partial\Delta}\Bigg]\\
&+ \frac{\partial \Xi_S(X_3,\theta)}{\partial\Delta}\cdot \Xi_S(X_1,\theta)\cdot [\widehat{\pi}(Z_1,\theta) - \pi(Z_1,\theta)]\cdot [\widehat{\pi}(Z_3,\theta) - \pi(Z_3,\theta)]\Bigg)\cdot \mathbb{1}_{\mathcal{Z}}\{U_1 \le U_2\}\cdot \mathbb{1}_{\mathcal{Z}}\{U_3 \le U_2\},\\
&g^f_{\Delta,n}(V_1, V_2;\theta) \equiv \Bigg(\widehat{\varphi}_S(V_1,\theta)\cdot \frac{\partial \widehat{\varphi}_S(V_1,\theta)}{\partial\Delta}\cdot \mathbb{1}_{\mathcal{Z}}\{U_1 \le U_2\} + \widehat{\varphi}_S(V_2,\theta)\cdot \frac{\partial \widehat{\varphi}_S(V_1,\theta)}{\partial\Delta}\cdot \mathbb{1}_{\mathcal{Z}}\{U_1 \le U_2\}\\
&\qquad\qquad + \widehat{\varphi}_S(V_1,\theta)\cdot \frac{\partial \widehat{\varphi}_S(V_2,\theta)}{\partial\Delta}\cdot \mathbb{1}_{\mathcal{Z}}\{U_1 \le U_2\}\Bigg),\\
&g^g_{\Delta,n}(V_1;\theta) \equiv \widehat{\varphi}_S(V_1,\theta)\cdot \frac{\partial \widehat{\varphi}_S(V_1,\theta)}{\partial\Delta}\cdot \mathbb{1}\{Z_1 \in \mathcal{Z}\}.
\end{aligned}
\tag{S1.57}
$$

Using these expressions, (S1.55) becomes[1],

$$\frac{\partial \widehat{Q}_{S,\mathcal{Z}}(\theta)}{\partial \Delta} = \frac{1}{n^3}\sum_{j=1}^{n}\sum_{i\neq j}\sum_{\ell\neq i,j} g^{a}_{\Delta}(V_i,V_j,V_\ell;\theta) - \frac{1}{n^4}\sum_{j=1}^{n}\sum_{i\neq j}\sum_{\ell\neq i,j}\sum_{k\neq i,j,\ell} g^{b}_{\Delta,n}(V_i,V_j,V_\ell,V_k;\theta)$$
$$- \frac{1}{n}\times\left(\frac{1}{n^3}\sum_{j=1}^{n}\sum_{i\neq j}\sum_{\ell\neq i,j} g^{c}_{\Delta,n}(V_i,V_j,V_\ell;\theta)\right) - \frac{1}{n^3}\sum_{j=1}^{n}\sum_{i\neq j}\sum_{\ell\neq i,j} g^{d}_{\Delta,n}(V_i,V_j,V_\ell;\theta)$$
$$+ \frac{1}{n^3}\sum_{j=1}^{n}\sum_{i\neq j}\sum_{\ell\neq i,j} g^{e}_{\Delta,n}(V_i,V_j,V_\ell;\theta) + \frac{1}{n}\times\left(\frac{1}{n^2}\sum_{j=1}^{n}\sum_{i\neq j} g^{f}_{\Delta,n}(V_i,V_j;\theta)\right) + \frac{1}{n^2}\times\left(\frac{1}{n}\sum_{j=1}^{n} g^{g}_{\Delta,n}(V_j;\theta)\right) \tag{S1.58}$$

Let

$$\mathcal{V}^{a}_{\Delta,n}(\theta) \equiv \frac{1}{n\cdot(n-1)\cdot(n-2)}\sum_{j=1}^{n}\sum_{i\neq j}\sum_{\ell\neq i,j} g^{a}_{\Delta}(V_i,V_j,V_\ell;\theta),$$
$$\mathcal{V}^{b}_{\Delta,n}(\theta) \equiv \frac{1}{n\cdot(n-1)\cdot(n-2)\cdot(n-3)}\sum_{j=1}^{n}\sum_{i\neq j}\sum_{\ell\neq i,j}\sum_{k\neq i,j,\ell} g^{b}_{\Delta,n}(V_i,V_j,V_\ell,V_k;\theta),$$
$$\mathcal{V}^{c}_{\Delta,n}(\theta) \equiv \frac{1}{n\cdot(n-1)\cdot(n-2)}\sum_{j=1}^{n}\sum_{i\neq j}\sum_{\ell\neq i,j} g^{c}_{\Delta,n}(V_i,V_j,V_\ell;\theta),$$
$$\mathcal{V}^{d}_{\Delta,n}(\theta) \equiv \frac{1}{n\cdot(n-1)\cdot(n-2)}\sum_{j=1}^{n}\sum_{i\neq j}\sum_{\ell\neq i,j} g^{d}_{\Delta,n}(V_i,V_j,V_\ell;\theta),$$
$$\mathcal{V}^{e}_{\Delta,n}(\theta) \equiv \frac{1}{n\cdot(n-1)\cdot(n-2)}\sum_{j=1}^{n}\sum_{i\neq j}\sum_{\ell\neq i,j} g^{e}_{\Delta,n}(V_i,V_j,V_\ell;\theta),$$
$$\mathcal{V}^{f}_{\Delta,n}(\theta) \equiv \frac{1}{n\cdot(n-1)}\sum_{j=1}^{n}\sum_{i\neq j} g^{f}_{\Delta,n}(V_i,V_j;\theta),$$
$$\mathcal{V}^{g}_{\Delta,n}(\theta) \equiv \frac{1}{n}\sum_{j=1}^{n}\sum_{i\neq j} g^{g}_{\Delta,n}(V_j;\theta).$$

Then (S1.58) can be re-expressed as,

$$\frac{\partial \widehat{Q}_{S,\mathcal{Z}}(\theta)}{\partial \Delta} = \left(\frac{(n-1)(n-2)}{n^2}\right)\cdot\mathcal{V}^{a}_{\Delta,n}(\theta) - \left(\frac{(n-1)(n-2)(n-3)}{n^3}\right)\cdot\mathcal{V}^{b}_{\Delta,n}(\theta) - \left(\frac{1}{n}\right)\times\left(\frac{(n-1)(n-2)}{n^2}\right)\cdot\mathcal{V}^{c}_{\Delta,n}(\theta)$$
$$- \left(\frac{(n-1)(n-2)}{n^2}\right)\cdot\mathcal{V}^{d}_{\Delta,n}(\theta) + \left(\frac{(n-1)(n-2)}{n^2}\right)\cdot\mathcal{V}^{e}_{\Delta,n}(\theta) + \left(\frac{1}{n}\right)\times\left(\frac{n-1}{n}\right)\cdot\mathcal{V}^{f}_{\Delta,n}(\theta) + \left(\frac{1}{n^2}\right)\times\mathcal{V}^{g}_{\Delta,n}(\theta) \tag{S1.59}$$

Next, we will analyze each term in (S1.59). Let us begin with $\mathcal{V}^{c}_{\Delta,n}(\theta)$. By the bounded-variation properties of our kernel function described in Assumption E1 and the Lipschitz-restrictions in Assumption E3, Nolan and Pollard (1987, Lemma 22), Pakes and Pollard (1989, Example 2.10, Lemma 2.13, Lemma 2.14) imply that the class of functions involved is *Euclidean* (see Sherman

[1] Recall that $\mathbb{1}\{U_j \le u\} \equiv \mathbb{1}\{U_j \in \mathcal{Z}\}\times\mathbb{1}\{U_j \le u\}$, so $\mathbb{1}_{\mathcal{Z}}\{U_j \le U_j\} = \mathbb{1}\{U_j \in \mathcal{Z}\}$, which explains the structure of $g^{g}_{\Delta,n}(V_j,\theta)$.

(1994, Definition 3)), and given the existence of $2+\delta$-moments in Assumptions I1 and E3, and the boundedness restrictions in part (iii) of Assumption G1, the corresponding envelope for the class also has finite $2+\delta$-moments. From here, Sherman (1994, Corollary 4) can be invoked to show that $\sup_{\theta\in\Theta}\left|\mathcal{V}^c_{\Delta,n}(\theta)\right| = O_p\left(\frac{1}{h_n^{d_Z}}\right)$. Therefore,

$$\sup_{\theta\in\Theta}\left|\left(\frac{1}{n}\right)\times\underbrace{\left(\frac{(n-1)(n-2)}{n^2}\right)}_{\to 1}\cdot\mathcal{V}^c_{\Delta,n}(\theta)\right| = O_p\left(\frac{1}{n\cdot h_n^{d_Z}}\right) = o_p\left(n^{-1/2}\right), \tag{S1.60}$$

where the last equality follows from the bandwidth convergence restrictions in Assumption E1. Next, from the asymptotic properties of the remainders $\vartheta_n^{\pi}(z,\theta)$ and $\vartheta_n^{\nabla\Delta\pi}(z,\theta)$ described in (S1.11) and (S1.27), and the integrability and boundedness restrictions in Assumptions G1 and E3, we have[2]

$$\begin{aligned}
&\sup_{\theta\in\Theta}\left|\mathcal{V}^d_{\Delta,n}(\theta)\right| \leq \\
&\underbrace{\sup_{\substack{\theta\in\Theta\\ z\in\mathcal{Z}}}\left|\vartheta_n^{\nabla\Delta\pi}(z,\theta)\right|}_{=o_p\left(n^{-1/2}\right)}\times\underbrace{\sup_{\theta\in\Theta}\left(\frac{1}{n\cdot(n-1)\cdot(n-2)}\sum_{j=1}^{n}\sum_{i\neq j}\sum_{\ell\neq i,j}|\varphi_S(V_i,\theta)|\cdot\mathbb{1}_{\mathcal{Z}}\{U_i\leq U_j\}\cdot\mathbb{1}_{\mathcal{Z}}\{U_\ell\leq U_j\}\right)}_{=O_p(1)} \\
&+\underbrace{\sup_{\substack{\theta\in\Theta\\ z\in\mathcal{Z}}}\left|\vartheta_n^{\pi}(z,\theta)\right|}_{=o_p\left(n^{-1/2}\right)}\times\underbrace{\sup_{\theta\in\Theta}\left(\frac{1}{n\cdot(n-1)\cdot(n-2)}\sum_{j=1}^{n}\sum_{i\neq j}\sum_{\ell\neq i,j}|\varphi_S(V_i,\theta)|\cdot\left\|\frac{\partial\Xi_S(X_\ell,\theta)}{\partial\Delta}\right\|\cdot\mathbb{1}_{\mathcal{Z}}\{U_i\leq U_j\}\cdot\mathbb{1}_{\mathcal{Z}}\{U_\ell\leq U_j\}\right)}_{=O_p(1)} \\
&+\underbrace{\sup_{\substack{\theta\in\Theta\\ z\in\mathcal{Z}}}\left|\vartheta_n^{\pi}(z,\theta)\right|}_{=o_p\left(n^{-1/2}\right)}\times\underbrace{\sup_{\theta\in\Theta}\left(\frac{1}{n\cdot(n-1)\cdot(n-2)}\sum_{j=1}^{n}\sum_{i\neq j}\sum_{\ell\neq i,j}\left\|\frac{\partial\varphi(X_\ell,\theta)}{\partial\Delta}\right\|\cdot\mathbb{1}_{\mathcal{Z}}\{U_i\leq U_j\}\cdot\mathbb{1}_{\mathcal{Z}}\{U_\ell\leq U_j\}\right)}_{=O_p(1)} \\
&=o_p\left(n^{-1/2}\right).
\end{aligned}$$

Thus,

$$\sup_{\theta\in\Theta}\left|\mathcal{V}^d_{\Delta,n}(\theta)\right| = o_p\left(n^{-1/2}\right) \quad\Longrightarrow\quad \sup_{\theta\in\Theta}\left|\underbrace{\left(\frac{(n-1)(n-2)}{n^2}\right)}_{\to 1}\cdot\mathcal{V}^d_{\Delta,n}(\theta)\right| = o_p\left(n^{-1/2}\right) \tag{S1.61}$$

[2]Recall that $|\Xi_S(x,\theta)|\leq 1\ \forall\ (x,\theta)$.

Combining the uniform rate of convergence results in equations (S1.12) and (S1.28) with the boundedness and existence-of-moments restrictions in Assumptions G1, I1 and E3, we have

$$\begin{aligned}
&\sup_{\theta\in\Theta}\left|\mathcal{V}^{e}_{\Delta,n}(\theta)\right| \leq \\
&\underbrace{\underbrace{\sup_{\substack{\theta\in\Theta\\ z\in\mathcal{Z}}}|\widehat{\pi}(z,\theta)-\pi(z,\theta)|}_{=o_p\left(n^{-1/4}\right)}\times\underbrace{\sup_{\substack{\theta\in\Theta\\ z\in\mathcal{Z}}}\left|\frac{\partial\widehat{\pi}(z,\theta)}{\partial\Delta}-\frac{\partial\pi(z,\theta)}{\partial\Delta}\right|}_{=o_p\left(n^{-1/4}\right)}}_{=o_p\left(n^{-1/2}\right)} \\
&+\underbrace{\underbrace{\sup_{\substack{\theta\in\Theta\\ z\in\mathcal{Z}}}|\widehat{\pi}(z,\theta)-\pi(z,\theta)|}_{=o_p\left(n^{-1/4}\right)}\times\underbrace{\sup_{\substack{\theta\in\Theta\\ z\in\mathcal{Z}}}|\widehat{\pi}(z,\theta)-\pi(z,\theta)|}_{=o_p\left(n^{-1/4}\right)}}_{=o_p\left(n^{-1/2}\right)} \\
&\times\underbrace{\sup_{\substack{\theta\in\Theta\\ z\in\mathcal{Z}}}\left(\frac{1}{n\cdot(n-1)\cdot(n-2)}\sum_{j=1}^{n}\sum_{i\neq j}\sum_{\ell\neq i,j}\left\|\frac{\partial\Xi_S(X_\ell,\theta)}{\partial\Delta}\right\|\cdot\mathbb{1}_{\mathcal{Z}}\{U_i\leq U_j\}\cdot\mathbb{1}_{\mathcal{Z}}\{U_\ell\leq U_j\}\right)}_{=O_p(1)} \\
&=o_p\left(n^{-1/2}\right).
\end{aligned}$$

Thus,

$$\sup_{\theta\in\Theta}\left|\underbrace{\left(\frac{(n-1)(n-2)}{n^2}\right)}_{\to 1}\cdot\mathcal{V}^{e}_{\Delta,n}(\theta)\right| \tag{S1.62}$$

Next we analyze $\mathcal{V}^{f}_{\Delta,n}(\theta)$. The results in equations (S1.42)-(S1.49) yield

$$\sup_{\theta\in\Theta}\left|\mathcal{V}^{f}_{\Delta,n}(\theta)\right| = \sup_{\theta\in\Theta}\left|\frac{1}{n\cdot(n-1)}\sum_{j=1}^{n}\sum_{i\neq j}g^{f}_{\Delta,n}(V_i,V_j;\theta)\right| = O_p(1).$$

$$\begin{aligned}
&\sup_{\theta\in\Theta}\left|\mathcal{V}^{f}_{\Delta,n}(\theta)\right| \\
&=\sup_{\theta\in\Theta}\left|\frac{1}{n\cdot(n-1)}\sum_{j=1}^{n}\sum_{i\neq j}\left(\widehat{\varphi}_S(V_i,\theta)\cdot\frac{\partial\widehat{\varphi}_S(V_i,\theta)}{\partial\Delta}\cdot\mathbb{1}_{\mathcal{Z}}\{U_i\leq U_j\}+\widehat{\varphi}_S(V_j,\theta)\cdot\frac{\partial\widehat{\varphi}_S(V_i,\theta)}{\partial\Delta}\cdot\mathbb{1}_{\mathcal{Z}}\{U_i\leq U_j\}\right.\right. \\
&\qquad\left.\left.+\,\widehat{\varphi}_S(V_i,\theta)\cdot\frac{\partial\widehat{\varphi}_S(V_j,\theta)}{\partial\Delta}\cdot\mathbb{1}_{\mathcal{Z}}\{U_i\leq U_j\}\right)\right| = O_p(1).
\end{aligned}$$

Therefore,

$$\sup_{\theta\in\Theta}\left|\left(\frac{1}{n}\right)\times\underbrace{\left(\frac{n-1}{n}\right)}_{\to 1}\cdot\mathcal{V}^{f}_{\Delta,n}(\theta)\right| = O_p\left(\frac{1}{n}\right) = o_p\left(n^{-1/2}\right), \tag{S1.63}$$

For $\mathcal{V}^{g}_{\Delta,n}(\theta)$, we note once again that the results in equations (S1.42)-(S1.49) yield,

$$\sup_{\theta\in\Theta}\left|\mathcal{V}^{g}_{\Delta,n}(\theta)\right| = \sup_{\theta\in\Theta}\left|\frac{1}{n}\sum_{j=1}^{n}\widehat{\varphi}_S(V_j,\theta)\cdot\frac{\partial\widehat{\varphi}_S(V_j,\theta)}{\partial\Delta}\cdot\mathbb{1}\{Z_j\in\mathcal{Z}\}\right| = O_p(1).$$

Thus,

$$\sup_{\theta\in\Theta}\left|\left(\frac{1}{n^2}\right)\times\mathcal{V}^{g}_{\Delta,n}(\theta)\right| = O_p\left(\frac{1}{n^2}\right) = o_p\left(n^{-1/2}\right) \tag{S1.64}$$

Next, we will analyze the U-statistics $\mathcal{V}^{a}_{\Delta,n}(\theta)$ and $\mathcal{V}^{b}_{\Delta,n}(\theta)$ through their *Hoeffding decompositions* (see Serfling (1980, pages 177-178) or Sherman (1994, equations (6)-(7))). Let us begin with $\mathcal{V}^{a}_{\Delta,n}(\theta)$. Let

$$\mathcal{T}(\theta) \equiv E\left[g^{a}_{\Delta}(V_1,V_2,V_3;\theta)\right].$$

Note that $\mathcal{T}(\theta_0) = 0$ since,

$$\begin{aligned}
\mathcal{T}(\theta_0) &= E\left[\varphi_S(V_1,\theta_0)\cdot\frac{\partial\varphi_S(V_3,\theta_0)}{\partial\Delta}\cdot\mathbb{1}_{\mathcal{Z}}\{U_1\le U_2\}\cdot\mathbb{1}_{\mathcal{Z}}\{U_3\le U_2\}\right]\\
&= E\Bigg[\underbrace{E\left[\varphi_S(V_1,\theta_0)\cdot\mathbb{1}_{\mathcal{Z}}\{U_1\le U_2\}\,\middle|U_2\right]}_{\equiv\tau_{\mathcal{Z}}(U_2,\theta_0)=0\ \text{ a.e } U_2}\cdot\frac{\partial\varphi_S(V_3,\theta_0)}{\partial\Delta}\cdot\mathbb{1}_{\mathcal{Z}}\{U_3\le U_2\}\Bigg]\\
&= E\left[0\cdot\frac{\partial\varphi_S(V_3,\theta_0)}{\partial\Delta}\cdot\mathbb{1}_{\mathcal{Z}}\{U_3\le U_2\}\right]\\
&= 0.
\end{aligned}$$

Next, define,

$$\widetilde{g}^{a}_{\Delta}(V_1,V_2,V_3;\theta) \equiv \frac{1}{3!}\sum_{p} g^{a}_{\Delta}(V_{m_1},V_{m_2},V_{m_3};\theta), \tag{S1.65}$$

where $\sum_{p}$ denotes the sum over the 3! permutations $\{m_1,m_2,m_3\}$ of $\{1,2,3\}$. By construction, $\widetilde{g}^{a}_{\Delta}(V_1,V_2,V_3;\theta)$ is symmetric in (V_1,V_2,V_3), and $E\left[\widetilde{g}^{a}_{\Delta}(V_1,V_2,V_3;\theta)\right] = \mathcal{T}(\theta)$. We can express,

$$\mathcal{V}^{a}_{\Delta,n}(\theta) = \binom{n}{3}^{-1}\sum_{c}\widetilde{g}^{a}_{\Delta}(V_i,V_j,V_\ell;\theta),$$

where $\sum_{c}$ denotes the sum over the $\binom{n}{3}$ distinct combinations $\{i,j,\ell\}$ from $\{1,\ldots,n\}$. Let

$$\overline{g}^{a}_{\Delta}(V_1;\theta) \equiv 3\cdot\left(E\left[\widetilde{g}^{a}_{\Delta}(V_1,V_2,V_3;\theta)\,\middle|V_1\right]-\mathcal{T}(\theta)\right) \tag{S1.66}$$

Note that, by construction, $E\left[\overline{g}^a_\Delta(V_1;\theta)\right] = 0 \; \forall \; \theta$. Also note that evaluated at θ_0, we have,

$$\overline{g}^a_\Delta(V_1;\theta_0) = \varphi_S(V_1,\theta_0) \cdot E\left[E\left[\frac{\partial \varphi_S(V_3,\theta_0)}{\partial \Delta} \cdot \mathbb{1}_{\mathcal{Z}}\{U_3 \le U_2\} \,\Big|\, U_2\right] \cdot \mathbb{1}\{U_1 \le U_2\} \,\Big|\, U_1\right] \tag{S1.67}$$

The Hoeffding decomposition of $\mathcal{V}^a_{\Delta,n}(\theta)$ (see Serfling (1980, pages 177-178) is given by,

$$\mathcal{V}^a_{\Delta,n}(\theta) = \mathcal{T}(\theta) + \frac{1}{n}\sum_{i=1}^{n} \overline{g}^a_\Delta(V_i;\theta) + \mathcal{U}^a_n(\theta),$$

where $\mathcal{U}^a_n(\theta)$ is a linear combination of *degenerate* U-statistics of orders 2 and 3. By the Lipschitz-restrictions in Assumption E3, Nolan and Pollard (1987, Lemma 22), Pakes and Pollard (1989, Example 2.10, Lemma 2.13, Lemma 2.14) imply that the class of functions involved is *Euclidean* (see Sherman (1994, Definition 3)), and given the existence of $2+\delta$-moments in Assumptions I1 and E3, and the boundedness restrictions in part (iii) of Assumption G1, the corresponding envelope for the class also has finite $2+\delta$-moments. From here, Sherman (1994, Corollary 4) yields,

$$\sup_{\theta\in\Theta} |\mathcal{U}^a_n(\theta)| = O_p\left(\frac{1}{n}\right) = o_p\left(n^{-1/2}\right).$$

Thus,

$$\mathcal{V}^a_{\Delta,n}(\theta) = \mathcal{T}(\theta) + \frac{1}{n}\sum_{i=1}^{n} \overline{g}^a_\Delta(V_i;\theta) + \mathcal{U}^a_n(\theta), \quad \text{where} \quad \sup_{\theta\in\Theta} |\mathcal{U}^a_n(\theta)| = o_p\left(n^{-1/2}\right). \tag{S1.68}$$

And evaluated at θ_0, we have $\mathcal{T}(\theta_0) = 0$, and

$$\begin{aligned} \mathcal{V}^a_{\Delta,n}(\theta_0) &= \frac{1}{n}\sum_{i=1}^{n} \overline{g}^a_\Delta(V_i;\theta_0) + o_p\left(n^{-1/2}\right), \quad \text{where} \\ \overline{g}^a_\Delta(V_1;\theta_0) &= \varphi_S(V_1,\theta_0) \cdot E\left[E\left[\frac{\partial \varphi_S(V_3,\theta_0)}{\partial \Delta} \cdot \mathbb{1}_{\mathcal{Z}}\{U_3 \le U_2\} \,\Big|\, U_2\right] \cdot \mathbb{1}\{U_1 \le U_2\} \,\Big|\, U_1\right] \end{aligned} \tag{S1.69}$$

$\sup_{\theta\in\Theta} |\mathcal{T}(\theta)| < \infty$ under our restrictions. Thus, from (S1.68), we have $\sup_{\theta\in\Theta} \left|\mathcal{V}^a_{\Delta,n}(\theta)\right| = O_p(1)$ and,

$$\sup_{\theta\in\Theta} \left| \underbrace{\left(\frac{2-3n}{n^2}\right)}_{=O\left(\frac{1}{n}\right)} \cdot \mathcal{V}^a_{\Delta,n}(\theta) \right| = O_p\left(\frac{1}{n}\right).$$

From here, it follows that,

$$\left(\frac{(n-1)(n-2)}{n^2}\right)\cdot \mathcal{V}^a_{\Delta,n}(\theta) = \mathcal{V}^a_{\Delta,n}(\theta) + \underbrace{\left(\frac{2-3n}{n^2}\right)\cdot \mathcal{V}^a_{\Delta,n}(\theta)}_{\equiv \vartheta^a_n(\theta)} = \mathcal{T}(\theta) + \frac{1}{n}\sum_{i=1}^{n} \overline{g}^a_{\Delta}(V_i;\theta) + \vartheta^a_n(\theta),$$

$$\text{where} \quad \sup_{\theta\in\Theta} |\vartheta^a_n(\theta)| = O_p\left(\frac{1}{n}\right) = o_p\left(n^{-1/2}\right).$$

In particular, evaluated at θ_0,

$$\left(\frac{(n-1)(n-2)}{n^2}\right)\cdot \mathcal{V}^a_{\Delta,n}(\theta_0) = \frac{1}{n}\sum_{i=1}^{n} \overline{g}^a_{\Delta}(V_i;\theta_0) + o_p\left(n^{-1/2}\right), \quad \text{where}$$

$$\overline{g}^a_{\Delta}(V_1;\theta_0) = \varphi_S(V_1,\theta_0)\cdot E\left[E\left[\frac{\partial \varphi_S(V_3,\theta_0)}{\partial \Delta}\cdot \mathbb{1}_{\mathcal{Z}}\{U_3 \le U_2\}\,\Big|\, U_2\right]\cdot \mathbb{1}_{\mathcal{Z}}\{U_1 \le U_2\}\,\Big|\, U_1\right] \tag{S1.70}$$

Next we analyze the Hoeffding decomposition of $\mathcal{V}^b_{\Delta,n}(\theta)$, the last remaining term to study in (S1.59). Note first that, from its definition in (S1.57), we have

$$E\left[g^b_{\Delta,n}(V_1,V_2,V_3,V_4;\theta)\,\Big|\, V_1,V_2,V_3\right] = 0 \;\forall\; \theta,$$

and therefore $E\left[g^b_{\Delta,n}(V_1,V_2,V_3,V_4;\theta)\right] = 0 \;\forall\; \theta$. This follows because

$$E\left[\psi^{\nabla\Delta\pi}_n(V_4;Z_3,\theta)\,\Big|\, Z_3\right] = E\left[\psi^{\pi}_n(V_4;Z_3,\theta)\,\Big|\, Z_3\right] = E\left[\psi^{\pi}_n(V_4;Z_1,\theta)\,\Big|\, Z_1\right] = 0 \;\forall\; \theta.$$

Define,

$$\widetilde{g}^b_{\Delta,n}(V_1,V_2,V_3,V_4;\theta) \equiv \frac{1}{4!}\sum_{p} g^b_{\Delta,n}(V_{m_1},V_{m_2},V_{m_3},V_{m_4};\theta), \tag{S1.71}$$

where $\sum_p$ denotes the sum over the 4! permutations $\{m_1,m_2,m_3,m_4\}$ of $\{1,2,3,4\}$. By construction, $\widetilde{g}^b_{\Delta,n}(V_1,V_2,V_3,V_4;\theta)$ is symmetric in (V_1,V_2,V_3,V_4), and $E\left[\widetilde{g}^b_{\Delta,n}(V_1,V_2,V_3,V_4;\theta)\right] = 0 \;\forall\; \theta$. We can express,

$$\mathcal{V}^b_{\Delta,n}(\theta) = \binom{n}{4}^{-1}\sum_{c} \widetilde{g}^b_{\Delta,n}(V_i,V_j,V_\ell,V_k;\theta),$$

where $\sum_c$ denotes the sum over the $\binom{n}{4}$ distinct combinations $\{i,j,\ell,k\}$ from $\{1,\dots,n\}$. Let

$$\overline{g}^b_{\Delta,n}(V_1;\theta) \equiv 4\cdot E\left[\widetilde{g}^b_{\Delta,n}(V_1,V_2,V_3,V_4;\theta)\,\Big|\, V_1\right].$$

By inspection, we can see that,

$$
\begin{aligned}
&E\left[\overline{g}^b_{\Delta,n}(V_1,V_2,V_3,V_4;\theta)\,\Big|V_1\right] = \frac{3!}{4!}E\left[g^b_{\Delta,n}(V_2,V_3,V_4,V_1;\theta)\,\Big|V_1\right] \\
=&\frac{1}{4}\times\Bigg(E\left[\varphi_S(V_2,\theta)\cdot\Xi_S(X_4,\theta)\cdot\frac{1}{h_n^{d_Z}}\psi_n^{\nabla\Delta\pi}(V_1;Z_4,\theta)\cdot\mathbb{1}_{\mathcal{Z}}\{U_2\le U_3\}\cdot\mathbb{1}_{\mathcal{Z}}\{U_4\le U_3\}\,\Big|V_1\right] \\
&+E\left[\frac{\partial\Xi_S(X_4,\theta)}{\partial\Delta}\cdot\varphi_S(V_2,\theta)\cdot\frac{1}{h_n^{d_Z}}\psi_n^{\pi}(V_1;Z_4,\theta)\cdot\mathbb{1}_{\mathcal{Z}}\{U_2\le U_3\}\cdot\mathbb{1}_{\mathcal{Z}}\{U_4\le U_3\}\,\Big|V_1\right] \\
&+E\left[\frac{\partial\varphi_S(V_4,\theta)}{\partial\Delta}\cdot\Xi_S(X_2,\theta)\cdot\frac{1}{h_n^{d_Z}}\psi_n^{\pi}(V_1;Z_2,\theta)\cdot\mathbb{1}_{\mathcal{Z}}\{U_2\le U_3\}\cdot\mathbb{1}_{\mathcal{Z}}\{U_4\le U_3\}\,\Big|V_1\right]\Bigg)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\overline{g}^b_{\Delta,n}(V_1;\theta) =& E\left[\varphi_S(V_2,\theta)\cdot\Xi_S(X_4,\theta)\cdot\frac{1}{h_n^{d_Z}}\psi_n^{\nabla\Delta\pi}(V_1;Z_4,\theta)\cdot\mathbb{1}_{\mathcal{Z}}\{U_2\le U_3\}\cdot\mathbb{1}_{\mathcal{Z}}\{U_4\le U_3\}\,\Big|V_1\right] \\
&+E\left[\frac{\partial\Xi_S(X_4,\theta)}{\partial\Delta}\cdot\varphi_S(V_2,\theta)\cdot\frac{1}{h_n^{d_Z}}\psi_n^{\pi}(V_1;Z_4,\theta)\cdot\mathbb{1}_{\mathcal{Z}}\{U_2\le U_3\}\cdot\mathbb{1}_{\mathcal{Z}}\{U_4\le U_3\}\,\Big|V_1\right] \\
&+E\left[\frac{\partial\varphi_S(V_4,\theta)}{\partial\Delta}\cdot\Xi_S(X_2,\theta)\cdot\frac{1}{h_n^{d_Z}}\psi_n^{\pi}(V_1;Z_2,\theta)\cdot\mathbb{1}_{\mathcal{Z}}\{U_2\le U_3\}\cdot\mathbb{1}_{\mathcal{Z}}\{U_4\le U_3\}\,\Big|V_1\right].
\end{aligned}
\tag{S1.72}
$$

Note that, evaluated at θ_0,

$$
\begin{aligned}
&E\left[\varphi_S(V_2,\theta_0)\cdot\Xi_S(X_4,\theta_0)\cdot\frac{1}{h_n^{d_Z}}\psi_n^{\nabla\Delta\pi}(V_1;Z_4,\theta_0)\cdot\mathbb{1}_{\mathcal{Z}}\{U_2\le U_3\}\cdot\mathbb{1}_{\mathcal{Z}}\{U_4\le U_3\}\,\Big|V_1\right] \\
=&E\Bigg[\underbrace{E\left[\varphi_S(V_2,\theta_0)\Big|U_2\right]}_{=0}\cdot\Xi_S(X_4,\theta_0)\cdot\frac{1}{h_n^{d_Z}}\psi_n^{\nabla\Delta\pi}(V_1;Z_4,\theta_0)\cdot\mathbb{1}_{\mathcal{Z}}\{U_2\le U_3\}\cdot\mathbb{1}_{\mathcal{Z}}\{U_4\le U_3\}\,\Big|V_1\Bigg]=0, \\
&E\left[\frac{\partial\Xi_S(X_4,\theta_0)}{\partial\Delta}\cdot\varphi_S(V_2,\theta_0)\cdot\frac{1}{h_n^{d_Z}}\psi_n^{\pi}(V_1;Z_4,\theta_0)\cdot\mathbb{1}_{\mathcal{Z}}\{U_2\le U_3\}\cdot\mathbb{1}_{\mathcal{Z}}\{U_4\le U_3\}\,\Big|V_1\right] \\
=&E\Bigg[\frac{\partial\Xi_S(X_4,\theta_0)}{\partial\Delta}\cdot\underbrace{E\left[\varphi_S(V_2,\theta_0)\Big|U_2\right]}_{=0}\cdot\frac{1}{h_n^{d_Z}}\psi_n^{\pi}(V_1;Z_4,\theta_0)\cdot\mathbb{1}_{\mathcal{Z}}\{U_2\le U_3\}\cdot\mathbb{1}_{\mathcal{Z}}\{U_4\le U_3\}\,\Big|V_1\Bigg]=0.
\end{aligned}
$$

Thus, evaluated at θ_0,

$$
\begin{aligned}
\overline{g}^b_{\Delta,n}(V_1;\theta_0) =& E\left[\frac{\partial\varphi_S(V_4,\theta_0)}{\partial\Delta}\cdot\Xi_S(X_2,\theta_0)\cdot\frac{1}{h_n^{d_Z}}\psi_n^{\pi}(V_1;Z_2,\theta_0)\cdot\mathbb{1}_{\mathcal{Z}}\{U_2\le U_3\}\cdot\mathbb{1}_{\mathcal{Z}}\{U_4\le U_3\}\,\Big|V_1\right] \\
=& E\left[\frac{\partial\varphi_S(V_4,\theta_0)}{\partial\Delta}\cdot E\left[\Xi_S(X_2,\theta_0)\cdot\frac{1}{h_n^{d_Z}}\psi_n^{\pi}(V_1;Z_2,\theta_0)\cdot\mathbb{1}_{\mathcal{Z}}\{U_2\le U_3\}\Big|V_1,U_3\right]\cdot\mathbb{1}_{\mathcal{Z}}\{U_4\le U_3\}\,\Big|V_1\right].
\end{aligned}
\tag{S1.73}
$$

The Hoeffding decomposition of $\mathcal{V}^b_{\Delta,n}(\theta)$ is given by,

$$\mathcal{V}^b_{\Delta,n}(\theta) = \frac{1}{n}\sum_{i=1}^{n} \overline{g}^b_{\Delta,n}(V_i;\theta) + \mathcal{U}^b_n(\theta),$$

where $\mathcal{U}^b_n(\theta)$ is a linear combination of degenerate U-statistics of orders 2, 3 and 4. By the bounded-variation properties of our kernel function described in Assumption E1 and the Lipschitz-restrictions in Assumption E3, Nolan and Pollard (1987, Lemma 22), Pakes and Pollard (1989, Example 2.10, Lemma 2.13, Lemma 2.14) imply that the class of functions involved is *Euclidean* (see Sherman (1994, Definition 3)), and given the existence of $2+\delta$-moments in Assumptions I1 and E3, and the boundedness restrictions in part (iii) of Assumption G1, the corresponding envelope for the class also has finite $2+\delta$-moments. From here, Sherman (1994, Corollary 4) yields

$$\sup_{\theta\in\Theta} |\mathcal{U}^b_n(\theta)| = O_p\left(\frac{1}{n\cdot h_n^{d_Z}}\right) = o_p\left(n^{-1/2}\right),$$

where the last equality follows from our bandwidth convergence restrictions since $n^{1/2}\cdot h_n^{d_Z} \longrightarrow \infty$. Thus,

$$\mathcal{V}^b_{\Delta,n}(\theta) = \frac{1}{n}\sum_{i=1}^{n} \overline{g}^b_{\Delta,n}(V_i;\theta) + \mathcal{U}^b_n(\theta), \quad \text{where} \quad \sup_{\theta\in\Theta} |\mathcal{U}^b_n(\theta)| = o_p\left(n^{-1/2}\right),$$

and where $\overline{g}^b_{\Delta,n}(V_1;\theta)$ is as described in (S1.72), and it satisfies $E\left[\overline{g}^b_{\Delta,n}(V_1;\theta)\right] = 0 \;\forall\; \theta$. Let us focus on $\theta = \theta_0$. Note first that,

$$\left(\frac{(n-1)(n-2)(n-3)}{n^3}\right)\cdot \mathcal{V}^b_{\Delta,n}(\theta_0) = \mathcal{V}^b_{\Delta,n}(\theta_0) + \underbrace{\underbrace{\left(\frac{-6n^2+11n-6}{n^3}\right)}_{=O\left(\frac{1}{n}\right)}\cdot \underbrace{\mathcal{V}^b_{\Delta,n}(\theta_0)}_{=O_p(1)}}_{=O_p\left(\frac{1}{n}\right)=o_p\left(n^{-1/2}\right)}$$

Thus,

$$\left(\frac{(n-1)(n-2)(n-3)}{n^3}\right)\cdot \mathcal{V}^b_{\Delta,n}(\theta_0) = \frac{1}{n}\sum_{i=1}^{n} \overline{g}^b_{\Delta,n}(V_i;\theta_0) + o_p(n^{-1/2}) \tag{S1.74}$$

where $\overline{g}^b_{\Delta,n}(V_1;\theta_0)$ is as described in (S1.73). We are now ready to put together our results for $\frac{\partial \widehat{Q}_{S,\mathcal{Z}}(\theta_0)}{\partial\Delta}$. As before, let $(V_1,V_2,V_3,V_4) \sim F_V \otimes F_V \otimes F_V \otimes F_V$ (four randomly drawn observations from our i.i.d sample $(V_i)_{i=1}^n$). Combining the results in (S1.60), (S1.61), (S1.62), (S1.63), (S1.64), (S1.70) and (S1.74), and plugging them into equation (S1.59), at $\theta = \theta_0$ we have,

$$\begin{aligned}
\frac{\partial \widehat{Q}_{S,\mathcal{Z}}(\theta_0)}{\partial \Delta} &= \frac{1}{n}\sum_{i=1}^{n}\Gamma_{\Delta,n}(V_i;\theta_0) + o_p\left(n^{-1/2}\right), \quad \text{where} \\
\Gamma_{\Delta,n}(V_1;\theta_0) &\equiv \overline{g}^a_{\Delta}(V_1;\theta_0) - \overline{g}^b_{\Delta,n}(V_1;\theta_0) \\
&= \varphi_S(V_1,\theta_0)\cdot E\left[E\left[\frac{\partial \varphi_S(V_3,\theta_0)}{\partial \Delta}\cdot \mathbb{1}_{\mathcal{Z}}\{U_3 \le U_2\}\,\middle|\, U_2\right]\cdot \mathbb{1}_{\mathcal{Z}}\{U_1 \le U_2\}\,\middle|\, U_1\right] \\
&\quad - E\left[\frac{\partial \varphi_S(V_4,\theta_0)}{\partial \Delta}\cdot E\left[\Xi_S(X_2,\theta_0)\cdot \frac{1}{h_n^{d_Z}}\psi_n^{\pi}(V_1;Z_2,\theta_0)\cdot \mathbb{1}_{\mathcal{Z}}\{U_2 \le U_3\}\,\middle|\, V_1,U_3\right]\cdot \mathbb{1}_{\mathcal{Z}}\{U_4 \le U_3\}\,\middle|\, V_1\right].
\end{aligned} \tag{S1.75}$$

Note, once again, that $E\left[\Gamma_{\Delta,n}(V_1;\theta_0)\right] = 0$.

S1.5.2 A linear representation result for $\widehat{\Delta}$ (final step in the proof of Proposition 1)

Let

$$\psi_{\Delta,n}(V_i;\theta_0) \equiv -\frac{\partial^2 Q_{S,\mathcal{Z}}(\theta_0)}{\partial \Delta \partial \Delta'}^{-1} \times \left(\Gamma_{\Delta,n}(V_i;\theta_0) + \frac{\partial^2 Q_{S,\mathcal{Z}}(\theta_0)}{\partial \Delta \partial \gamma'}\cdot \psi_{\gamma}(V_i;\gamma_0)\right) \tag{S1.76}$$

Note that $E\left[\psi_{\Delta,n}(V;\theta_0)\right] = 0$. From equation (S1.54), our result in (S1.75) yields,

$$\widehat{\Delta} - \Delta_0 = \frac{1}{n}\sum_{i=1}^{n}\psi_{\Delta,n}(V_i;\theta_0) + o_p\left(n^{-1/2}\right), \tag{S1.77}$$

and $\sqrt{n}\cdot\left(\widehat{\Delta}-\Delta_0\right) \xrightarrow{d} \mathcal{N}(0,\Omega_\Delta)$, where $\Omega_\Delta = \lim_{n\to\infty} E\left[\psi_{\Delta,n}(V;\theta_0)\cdot \psi_{\Delta,n}(V;\theta_0)'\right]$. This is the result described in Proposition 1. ■

S1.6 An estimator for Ω_Δ, the asymptotic variance of $\sqrt{n}\cdot(\widehat{\Delta}-\Delta_0)$

Based on our result in Section S1.5.2, our proposal is to estimate Ω_Δ, the asymptotic variance of $\sqrt{n}\cdot(\widehat{\Delta}-\Delta_0)$, as $\widehat{\Omega}_\Delta \equiv \frac{1}{n}\sum_{i=1}^{n}\widehat{\psi}_{\Delta,n}(V_i;\widehat{\theta})\cdot \widehat{\psi}_{\Delta,n}(V_i;\widehat{\theta})'$, where $\widehat{\psi}_{\Delta,n}(V_i;\widehat{\theta})$ is an estimator of the influence function $\psi_{\Delta,n}(V_i;\widehat{\theta})$. We estimate the influence function $\psi^{\pi}_n(V_i;z,\theta)$ described in equations (S1.7)-(S1.11) as follows. Let,

$$\begin{aligned}
\widehat{R}^{\pi}_a(z,\theta) &\equiv \frac{1}{n\cdot h_n^{d_Z}}\sum_{i=1}^{n}\left(S_i - m_S^{NC}(X_i,\theta)\right)\cdot \Xi_S(X_i,\theta)\cdot K\left(\frac{Z_i - z}{h_n}\right), \\
\widehat{R}^{\pi}_b(z,\theta) &\equiv \frac{1}{n\cdot h_n^{d_Z}}\sum_{i=1}^{n}\Xi_S(X_i,\theta)^2\cdot K\left(\frac{Z_i - z}{h_n}\right), \\
\widehat{\psi}^{\pi}_{a,n}(V_i;z,\theta) &\equiv \left(S_i - m_S^{NC}(X_i,\theta)\right)\cdot \Xi_S(X_i,\theta)\cdot K\left(\frac{Z_i - z}{h_n}\right) - h_n^{d_Z}\cdot \widehat{R}^{\pi}_a(z,\theta), \\
\widehat{\psi}^{\pi}_{b,n}(U_i;z,\theta) &\equiv \Xi_S(X_i,\theta)^2\cdot K\left(\frac{Z_i - z}{h_n}\right) - h_n^{d_Z}\cdot \widehat{R}^{\pi}_b(z,\theta), \\
\widehat{\psi}^{\pi}_{n}(V_i;z,\theta) &\equiv \frac{1}{\widehat{R}^{\pi}_b(z,\theta)}\cdot \widehat{\psi}^{\pi}_{a,n}(V_i;z,\theta) - \frac{\widehat{\pi}(z,\theta)}{\widehat{R}^{\pi}_b(z,\theta)}\cdot \widehat{\psi}^{\pi}_{b,n}(U_i;z,\theta)
\end{aligned}$$

Next, we have[3]

$$\frac{\partial^2 \widehat{Q}_{S,\mathcal{Z}}(\theta)}{\partial\theta\partial\theta'} = \frac{1}{n}\sum_{j=1}^{n}\left(\frac{\partial^2\widehat{\tau}_{\mathcal{Z}}(U_j,\theta)}{\partial\theta\partial\theta'}\cdot\widehat{\tau}_{\mathcal{Z}}(U_j,\theta)+\frac{\partial\widehat{\tau}_{\mathcal{Z}}(U_j,\theta)}{\partial\theta}\cdot\frac{\partial\widehat{\tau}_{\mathcal{Z}}(U_j,\theta)'}{\partial\theta}\right).$$

Where,

$$\frac{\partial\widehat{\tau}_{\mathcal{Z}}(u,\theta)}{\partial\theta} = \frac{1}{n}\sum_{i=1}^{n}\frac{\partial\widehat{\varphi}_S(V_i,\theta)}{\partial\theta}\cdot\mathbb{1}_{\mathcal{Z}}\{U_i\le u\},\quad \frac{\partial^2\widehat{\tau}_{\mathcal{Z}}(u,\theta)}{\partial\theta\partial\theta'} = \frac{1}{n}\sum_{i=1}^{n}\frac{\partial^2\widehat{\varphi}_S(V_i,\theta)}{\partial\theta\partial\theta'}\cdot\mathbb{1}_{\mathcal{Z}}\{U_i\le u\}.$$

And,

$$\begin{aligned}
\widehat{\varphi}_S(V_i,\theta) &\equiv S_i - m_S^{NC}(X_i,\theta)-\widehat{\pi}(Z_i,\theta)\cdot\Xi_S(X_i,\theta),\\
\frac{\partial\widehat{\varphi}_S(V_i,\theta)}{\partial\theta} &= -\left(\frac{\partial m_S^{NC}(X_i,\theta)}{\partial\theta}+\frac{\partial\widehat{\pi}(Z_i,\theta)}{\partial\theta}\cdot\Xi_S(X_i,\theta)+\frac{\partial\Xi_S(X_i,\theta)}{\partial\theta}\cdot\widehat{\pi}(Z_i,\theta)\right],\\
\frac{\partial^2\widehat{\varphi}_S(V_i,\theta)}{\partial\theta\partial\theta'} &= -\left(\frac{\partial^2 m_S^{NC}(X_i,\theta)}{\partial\theta\partial\theta'}+\frac{\partial\widehat{\pi}(Z_i,\theta)}{\partial\theta}\cdot\frac{\partial\Xi_S(X_i,\theta)'}{\partial\theta}+\frac{\partial^2\widehat{\pi}(Z_i,\theta)}{\partial\theta\partial\theta'}\cdot\Xi_S(X_i,\theta)\right.\\
&\qquad\left.+\frac{\partial\Xi_S(X_i,\theta)}{\partial\theta}\cdot\frac{\partial\widehat{\pi}(Z_i,\theta)'}{\partial\theta}+\frac{\partial^2\Xi_S(X_i,\theta)}{\partial\theta\partial\theta'}\cdot\widehat{\pi}(Z_i,\theta)\right).
\end{aligned}$$

The Jacobian $\frac{\partial\widehat{\pi}(Z_i,\theta)}{\partial\theta}$ and Hessian $\frac{\partial^2\widehat{\pi}(Z_i,\theta)}{\partial\theta\partial\theta'}$ of our estimated weights $\widehat{\pi}(Z_i,\theta)$ are as described in equations (S1.13)-(S1.33). For $m\in\mathbb{N}$ let $(m)_k\equiv m\cdot(m-1)\cdots(m-k)$. From the definition in (A8), and the definition of $\Gamma_{\Delta,n}(V_1;\theta)$ in (S1.75), we estimate

$$\begin{aligned}
\widehat{\Gamma}_{\Delta,n}(V_i;\theta) &= \widehat{\varphi}_S(V_i,\theta)\cdot\frac{1}{(n-1)_1}\sum_{j\neq i}\sum_{k\neq i,j}\frac{\partial\widehat{\varphi}_S(V_k,\theta)}{\partial\Delta}\cdot\mathbb{1}_{\mathcal{Z}}\{U_k\le U_j\}\cdot\mathbb{1}_{\mathcal{Z}}\{U_i\le U_j\}\\
&-\frac{1}{(n-1)_2}\sum_{j\neq i}\sum_{k\neq i,j}\sum_{\ell\neq i,j,k}\frac{\partial\widehat{\varphi}_S(V_\ell,\theta)}{\partial\Delta}\cdot\Xi_S(X_j,\theta)\cdot\frac{1}{h_n^{d_Z}}\widehat{\psi}_n^{\pi}(V_i;Z_j,\theta)\cdot\mathbb{1}_{\mathcal{Z}}\{U_j\le U_k\}\cdot\mathbb{1}_{\mathcal{Z}}\{U_\ell\le U_k\}
\end{aligned}$$

Let $\widehat{\psi}_\gamma(V_i;\widehat{\gamma})$ be the estimated MLE influence function for $\widehat{\gamma}$. Using (A9), we estimate $\psi_{\Delta,n}(V_i;\theta_0)$ with,

$$\widehat{\psi}_{\Delta,n}(V_i;\widehat{\theta})\equiv -\frac{\partial^2\widehat{Q}_{S,\mathcal{Z}}(\widehat{\theta})}{\partial\Delta\partial\Delta'}^{-1}\times\left(\widehat{\Gamma}_{\Delta,n}(V_i;\widehat{\theta})+\frac{\partial^2\widehat{Q}_{S,\mathcal{Z}}(\widehat{\theta})}{\partial\Delta\partial\gamma'}\cdot\widehat{\psi}_\gamma(V_i;\widehat{\gamma})\right) \tag{S1.78}$$

From here, we estimate Ω_Δ, the asymptotic variance of $\sqrt{n}\cdot(\widehat{\Delta}-\Delta_0)$, as $\widehat{\Omega}_\Delta\equiv\frac{1}{n}\sum_{i=1}^n\widehat{\psi}_{\Delta,n}(V_i;\widehat{\theta})\cdot\widehat{\psi}_{\Delta,n}(V_i;\widehat{\theta})'$. Using the results in Sections S1.1, S1.2, S1.3, S1.4, we see that under the restrictions of Proposition 1, we have $\left\|\frac{1}{n}\sum_{i=1}^n\widehat{\psi}_{\Delta,n}(V_i;\widehat{\theta})\cdot\widehat{\psi}_{\Delta,n}(V_i;\widehat{\theta})'-\frac{1}{n}\sum_{i=1}^n\psi_{\Delta,n}(V_i;\theta_0)\cdot\psi_{\Delta,n}(V_i;\theta_0)'\right\|\xrightarrow{p}0$, so $\left\|\widehat{\Omega}_\Delta-\Omega_\Delta\right\|\xrightarrow{p}0$.

---

[3]Note that $\frac{\partial^2 Q_{S,\mathcal{Z}}(\theta_0)}{\partial\theta\partial\theta'}=E\left[\frac{\partial\tau_{\mathcal{Z}}(U,\theta_0)}{\partial\theta}\cdot\frac{\partial\tau_{\mathcal{Z}}(U,\theta_0)'}{\partial\theta}\right]$ (since $\tau_{\mathcal{Z}}(U,\theta_0)=0$ a.s), and we can estimate $\frac{\partial^2 Q_{S,\mathcal{Z}}(\theta_0)}{\partial\theta\partial\theta'}$ as

$$\frac{\partial^2\widehat{Q}_{S,\mathcal{Z}}(\widehat{\theta})}{\partial\theta\partial\theta'}=\frac{1}{n}\sum_{j=1}^{n}\frac{\partial\widehat{\tau}_{\mathcal{Z}}(U_j,\widehat{\theta})}{\partial\theta}\cdot\frac{\partial\widehat{\tau}_{\mathcal{Z}}(U_j,\widehat{\theta})}{\partial\theta}'.$$

S2 A linear representation result for $\widehat{M}_g(\widehat{\gamma},\Delta)$ (the statistic proposed in Section 2.7) and the proof of Proposition 2

In this section we describe the details of the asymptotic results mentioned in Section 2.7. There, we allowed for the possibility that players cooperate almost surely, leading to the possibility that Δ_0 is no longer point-identified. Under the maintained assumption that players always choose pure strategies, γ_0 is identified and estimable using MLE. Our proposal was to construct a confidence set (CS) for Δ_0 as follows. Let $\mathcal{Z}$ be our pre-specified inference range for U and let $\mathbb{1}_{\mathcal{Z}}\{U \leq u\}$ be as defined previously. Next, let $g : \mathbb{R}^{d_U} \to \mathbb{R}$ be a real-valued, pre-specified function of U. In general, our instrument function g would satisfy $g(u) > 0 \ \forall \ u$. For a given $u \in \mathbb{R}^{d_U}$ we defined,

$$\tau_g(u,\theta) \equiv E\left[\varphi_S(V,\theta)\cdot g(U)\cdot \mathbb{1}_{\mathcal{Z}}\{U \leq u\}\right], \quad M_g(\theta) \equiv E[\tau_g(U,\theta)].$$

By iterated expectations, $M_g(\theta_0) = 0$. Since γ_0 is identified, our proposal was to use the population statistic $M_g(\gamma_0,\Delta)$ to construct a CS for Δ. Let $\Theta_g^I = \{\Delta \in \Theta : M_g(\gamma_0,\Delta) = 0\}$ be our target identified set for Δ based on the moment restriction $M_g(\theta_0) = 0$. Our sample statistic was

$$\widehat{M}_g(\widehat{\gamma},\Delta) = \frac{1}{n}\sum_{j=1}^{n}\widehat{\tau}_g(U_j,\widehat{\gamma},\Delta) = \frac{1}{n^2}\sum_{j=1}^{n}\sum_{i=1}^{n}\widehat{\varphi}_S(V_i,\widehat{\gamma},\Delta)\cdot g(U_i)\cdot \mathbb{1}_{\mathcal{Z}}\{U_i \leq U_j\}$$

In this section we will show that, under our restrictions, our statistic $\widehat{M}_g(\widehat{\gamma},\Delta)$ satisfies a linear representation result. Note first that,

$$\begin{aligned}
\widehat{M}_g(\widehat{\gamma},\Delta) &= \frac{1}{n}\sum_{j=1}^{n}\widehat{\tau}_g(U_j,\widehat{\gamma},\Delta) \\
&= \frac{1}{n^2}\sum_{j=1}^{n}\sum_{i=1}^{n}\widehat{\varphi}_S(V_i,\widehat{\gamma},\Delta)\cdot g(U_i)\cdot \mathbb{1}_{\mathcal{Z}}\{U_i \leq U_j\} \\
&= \frac{1}{n^2}\sum_{j=1}^{n}\sum_{i\neq j}\varphi_S(V_i,\widehat{\gamma},\Delta)\cdot g(U_i)\cdot \mathbb{1}_{\mathcal{Z}}\{U_i \leq U_j\} \\
&+ \frac{1}{n}\times\left(\frac{1}{n}\sum_{j=1}^{n}\widehat{\varphi}_S(V_j,\widehat{\gamma},\Delta)\cdot g(U_j)\cdot \mathbb{1}\{Z_j \in \mathcal{Z}\}\right).
\end{aligned}$$

From the asymptotic properties of $\widehat{\pi}(z,\theta)$ described in Section S1.1 (see equation S1.12) of this supplement and the integrability and boundedness restrictions in Assumptions G1 and E3, we have

$$\sup_{\theta\in\Theta}\left|\frac{1}{n}\sum_{j=1}^{n}\widehat{\varphi}_S(V_j,\theta)\cdot g(U_j)\cdot \mathbb{1}\{Z_j \in \mathcal{Z}\}\right| = O_p(1).$$

Therefore,

$$\sup_{\Delta\in\Theta}\left|\frac{1}{n}\times\left(\frac{1}{n}\sum_{j=1}^{n}\widehat{\varphi}_S(V_j,\widehat{\gamma},\Delta)\cdot g(U_j)\cdot \mathbb{1}\{Z_j \in \mathcal{Z}\}\right)\right| = O_p\left(\frac{1}{n}\right) = o_p\left(n^{-1/2}\right).$$

Thus,

$$\widehat{M}_g(\widehat{\gamma},\Delta) = \frac{1}{n^2}\sum_{j=1}^{n}\sum_{i\neq j}\widehat{\varphi}_S(V_i,\widehat{\gamma},\Delta)\cdot g(U_i)\cdot \mathbb{1}_{\mathcal{Z}}\{U_i \le U_j\} + \varsigma_{a,n}^{M_g}(\Delta), \quad \text{where} \quad \sup_{\Delta\in\Theta}\left|\varsigma_{a,n}^{M_g}(\Delta)\right| = o_p\left(n^{-1/2}\right).$$

Since $\widehat{\varphi}_S(V_i,\widehat{\gamma},\Delta) - \varphi_S(V_i,\widehat{\gamma},\Delta) = -[\widehat{\pi}(Z_i,\widehat{\gamma},\Delta) - \pi(Z_i,\widehat{\gamma},\Delta)]\cdot \Xi_S(X_i,\widehat{\gamma},\Delta)$, we obtain,

$$\begin{aligned}\widehat{M}_g(\widehat{\gamma},\Delta) &= \frac{1}{n^2}\sum_{j=1}^{n}\sum_{i\neq j}\varphi_S(V_i,\widehat{\gamma},\Delta)\cdot g(U_i)\cdot \mathbb{1}_{\mathcal{Z}}\{U_i \le U_j\}\\ &- \frac{1}{n^2}\sum_{j=1}^{n}\sum_{i\neq j}[\widehat{\pi}(Z_i,\widehat{\gamma},\Delta) - \pi(Z_i,\widehat{\gamma},\Delta)]\cdot \Xi_S(X_i,\widehat{\gamma},\Delta)\cdot g(U_i)\cdot \mathbb{1}_{\mathcal{Z}}\{U_i \le U_j\}\\ &+ \varsigma_{a,n}^{M_g}(\Delta), \quad \text{where} \quad \sup_{\Delta\in\Theta}\left|\varsigma_{a,n}^{M_g}(\Delta)\right| = o_p\left(n^{-1/2}\right).\end{aligned} \tag{S2.1}$$

Let us begin with the first term on the right hand side of (S2.1). Using a second-order approximation, the first term becomes,

$$\begin{aligned}\frac{1}{n^2}\sum_{j=1}^{n}\sum_{i\neq j}\varphi_S(V_i,\widehat{\gamma},\Delta)\cdot g(U_i)\cdot \mathbb{1}_{\mathcal{Z}}\{U_i \le U_j\} &= \frac{1}{n^2}\sum_{j=1}^{n}\sum_{i\neq j}\varphi_S(V_i,\gamma_0,\Delta)\cdot g(U_i)\cdot \mathbb{1}_{\mathcal{Z}}\{U_i \le U_j\}\\ + \left(\frac{1}{n^2}\sum_{j=1}^{n}\sum_{i\neq j}\frac{\partial \varphi_S(V_i,\gamma_0,\Delta)}{\partial\gamma}\cdot g(U_i)\cdot \mathbb{1}_{\mathcal{Z}}\{U_i \le U_j\}\right)'(\widehat{\gamma}-\gamma_0)\\ + \frac{1}{2}(\widehat{\gamma}-\gamma_0)'\left(\frac{1}{n^2}\sum_{j=1}^{n}\sum_{i\neq j}\frac{\partial^2 \varphi_S(V_i,\overline{\gamma},\Delta)}{\partial\gamma\partial\gamma'}\cdot g(U_i)\cdot \mathbb{1}_{\mathcal{Z}}\{U_i \le U_j\}\right)(\widehat{\gamma}-\gamma_0),\end{aligned} \tag{S2.2}$$

where, as usual, $\overline{\gamma}$ is a point in the line segment connecting $\widehat{\gamma}$ and γ_0. Under our restrictions, we have

$$\sup_{\theta\in\Theta}\left\|\frac{1}{n\cdot(n-1)}\sum_{j=1}^{n}\sum_{i\neq j}\frac{\partial^2 \varphi_S(V_i,\theta)}{\partial\gamma\partial\gamma'}\cdot g(U_i)\cdot \mathbb{1}_{\mathcal{Z}}\{U_i \le U_j\}\right\| = O_p(1).$$

Thus,

$$\begin{aligned}&\sup_{\Delta\in\Theta}\left|(\widehat{\gamma}-\gamma_0)'\left(\frac{1}{n^2}\sum_{j=1}^{n}\sum_{i\neq j}\frac{\partial^2 \varphi_S(V_i,\overline{\gamma},\Delta)}{\partial\gamma\partial\gamma'}\cdot g(U_i)\cdot \mathbb{1}_{\mathcal{Z}}\{U_i \le U_j\}\right)(\widehat{\gamma}-\gamma_0)\right|\\ &\le \underbrace{\left(\frac{n\cdot(n-1)}{n^2}\right)}_{\to 1}\times\underbrace{\left\|\widehat{\gamma}-\gamma_0\right\|^2}_{=O_p\left(\frac{1}{n}\right)}\times\underbrace{\sup_{\theta\in\Theta}\left\|\frac{1}{n\cdot(n-1)}\sum_{j=1}^{n}\sum_{i\neq j}\frac{\partial^2 \varphi_S(V_i,\theta)}{\partial\gamma\partial\gamma'}\cdot g(U_i)\cdot \mathbb{1}_{\mathcal{Z}}\{U_i \le U_j\}\right\|}_{=O_p(1)}\\ &= O_p\left(\frac{1}{n}\right) = o_p\left(n^{-1/2}\right)\end{aligned}$$

Plugging back into (S2.2),

$$\frac{1}{n^2}\sum_{j=1}^{n}\sum_{i\neq j}\varphi_S(V_i,\widehat{\gamma},\Delta)\cdot g(U_i)\cdot \mathbb{1}_{\mathcal{Z}}\{U_i\leq U_j\} = \frac{1}{n^2}\sum_{j=1}^{n}\sum_{i\neq j}\varphi_S(V_i,\gamma_0,\Delta)\cdot g(U_i)\cdot \mathbb{1}_{\mathcal{Z}}\{U_i\leq U_j\}$$

$$+\left(\frac{1}{n^2}\sum_{j=1}^{n}\sum_{i\neq j}\frac{\partial\varphi_S(V_i,\gamma_0,\Delta)}{\partial\gamma}\cdot g(U_i)\cdot \mathbb{1}_{\mathcal{Z}}\{U_i\leq U_j\}\right)'(\widehat{\gamma}-\gamma_0)+\varsigma_{b,n}^{M_g}(\Delta), \tag{S2.3}$$

where $\sup_{\Delta\in\Theta}\left|\varsigma_{b,n}^{M_g}(\Delta)\right| = o_p\left(n^{-1/2}\right)$.

Let $(V_1,V_2)\sim F_V\otimes F_V$ (two randomly drawn observations from our i.i.d sample $(V_i)_{i=1}^n$, and define

$$H_\gamma^{M_g}(\theta)\equiv E\left[\frac{\partial\varphi_S(V_1,\theta)}{\partial\gamma}\cdot g(U_1)\cdot \mathbb{1}_{\mathcal{Z}}\{U_1\leq U_2\}\right]. \tag{S2.4}$$

By the Lipschitz-restrictions in Assumption E3, Nolan and Pollard (1987, Lemma 22), Pakes and Pollard (1989, Example 2.10, Lemma 2.13, Lemma 2.14) imply that the class of functions involved is *Euclidean* (see Sherman (1994, Definition 3)), and given the existence of $2+\delta$-moments in Assumptions I1 and E3, and the boundedness restrictions in part (iii) of Assumption G1, the corresponding envelope for the class also has finite $2+\delta$-moments. From here, Sherman (1994, Corollary 4) yields,

$$\sup_{\theta\in\Theta}\left|\frac{1}{n\cdot(n-1)}\sum_{j=1}^{n}\sum_{i\neq j}\frac{\partial\varphi_S(V_i,\theta)}{\partial\gamma}\cdot g(U_i)\cdot \mathbb{1}_{\mathcal{Z}}\{U_i\leq U_j\}\right| = O_p(1),$$

$$\sup_{\theta\in\Theta}\left|\frac{1}{n\cdot(n-1)}\sum_{j=1}^{n}\sum_{i\neq j}\frac{\partial\varphi_S(V_i,\theta)}{\partial\gamma}\cdot g(U_i)\cdot \mathbb{1}_{\mathcal{Z}}\{U_i\leq U_j\}-H_\gamma^{M_g}(\theta)\right| = o_p(1).$$

Our restrictions also imply that $\left\|H_\gamma^{M_g}(\theta)\right\|$ is bounded over Θ. Combined with the linear representation of $\widehat{\gamma}-\gamma_0$, these results imply,

$$\begin{aligned}&\left(\frac{1}{n^2}\sum_{j=1}^{n}\sum_{i\neq j}\frac{\partial\varphi_S(V_i,\gamma_0,\Delta)}{\partial\gamma}\cdot g(U_i)\cdot \mathbb{1}_{\mathcal{Z}}\{U_i\leq U_j\}\right)'(\widehat{\gamma}-\gamma_0)\\ &=\underbrace{\left(\frac{n\cdot(n-1)}{n^2}\right)}_{\to 1}\times\left(\frac{1}{n\cdot(n-1)}\sum_{j=1}^{n}\sum_{i\neq j}\frac{\partial\varphi_S(V_i,\gamma_0,\Delta)}{\partial\gamma}\cdot g(U_i)\cdot \mathbb{1}_{\mathcal{Z}}\{U_i\leq U_j\}\right)'(\widehat{\gamma}-\gamma_0)\\ &=\frac{1}{n}\sum_{i=1}^{n}H_\gamma^{M_g}(\gamma_0,\Delta)'\psi_\gamma(V_i;\gamma_0)+\varsigma_{c,n}^{M_g}(\Delta),\quad\text{where}\quad \sup_{\Delta\in\Theta}\left|\varsigma_{c,n}^{M_g}(\Delta)\right| = o_p\left(n^{-1/2}\right).\end{aligned} \tag{S2.5}$$

Note that $E\left[H_\gamma^{M_g}(\gamma_0,\Delta)'\psi_\gamma(V_1;\gamma_0)\right]=0\ \forall\ \Delta\in\Theta$. Now let us analyze the first term on the right hand side of (S2.3). Let

$$q_{M_g}^a(V_1,V_2;\theta)\equiv\varphi_S(V_1,\theta)\cdot g(U_1)\cdot \mathbb{1}_{\mathcal{Z}}\{U_1\leq U_2\}.$$

Note that,

$$M_g(\theta) = E\Big[q^a_{M_g}(V_1, V_2; \theta)\Big].$$

Note that,

$$\frac{1}{n^2}\sum_{j=1}^{n}\sum_{i\neq j}\varphi_S(V_i,\gamma_0,\Delta)\cdot g(U_i)\cdot \mathbb{1}_{\mathcal{Z}}\{U_i \le U_j\} = \frac{1}{n^2}\sum_{j=1}^{n}\sum_{i\neq j} q^a_{M_g}(V_i, V_j;\theta).$$

Recall that the identified set we are focusing on, Θ^I_g for the strategic-interaction parameters Δ, is defined as,

$$\Theta^I_g \equiv \Big\{\Delta \in \Theta : E\left[\varphi_S(V,\gamma_0,\Delta)\cdot g(U)\cdot \mathbb{1}_{\mathcal{Z}}\{U \le u\}\right] = 0,\ F_U\text{–a.e } u \in \mathbb{R}^{d_U}\Big\}.$$

And, as a result, $M_g(\gamma_0,\Delta) = 0\ \forall\ \theta \in \Theta^I_g$. Let

$$\widetilde{q}^a_{M_g}(V_1,V_2;\theta) = \frac{1}{2}\cdot\Big(q^a_{M_g}(V_1,V_2;\theta) + q^a_{M_g}(V_2,V_1;\theta)\Big),$$

By construction, $\widetilde{q}^a_{M_g}(V_1,V_2;\theta)$ is symmetric in (V_1,V_2) and $E\Big[\widetilde{q}^a_{M_g}(V_1,V_2;\theta)\Big] = M_g(\theta)$. Let

$$\mathcal{V}^a_{M_g,n}(\theta) \equiv \binom{n}{2}^{-1}\sum_{c}\widetilde{q}^a_{M_g}(V_i,V_j;\theta),$$

where $\sum_c$ denotes the sum over the $\binom{n}{2}$ distinct combinations $\{i,j\}$ from $\{1,\ldots,n\}$. Note that,

$$\frac{1}{n^2}\sum_{j=1}^{n}\sum_{i\neq j}\varphi_S(V_i,\gamma_0,\Delta)\cdot g(U_i)\cdot \mathbb{1}_{\mathcal{Z}}\{U_i \le U_j\} = \left(\frac{n\cdot(n-1)}{n^2}\right)\cdot \mathcal{V}^a_{M_g,n}(\gamma_0,\Delta) \tag{S2.6}$$

We proceed by analyzing the Hoeffding decomposition of $\mathcal{V}^a_{M_g,n}(\theta)$. Let

$$\overline{q}^a_{M_g}(V_1;\theta) \equiv 2\cdot\Big(E\Big[\widetilde{q}^a_{M_g}(V_1,V_2;\theta)\ \Big|V_1\Big] - M_g(\theta)\Big) \tag{S2.7}$$

Note that, by construction, $E[\overline{q}^a_{M_g}(V_1;\theta)] = 0\ \forall\ \theta$. By inspection, we can also see that,

$$\overline{q}^a_{M_g}(V_1;\gamma_0,\Delta) = \varphi_S(V_1,\gamma_0,\Delta)\cdot g(U_1)\cdot E\left[\mathbb{1}_{\mathcal{Z}}\{U_1 \le U_2\}\Big|U_1\right] \quad \forall\ \Delta \in \Theta^I_g. \tag{S2.8}$$

The Hoeffding decomposition of $\mathcal{V}^a_{M_g,n}(\theta)$ (see Serfling (1980, pages 177-178) is given by,

$$\mathcal{V}^a_{M_g,n}(\theta) = M_g(\theta) + \frac{1}{n}\sum_{i=1}^{n}\overline{q}^a_{M_g}(V_i;\theta) + \mathcal{R}^a_n(\theta),$$

where $\mathcal{R}^a_n(\theta)$ is a degenerate U-statistic of order 2. By the Lipschitz-restrictions in Assumption E3, Nolan and Pollard (1987, Lemma 22), Pakes and Pollard (1989, Example 2.10, Lemma 2.13, Lemma 2.14) imply that the class of functions involved is *Euclidean* (see Sherman (1994, Definition 3)), and given the existence of $2+\delta$-moments in Assumptions I1 and E3, and the boundedness restrictions in part (iii) of Assumption G1, the corresponding envelope for the class also has finite $2+\delta$-moments. From here, Sherman (1994, Corollary 4) yields,

$$\sup_{\theta\in\Theta}|\mathcal{R}_n^a(\theta)| = O_p\left(\frac{1}{n}\right) = o_p\left(n^{-1/2}\right).$$

Thus, $\forall\ \theta\in\Theta$,

$$\mathcal{V}^a_{M_g,n}(\theta) = M_g(\theta) + \frac{1}{n}\sum_{i=1}^{n}\overline{q}^a_{M_g}(V_i;\theta) + \mathcal{R}_n^a(\theta), \quad \text{where} \quad \sup_{\theta\in\Theta}|\mathcal{R}_n^a(\theta)| = o_p\left(n^{-1/2}\right). \tag{S2.9}$$

And, in particular, over our identified set Θ_g^I, the above representation simplifies to,

$$\left.\begin{array}{l}\mathcal{V}^a_{M_g,n}(\gamma_0,\Delta) = \frac{1}{n}\sum_{i=1}^{n}\overline{q}^a_{M_g}(V_i;\gamma_0,\Delta) + \mathcal{R}_n^a(\gamma_0,\Delta), \quad \text{where} \quad \sup_{\Delta\in\Theta_g^I}\left|\mathcal{R}_n^a(\gamma_0,\Delta)\right| = o_p\left(n^{-1/2}\right)\\ \overline{q}^a_{M_g}(V_1;\gamma_0,\Delta) = \varphi_S(V_1,\gamma_0,\Delta)\cdot g(U_1)\cdot E\left[\mathbb{1}_{\mathcal{Z}}\{U_1\le U_2\}\middle|U_1\right]\end{array}\right\}\ \forall\ \Delta\in\Theta_g^I \tag{S2.10}$$

Our restrictions yield $\sup_{\theta\in\Theta}\left|\mathcal{V}^a_{M_g,n}(\theta)\right| = O_p(1)$. Thus, since

$$\left(\frac{n\cdot(n-1)}{n^2}\right)\cdot\mathcal{V}^a_{M_g,n}(\gamma_0,\Delta) = \mathcal{V}^a_{M_g,n}(\gamma_0,\Delta) - \frac{1}{n}\cdot\mathcal{V}^a_{M_g,n}(\gamma_0,\Delta),$$

and $\sup_{\Delta\in\Theta}\left|\frac{1}{n}\cdot\mathcal{V}^a_{M_g,n}(\gamma_0,\Delta)\right| \le \sup_{\theta\in\Theta}\left|\frac{1}{n}\cdot\mathcal{V}^a_{M_g,n}(\theta)\right| = O_p\left(\frac{1}{n}\right) = o_p\left(n^{-1/2}\right)$, combining (S2.9)-(S2.10) with (S2.6), $\forall\ \Delta\in\Theta$ we have,

$$\begin{array}{l}\displaystyle\frac{1}{n^2}\sum_{j=1}^{n}\sum_{i\neq j}\varphi_S(V_i,\gamma_0,\Delta)\cdot g(U_i)\cdot\mathbb{1}_{\mathcal{Z}}\{U_i\le U_j\} = M_g(\gamma_0,\Delta) + \frac{1}{n}\sum_{i=1}^{n}\overline{q}^a_{M_g}(V_i;\gamma_0,\Delta) + \varsigma_{d,n}^{M_g}(\Delta),\\ \quad\text{where}\quad \sup_{\Delta\in\Theta}\left|\varsigma_{d,n}^{M_g}(\Delta)\right| = o_p\left(n^{-1/2}\right),\end{array} \tag{S2.11}$$

and where $\overline{q}^a_{M_g}(V_1;\theta)$ is as defined in (S2.7). In particular, over our identified set Θ_g^I, the result in (S2.12) simplifies to,

$$\left.\begin{array}{l}\frac{1}{n^2}\sum_{j=1}^{n}\sum_{i\neq j}\varphi_S(V_i,\gamma_0,\Delta)\cdot g(U_i)\cdot\mathbb{1}_{\mathcal{Z}}\{U_i\le U_j\} = \frac{1}{n}\sum_{i=1}^{n}\overline{q}^a_{M_g}(V_i;\gamma_0,\Delta) + \varsigma_{d,n}^{M_g}(\Delta),\\ \text{where}\quad \sup_{\Delta\in\Theta}\left|\varsigma_{d,n}^{M_g}(\Delta)\right| = o_p\left(n^{-1/2}\right),\\ \overline{q}^a_{M_g}(V_1;\gamma_0,\Delta) = \varphi_S(V_1,\gamma_0,\Delta)\cdot g(U_1)\cdot E\left[\mathbb{1}_{\mathcal{Z}}\{U_1\le U_2\}\middle|U_1\right]\end{array}\right\}\ \forall\ \Delta\in\Theta_g^I \tag{S2.12}$$

Plugging in the results in (S2.5) and (S2.12) into (S2.3), we obtain,

$$\left.\begin{array}{l}\frac{1}{n^2}\sum_{j=1}^n\sum_{i\neq j}\varphi_S(V_i,\widehat{\gamma},\Delta)\cdot g(U_i)\cdot\mathbb{1}_{\mathcal{Z}}\{U_i\leq U_j\}=\\ M_g(\gamma_0,\Delta)+\frac{1}{n}\sum_{i=1}^n\left[H_\gamma^{M_g}(\gamma_0,\Delta)'\psi_\gamma(V_i;\gamma_0)+\overline{q}^a_{M_g}(V_i;\gamma_0,\Delta)\right]+\varsigma^{M_g}_{e,n}(\Delta),\\ \text{where}\quad\sup_{\Delta\in\Theta}\left|\varsigma^{M_g}_{e,n}(\Delta)\right|=o_p\left(n^{-1/2}\right),\end{array}\right\}\ \forall\ \Delta\in\Theta \tag{S2.13}$$

where $\overline{q}^a_{M_g}(V_1;\theta)$ is as defined in (S2.7). In particular, over our identified set Θ_g^I, the result in (S2.13) simplifies to,

$$\left.\begin{array}{l}\frac{1}{n^2}\sum_{j=1}^n\sum_{i\neq j}\varphi_S(V_i,\widehat{\gamma},\Delta)\cdot g(U_i)\cdot\mathbb{1}_{\mathcal{Z}}\{U_i\leq U_j\}=\\ \frac{1}{n}\sum_{i=1}^n\left[H_\gamma^{M_g}(\gamma_0,\Delta)'\psi_\gamma(V_i;\gamma_0)+\overline{q}^a_{M_g}(V_i;\gamma_0,\Delta)\right]+\varsigma^{M_g}_{e,n}(\Delta),\\ \text{where}\quad\sup_{\Delta\in\Theta}\left|\varsigma^{M_g}_{e,n}(\Delta)\right|=o_p\left(n^{-1/2}\right),\\ \overline{q}^a_{M_g}(V_1;\gamma_0,\Delta)=\varphi_S(V_1,\gamma_0,\Delta)\cdot g(U_1)\cdot E\left[\mathbb{1}_{\mathcal{Z}}\{U_1\leq U_2\}\middle|U_1\right]\end{array}\right\}\ \forall\ \Delta\in\Theta_g^I \tag{S2.14}$$

Now we analyze the second term in (S2.1). Using a linear approximation,

$$\begin{aligned}&\frac{1}{n^2}\sum_{j=1}^n\sum_{i\neq j}[\widehat{\pi}(Z_i,\widehat{\gamma},\Delta)-\pi(Z_i,\widehat{\gamma},\Delta)]\cdot\Xi_S(X_i,\widehat{\gamma},\Delta)\cdot g(U_i)\cdot\mathbb{1}_{\mathcal{Z}}\{U_i\leq U_j\}\\ =&\frac{1}{n^2}\sum_{j=1}^n\sum_{i\neq j}[\widehat{\pi}(Z_i,\gamma_0,\Delta)-\pi(Z_i,\gamma_0,\Delta)]\cdot\Xi_S(X_i,\gamma_0,\Delta)\cdot g(U_i)\cdot\mathbb{1}_{\mathcal{Z}}\{U_i\leq U_j\}\\ +&\Bigg(\frac{1}{n^2}\sum_{j=1}^n\sum_{i\neq j}\Bigg\{\left[\frac{\partial\widehat{\pi}(Z_i,\overline{\gamma},\Delta)}{\partial\gamma}-\frac{\pi(Z_i,\overline{\gamma},\Delta)}{\partial\gamma}\right]\cdot\Xi_S(X_i,\overline{\gamma},\Delta)\cdot g(U_i)\cdot\mathbb{1}_{\mathcal{Z}}\{U_i\leq U_j\}\\ &\qquad+[\widehat{\pi}(Z_i,\overline{\gamma},\Delta)-\pi(Z_i,\overline{\gamma},\Delta)]\cdot\frac{\partial\Xi_S(X_i,\overline{\gamma},\Delta)}{\partial\gamma}\cdot g(U_i)\cdot\mathbb{1}_{\mathcal{Z}}\{U_i\leq U_j\}\Bigg\}\Bigg)'(\widehat{\gamma}-\gamma_0)\end{aligned} \tag{S2.15}$$

where, once again, $\overline{\gamma}$ is a point in the line segment connecting $\widehat{\gamma}$ and γ_0. Recall that $|\Xi_S(x,\theta)| \leq 1$ $\forall\ x,\theta$. Using the results in (S1.12) and (S1.28),

$$\begin{aligned}
&\sup_{\theta\in\Theta}\left\|\frac{1}{n\cdot(n-1)}\sum_{j=1}^{n}\sum_{i\neq j}\left\{\left[\frac{\partial\widehat{\pi}(Z_i,\theta)}{\partial\gamma}-\frac{\pi(Z_i,\theta)}{\partial\gamma}\right]\cdot\Xi_S(X_i,\theta)\cdot g(U_i)\cdot\mathbb{1}_{\mathcal{Z}}\{U_i\leq U_j\}\right.\right.\\
&\qquad\left.\left.+\left[\widehat{\pi}(Z_i,\theta)-\pi(Z_i,\theta)\right]\cdot\frac{\partial\Xi_S(X_i,\theta)}{\partial\gamma}\cdot g(U_i)\cdot\mathbb{1}_{\mathcal{Z}}\{U_i\leq U_j\}\right\}\right\|\\
&\leq\underbrace{\sup_{\substack{\theta\in\Theta\\ z\in\mathcal{Z}}}\left\|\frac{\partial\widehat{\pi}(z,\theta)}{\partial\gamma}-\frac{\pi(z,\theta)}{\partial\gamma}\right\|}_{=o_p(n^{-1/4})}\times\underbrace{\frac{1}{n\cdot(n-1)}\sum_{j=1}^{n}\sum_{i\neq j}\left|g(U_i)\right|\cdot\mathbb{1}_{\mathcal{Z}}\{U_i\leq U_j\}}_{=O_p(1)}\\
&\quad+\underbrace{\sup_{\substack{\theta\in\Theta\\ z\in\mathcal{Z}}}\left\|\widehat{\pi}(z,\theta)-\pi(z,\theta)\right\|}_{=o_p(n^{-1/4})}\times\underbrace{\sup_{\theta\in\Theta}\left(\frac{1}{n\cdot(n-1)}\sum_{j=1}^{n}\sum_{i\neq j}\left\|\frac{\partial\Xi_S(X_i,\theta)}{\partial\gamma}\right\|\cdot\left|g(U_i)\right|\cdot\mathbb{1}_{\mathcal{Z}}\{U_i\leq U_j\}\right)}_{=O_p(1)}\\
&=o_p\left(n^{-1/4}\right).
\end{aligned}$$

From here,

$$\begin{aligned}
&\sup_{\Delta\in\Theta}\left|\left(\frac{1}{n^2}\sum_{j=1}^{n}\sum_{i\neq j}\left\{\left[\frac{\partial\widehat{\pi}(Z_i,\overline{\gamma},\Delta)}{\partial\gamma}-\frac{\pi(Z_i,\overline{\gamma},\Delta)}{\partial\gamma}\right]\cdot\Xi_S(X_i,\overline{\gamma},\Delta)\cdot g(U_i)\cdot\mathbb{1}_{\mathcal{Z}}\{U_i\leq U_j\}\right.\right.\right.\\
&\qquad\left.\left.\left.+\left[\widehat{\pi}(Z_i,\overline{\gamma},\Delta)-\pi(Z_i,\overline{\gamma},\Delta)\right]\cdot\frac{\partial\Xi_S(X_i,\overline{\gamma},\Delta)}{\partial\gamma}\cdot g(U_i)\cdot\mathbb{1}_{\mathcal{Z}}\{U_i\leq U_j\}\right\}\right)'(\widehat{\gamma}-\gamma_0)\right|\\
&\leq\underbrace{\underbrace{\left(\frac{n-1}{n^2}\right)}_{\to 1}\times\sup_{\theta\in\Theta}\left\|\frac{1}{n\cdot(n-1)}\sum_{j=1}^{n}\sum_{i\neq j}\left\{\left[\frac{\partial\widehat{\pi}(Z_i,\theta)}{\partial\gamma}-\frac{\pi(Z_i,\theta)}{\partial\gamma}\right]\cdot\Xi_S(X_i,\theta)\cdot g(U_i)\cdot\mathbb{1}_{\mathcal{Z}}\{U_i\leq U_j\}\right.\right.}_{}\\
&\qquad\underbrace{\left.\left.+\left[\widehat{\pi}(Z_i,\theta)-\pi(Z_i,\theta)\right]\cdot\frac{\partial\Xi_S(X_i,\theta)}{\partial\gamma}\cdot g(U_i)\cdot\mathbb{1}_{\mathcal{Z}}\{U_i\leq U_j\}\right\}\right\|}_{=o_p(n^{-1/4})}\times\underbrace{\left\|\widehat{\gamma}-\gamma_0\right\|}_{=O_p\left(n^{-1/2}\right)}\\
&=o_p\left(n^{-3/4}\right)=o_p\left(n^{-1/2}\right).
\end{aligned}$$

Plugging back into (S2.15),

$$\begin{aligned}&\frac{1}{n\cdot(n-1)}\sum_{j=1}^{n}\sum_{i\neq j}\left[\widehat{\pi}(Z_i,\widehat{\gamma},\Delta)-\pi(Z_i,\widehat{\gamma},\Delta)\right]\cdot\Xi_S(X_i,\widehat{\gamma},\Delta)\cdot g(U_i)\cdot\mathbb{1}_{\mathcal{Z}}\{U_i\leq U_j\}\\ =&\frac{1}{n\cdot(n-1)}\sum_{j=1}^{n}\sum_{i\neq j}\left[\widehat{\pi}(Z_i,\gamma_0,\Delta)-\pi(Z_i,\gamma_0,\Delta)\right]\cdot\Xi_S(X_i,\gamma_0,\Delta)\cdot g(U_i)\cdot\mathbb{1}_{\mathcal{Z}}\{U_i\leq U_j\}+\varsigma_{f,n}^{M_g}(\Delta),\\ &\text{where}\quad\sup_{\Delta\in\Theta}\left|\varsigma_{f,n}^{M_g}(\Delta)\right|=o_p\left(n^{-1/2}\right).\end{aligned}\tag{S2.16}$$

Next, using the linear representation result in (S1.11),

$$\begin{aligned}&\frac{1}{n^2}\sum_{j=1}^{n}\sum_{i\neq j}\left[\widehat{\pi}(Z_i,\gamma_0,\Delta)-\pi(Z_i,\gamma_0,\Delta)\right]\cdot\Xi_S(X_i,\gamma_0,\Delta)\cdot g(U_i)\cdot\mathbb{1}_{\mathcal{Z}}\{U_i\leq U_j\}\\ =&\frac{1}{n^2}\sum_{j=1}^{n}\sum_{i\neq j}\left\{\frac{1}{n\cdot h_n^{d_Z}}\sum_{k=1}^{n}\psi_n^{\pi}(V_k;Z_i,\gamma_0,\Delta)+\vartheta_n^{\pi}(Z_i,\gamma_0,\Delta)\right\}\cdot\Xi_S(X_i,\gamma_0,\Delta)\cdot g(U_i)\cdot\mathbb{1}_{\mathcal{Z}}\{U_i\leq U_j\}\\ =&\frac{1}{n^3}\sum_{j=1}^{n}\sum_{i\neq j}\sum_{k\neq i,j}\frac{1}{h_n^{d_Z}}\psi_n^{\pi}(V_k;Z_i,\gamma_0,\Delta)\cdot\Xi_S(X_i,\gamma_0,\Delta)\cdot g(U_i)\cdot\mathbb{1}_{\mathcal{Z}}\{U_i\leq U_j\}\\ &+\frac{1}{n}\times\left(\frac{1}{n^2}\sum_{j=1}^{n}\sum_{i\neq j}\frac{1}{h_n^{d_Z}}\cdot\left[\psi_n^{\pi}(V_i;Z_i,\gamma_0,\Delta)+\psi_n^{\pi}(V_j;Z_i,\gamma_0,\Delta)\right]\cdot\Xi_S(X_i,\gamma_0,\Delta)\cdot g(U_i)\cdot\mathbb{1}_{\mathcal{Z}}\{U_i\leq U_j\}\right)\\ &+\frac{1}{n^2}\sum_{j=1}^{n}\sum_{i\neq j}\vartheta_n^{\pi}(Z_i,\gamma_0,\Delta)\cdot\Xi_S(X_i,\gamma_0,\Delta)\cdot g(U_i)\cdot\mathbb{1}_{\mathcal{Z}}\{U_i\leq U_j\}\end{aligned}\tag{S2.17}$$

Recall that $|\Xi_S(x,\theta)|\leq 1\ \forall\ x,\theta$. Thus, under our restrictions, we have

$$\begin{aligned}&\sup_{\Delta\in\Theta}\left|\frac{1}{n}\times\left(\frac{1}{n^2}\sum_{j=1}^{n}\sum_{i\neq j}\frac{1}{h_n^{d_Z}}\cdot\left[\psi_n^{\pi}(V_i;Z_i,\gamma_0,\Delta)+\psi_n^{\pi}(V_j;Z_i,\gamma_0,\Delta)\right]\cdot\Xi_S(X_i,\gamma_0,\Delta)\cdot g(U_i)\cdot\mathbb{1}_{\mathcal{Z}}\{U_i\leq U_j\}\right)\right|\\ \leq&\frac{1}{n}\times\underbrace{\left(\frac{n\cdot(n-1)}{n^2}\right)}_{\to 1}\times\underbrace{\sup_{\theta\in\Theta}\frac{1}{n\cdot(n-1)}\sum_{j=1}^{n}\sum_{i\neq j}\frac{1}{h_n^{d_Z}}\cdot\left|\psi_n^{\pi}(V_i;Z_i,\theta)+\psi_n^{\pi}(V_j;Z_i,\theta)\right|\cdot\left|g(U_i)\right|\cdot\mathbb{1}_{\mathcal{Z}}\{U_i\leq U_j\}}_{=O_p\left(\frac{1}{h_n^{d_Z}}\right)}\\ =&O_p\left(\frac{1}{n\cdot h_n^{d_Z}}\right)=o_p\left(n^{-1/2}\right),\end{aligned}$$

And using the asymptotic properties of the remainder $\vartheta_n^{\pi}(z,\theta)$ described in equation (S1.11),

$$\begin{aligned}
&\sup_{\Delta\in\Theta}\left|\frac{1}{n^2}\sum_{j=1}^{n}\sum_{i\neq j}\vartheta_n^{\pi}(Z_i,\gamma_0,\Delta)\cdot\Xi_S(X_i,\gamma_0,\Delta)\cdot g(U_i)\cdot\mathbb{1}_{\mathcal{Z}}\{U_i\leq U_j\}\right| \\
&\leq \underbrace{\sup_{\substack{\theta\in\Theta\\ z\in\mathcal{Z}}}\left|\vartheta_n^{\pi}(z,\theta)\right|}_{=o_p\left(n^{-1/2}\right)}\times\underbrace{\left(\frac{n\cdot(n-1)}{n^2}\right)}_{\to 1}\times\underbrace{\frac{1}{n\cdot(n-1)}\sum_{j=1}^{n}\sum_{i\neq j}\left|g(U_i)\right|\cdot\mathbb{1}_{\mathcal{Z}}\{U_i\leq U_j\}}_{=O_p(1)} \\
&= o_p\left(n^{-1/2}\right).
\end{aligned}$$

Plugging these results back into (S2.17),

$$\begin{aligned}
&\frac{1}{n^2}\sum_{j=1}^{n}\sum_{i\neq j}\left[\widehat{\pi}(Z_i,\gamma_0,\Delta)-\pi(Z_i,\gamma_0,\Delta)\right]\cdot\Xi_S(X_i,\gamma_0,\Delta)\cdot g(U_i)\cdot\mathbb{1}_{\mathcal{Z}}\{U_i\leq U_j\} \\
=&\frac{1}{n^3}\sum_{j=1}^{n}\sum_{i\neq j}\sum_{k\neq i,j}\frac{1}{h_n^{d_Z}}\psi_n^{\pi}(V_k;Z_i,\gamma_0,\Delta)\cdot\Xi_S(X_i,\gamma_0,\Delta)\cdot g(U_i)\cdot\mathbb{1}_{\mathcal{Z}}\{U_i\leq U_j\}+\varsigma_{g,n}^{M_g}(\Delta), \\
&\text{where}\quad \sup_{\Delta\in\Theta}\left|\varsigma_{g,n}^{M_g}(\Delta)\right| = o_p\left(n^{-1/2}\right).
\end{aligned} \tag{S2.18}$$

Next we analyze the leading term in (S2.18). As we have done before, let $(V_1,V_2,V_3)\sim F_V\otimes F_V\otimes F_V$ (three randomly drawn observations from our i.i.d sample $(V_i)_{i=1}^n$. Let

$$q_{M_g,n}^{b}(V_1,V_2,V_3;\theta)\equiv\frac{1}{h_n^{d_Z}}\cdot\psi_n^{\pi}(V_3;Z_1,\theta)\cdot\Xi_S(X_1,\theta)\cdot g(U_1)\cdot\mathbb{1}_{\mathcal{Z}}\{U_1\leq U_2\}.$$

Note that $E\left[q_{M_g,n}^{b}(V_1,V_2,V_3;\theta)\middle|V_1,V_2\right]=0\ \forall\ \theta$, and therefore $E\left[q_{M_g,n}^{b}(V_1,V_2,V_3;\theta)\right]=0\ \forall\ \theta$. Let,

$$\mathcal{V}_{M_g,n}^{b}(\theta)\equiv\frac{1}{n\cdot(n-1)\cdot(n-2)}\sum_{j=1}^{n}\sum_{i\neq j}\sum_{k\neq i,j}q_{M_g,n}^{b}(V_i,V_j,V_k;\theta).$$

Note that,

$$\begin{aligned}
&\frac{1}{n^3}\sum_{j=1}^{n}\sum_{i\neq j}\sum_{k\neq i,j}\frac{1}{h_n^{d_Z}}\psi_n^{\pi}(V_k;Z_i,\gamma_0,\Delta)\cdot\Xi_S(X_i,\gamma_0,\Delta)\cdot g(U_i)\cdot\mathbb{1}_{\mathcal{Z}}\{U_i\leq U_j\} \\
=&\left(\frac{n\cdot(n-1)\cdot(n-2)}{n^3}\right)\cdot\mathcal{V}_{M_g,n}^{b}(\gamma_0,\Delta)
\end{aligned} \tag{S2.19}$$

As we have done before, we proceed now to analyze the Hoeffding decomposition of $\mathcal{V}_{M_g,n}^{b}(\theta)$. Define,

$$\widetilde{q}_{M_g,n}^{b}(V_1,V_2,V_3;\theta)\equiv\frac{1}{3!}\sum_{p}q_{M_g,n}^{b}(V_{m_1},V_{m_2},V_{m_3};\theta), \tag{S2.20}$$

where $\sum_p$ denotes the sum over the 3! permutations $\{m_1, m_2, m_3\}$ of $\{1,2,3\}$. By construction, $\widetilde{q}^b_{M_g,n}(V_1, V_2, V_3;\theta)$ is symmetric in (V_1, V_2, V_3), and $E\left[\widetilde{q}^b_{M_g,n}(V_1, V_2, V_3;\theta)\right] = 0$. We can express,

$$\mathcal{V}^b_{M_g,n}(\theta) = \binom{n}{3}^{-1} \sum_c \widetilde{q}^b_{M_g,n}(V_i, V_j, V_k;\theta),$$

where $\sum_c$ denotes the sum over the $\binom{n}{3}$ distinct combinations $\{i,j,k\}$ from $\{1,\ldots,n\}$. Let

$$\overline{q}^b_{M_g,n}(V_1;\theta) \equiv 3 \cdot E\left[\widetilde{q}^b_{M_g,n}(V_1, V_2, V_3;\theta) \,\middle|\, V_1\right]$$

Note that, by iterated expectations, $E\left[\overline{q}^b_{M_g,n}(V_1;\theta)\right] = 0 \;\forall\; \theta$. By inspection, we can see that,

$$\begin{aligned} E\left[\widetilde{q}^b_{M_g,n}(V_1, V_2, V_3;\theta) \,\middle|\, V_1\right] &= \frac{2}{3!} E\left[q^b_{M_g,n}(V_2, V_3, V_1;\theta) \,\middle|\, V_1\right] \\ &= \frac{1}{3} E\left[\frac{1}{h_n^{d_Z}} \cdot \psi^\pi_n(V_1; Z_2, \theta) \cdot \Xi_S(X_2,\theta) \cdot g(U_2) \cdot \mathbb{1}_{\mathcal{Z}}\{U_2 \leq U_3\} \,\middle|\, V_1\right]. \end{aligned}$$

Therefore,

$$\overline{q}^b_{M_g,n}(V_1;\theta) = E\left[\frac{1}{h_n^{d_Z}} \cdot \psi^\pi_n(V_1; Z_2, \theta) \cdot \Xi_S(X_2,\theta) \cdot g(U_2) \cdot \mathbb{1}_{\mathcal{Z}}\{U_2 \leq U_3\} \,\middle|\, V_1\right]. \quad \text{(S2.21)}$$

The Hoeffding decomposition of $\mathcal{V}^b_{M_g,n}(\theta)$ (see Serfling (1980, pages 177-178) is given by,

$$\mathcal{V}^b_{M_g,n}(\theta) = \frac{1}{n}\sum_{i=1}^n \overline{q}^b_{M_g,n}(V_i;\theta) + \mathcal{R}^b_n(\theta),$$

where $\mathcal{R}^b_n(\theta)$ is a linear combination of *degenerate* U-statistics of orders 2 and 3. By the bounded-variation properties of our kernel function described in Assumption E1 and the Lipschitz-restrictions in Assumption E3, Nolan and Pollard (1987, Lemma 22), Pakes and Pollard (1989, Example 2.10, Lemma 2.13, Lemma 2.14) imply that the class of functions involved is *Euclidean* (see Sherman (1994, Definition 3)), and given the existence of $2+\delta$-moments in Assumptions I1 and E3, and the boundedness restrictions in part (iii) of Assumption G1, the corresponding envelope for the class also has finite $2+\delta$-moments. From here, Sherman (1994, Corollary 4) yields

$$\sup_{\theta\in\Theta} |\mathcal{R}^b_n(\theta)| = O_p\left(\frac{1}{n \cdot h_n^{d_Z}}\right) = o_p\left(n^{-1/2}\right),$$

where the last equality follows from our bandwidth convergence restrictions since $n^{1/2} \cdot h_n^{d_Z} \longrightarrow \infty$. Thus,

$$\mathcal{V}^b_{M_g,n}(\theta) = \frac{1}{n}\sum_{i=1}^n \overline{q}^b_{M_g,n}(V_i;\theta) + \mathcal{R}^b_n(\theta), \quad \text{where} \quad \sup_{\theta\in\Theta} |\mathcal{R}^b_n(\theta)| = o_p\left(n^{-1/2}\right), \quad \text{(S2.22)}$$

and where $\overline{q}^b_{M_g,n}(V_1;\theta)$ is as described in (S2.21), and it satisfies $E\left[\overline{q}^b_{M_g,n}(V_1;\theta)\right] = 0 \;\forall\; \theta$. Next, note that,

$$\left(\frac{n\cdot(n-1)\cdot(n-2)}{n^3}\right)\cdot \mathcal{V}^b_{M_g,n}(\gamma_0,\Delta) = \mathcal{V}^b_{M_g,n}(\gamma_0,\Delta) + \underbrace{\left(\frac{2n-3n^2}{n^3}\right)}_{=O(\frac{1}{n})}\cdot \underbrace{\mathcal{V}^b_{M_g,n}(\gamma_0,\Delta)}_{=O_p(1)}$$

Since

$$\sup_{\Delta\in\Theta}\left|\mathcal{V}^b_{M_g,n}(\gamma_0,\Delta)\right| \le \sup_{\theta\in\Theta}\left|\mathcal{V}^b_{M_g,n}(\theta)\right| = O_p(1),$$

it follows from (S2.19) and (S2.22) that,

$$\begin{aligned}
&\frac{1}{n^3}\sum_{j=1}^{n}\sum_{i\neq j}\sum_{k\neq i,j}\frac{1}{h_n^{d_Z}}\psi_n^{\pi}(V_k;Z_i,\gamma_0,\Delta)\cdot\Xi_S(X_i,\gamma_0,\Delta)\cdot g(U_i)\cdot\mathbb{1}_{\mathcal{Z}}\{U_i\le U_j\}\\
=&\left(\frac{n\cdot(n-1)\cdot(n-2)}{n^3}\right)\cdot\mathcal{V}^b_{M_g,n}(\gamma_0,\Delta)\\
=&\frac{1}{n}\sum_{i=1}^{n}\overline{q}^b_{M_g,n}(V_i;\gamma_0,\Delta)+\varsigma^{M_g}_{h,n}(\Delta),\quad\text{where}\quad \sup_{\Delta\in\Theta}\left|\varsigma^{M_g}_{h,n}(\Delta)\right| = o_p\left(n^{-1/2}\right),
\end{aligned} \tag{S2.23}$$

and where $\overline{q}^b_{M_g,n}(V_1;\theta)$ is as described in (S2.21) and it satisfies $E\left[\overline{q}^b_{M_g,n}(V_1;\theta)\right] = 0 \;\forall\; \theta$. Let

$$\psi_{M_g,n}(V_i;\gamma_0,\Delta) \equiv H_\gamma^{M_g}(\gamma_0,\Delta)'\psi_\gamma(V_i;\gamma_0) + \overline{q}^a_{M_g}(V_i;\gamma_0,\Delta) - \overline{q}^b_{M_g,n}(V_i;\gamma_0,\Delta). \tag{S2.24}$$

$H_\gamma^{M_g}(\gamma_0,\Delta)$ is as defined in (S2.4), $\psi_\gamma(V_i;\gamma_0)$ is the influence function of our MLE estimator $\widehat{\gamma}$, $\overline{q}^a_{M_g}(V_i;\gamma_0,\Delta)$ is defined in (S2.7) and $\overline{q}^b_{M_g,n}(V_i;\gamma_0,\Delta)$ is defined in (S2.21). Note that $E[\psi_\gamma(V_i;\gamma_0)] = 0$, and that $E[\overline{q}^a_{M_g}(V_i;\gamma_0,\Delta)] = 0$ and $E[\overline{q}^b_{M_g,n}(V_i;\gamma_0,\Delta)] = 0 \;\forall\; \Delta\in\Theta$. Therefore, $E[\psi_{M_g,n}(V_i;\gamma_0,\Delta)] = 0 \;\forall\; \Delta\in\Theta$. Combining our results in (S2.13) and (S2.23) with (S2.1), we obtain our final result. Under our restrictions,

$$\begin{aligned}
&\widehat{M}_g(\widehat{\gamma},\Delta) = M_g(\gamma_0,\Delta) + \frac{1}{n}\sum_{i=1}^{n}\psi_{M_g,n}(V_i;\gamma_0,\Delta) + \varsigma_n^{M_g}(\Delta),\;\forall\;\Delta\in\Theta,\\
&\text{where}\quad \sup_{\Delta\in\Theta}\left|\varsigma_n^{M_g}(\Delta)\right| = o_p\left(n^{-1/2}\right).
\end{aligned} \tag{S2.25}$$

And over our target identified set Θ^I_g, the result in (S2.25) simplifies to,

$$\begin{aligned}
&\widehat{M}_g(\widehat{\gamma},\Delta) = \frac{1}{n}\sum_{i=1}^{n}\psi_{M_g,n}(V_i;\gamma_0,\Delta) + \varsigma_n^{M_g}(\Delta),\;\forall\;\Delta\in\Theta^I_g,\\
&\text{where}\quad \sup_{\Delta\in\Theta}\left|\varsigma_n^{M_g}(\Delta)\right| = o_p\left(n^{-1/2}\right).
\end{aligned} \tag{S2.26}$$

Combined, S2.25-S2.26 are the statements in Proposition 2. ■

S2.1 Uniform asymptotic coverage probability of the CS described in Section 2.7

Let $\mathcal{F}$ denote the space of distributions that contains the true DGP. If the conditions leading to (S2.26) are satisfied jointly with Assumption E4, then[4] the linear representation result described in (S2.26) is satisfied uniformly over $\mathcal{F}$, so

$$\widehat{M}_g(\widehat{\gamma}, \Delta) = M_{g,F}(\gamma_0, \Delta) + \frac{1}{n}\sum_{i=1}^{n} \psi^F_{M_g,n}(V_i;\gamma_0,\Delta) + \varsigma^{M_g}_{F,n}(\Delta), \ \forall\ \Delta \in \Theta, \quad \text{where}$$

$$\begin{cases} E_F[\psi^F_{M_g,n}(V;\gamma_0,\Delta)] = 0 \ \forall\ (F,\Delta) \in \mathcal{F}\times\Theta, \\ \underline{\lim}_{n\to 0} E_F\left[\psi^F_{M_g,n}(V;\gamma_0,\Delta)^2\right] \geq \underline{B} > 0 \ \forall\ (F,\Delta) \in \mathcal{F}\times\Theta, \\ \overline{\lim}_{n\to\infty} E_F\left[\left|\psi^F_{M_g,n}(V;\gamma_0,\Delta)\right|^3\right] \leq \overline{D} < \infty \ \forall\ (F,\Delta) \in \mathcal{F}\times\Theta, \\ \overline{\lim}_{n\to\infty} \frac{E_F\left[\left|\psi^F_{M_g,n}(V;\gamma_0,\Delta)\right|^3\right]}{\left(E_F\left[\psi^F_{M_g,n}(V;\gamma_0,\Delta)^2\right]\right)^{3/2}} \leq \overline{C} < \infty \ \forall\ (F,\Delta) \in \mathcal{F}\times\Theta, \\ \sup_{\Delta\in\Theta}\left|\varsigma^{M_g}_{F,n}(\Delta)\right| = o_p\left(n^{-1/2}\right) \ \text{uniformly over } \mathcal{F} \text{ (i.e, } \sup_{F\in\mathcal{F}} P_F\left(n^{1/2}\cdot\sup_{\Delta\in\Theta}\left|\varsigma^{M_g}_{F,n}(\Delta)\right| > \epsilon\right) \to 0 \ \forall\ \epsilon > 0) \end{cases} \tag{S2.27}$$

From (S2.27),

$$\overline{\lim}_{n\to\infty} \frac{E_F\left[\left|\psi^F_{M_g,n}(V;\gamma_0,\Delta)\right|^3\right]}{\left(E_F\left[\psi^F_{M_g,n}(V;\gamma_0,\Delta)^2\right]\right)^{3/2}} \leq \overline{C} < \infty \ \forall\ (F,\Delta) \in \mathcal{F}\times\Theta \tag{S2.28}$$

Letting

$$\sigma^F_{M_g,n}(\Delta)^2 \equiv E_F\left[\psi^F_{M_g,n}(V;\gamma_0,\Delta)^2\right],$$

the condition in (S2.28) and the Berry-Esseen Theorem (Lehmann and Romano (2005, Theorem 11.2.7)) imply that there exists a $\overline{M} < \infty$ (not depending on x, n, Δ or F) such that,

$$\left| P_F\left(n^{1/2}\cdot\frac{\frac{1}{n}\sum_{i=1}^{n}\psi^F_{M_g,n}(V_i;\gamma_0,\Delta)}{\sigma^F_{M_g,n}(\Delta)} \leq x\right) - \Phi(x)\right| \leq \frac{\overline{M}}{n^{1/2}} \quad \forall\ (F,\Delta) \in \mathcal{F}\times\Theta$$

For each $F \in \mathcal{F}$ let $\Theta^I_{g,F} \equiv \left\{\Delta \in \Theta : M_{g,F}(\gamma_0,\Delta) = 0\right\}$. For a target coverage probability $1-\alpha$, let $z_{1-\frac{\alpha}{2}}$ denote the $1-\frac{\alpha}{2}$ quantile of the $N(0,1)$ distribution. From the previous result we would obtain,

$$\lim_{n\to\infty} \underset{(F,\Delta)\in\mathcal{F}\times\Theta}{sup} \left| P_F\left(\left| n^{1/2}\cdot\frac{\frac{1}{n}\sum_{i=1}^{n}\psi^F_{M_g,n}(V_i;\gamma_0,\Delta)}{\sigma^F_{M_g,n}(\Delta)}\right| \leq z_{1-\frac{\alpha}{2}}\right) - (1-\alpha)\right| = 0.$$

[4]Note that we are explicitly denoting the dependence if each functional on F.

From here, under the restrictions leading to (S2.27), if $\sigma^F_{M_g,n}(\Delta)$ were known, a CS for Δ with target asymptotic coverage $1-\alpha$ could be constructed as

$$\left\{\Delta \in \Theta : \left|\frac{n^{1/2}\cdot \widehat{M}_g(\widehat{\gamma},\Delta)}{\sigma^F_{M_g,n}(\Delta)}\right| \leq z_{1-\frac{\alpha}{2}}\right\}.$$

Our proposed estimator for $\sigma^F_{M_g,n}(\Delta)^2$ would be of the form,

$$\widehat{\sigma}_{M_g,n}(\Delta)^2 = \frac{1}{n}\sum_{i=1}^{n} \widehat{\psi}_{M_g,n}(V_i;\widehat{\gamma},\Delta)^2,$$

where $\widehat{\psi}_{M_g,n}(v;\gamma,\Delta)$ is an estimator of the influence function $\psi^F_{M_g,n}(v;\gamma,\Delta)$. A CS for Δ_0 with target asymptotic coverage probability $1-\alpha$ can be constructed as,

$$CS^{\Delta}_n(1-\alpha) = \left\{\Delta \in \Theta : \left|\frac{n^{1/2}\cdot \widehat{M}_g(\widehat{\gamma},\Delta)}{\widehat{\sigma}_{M_g,n}(\Delta)}\right| \leq \Phi^{-1}\left(1-\frac{\alpha}{2}\right)\right\} \tag{S2.29}$$

Under conditions such that,

$$\liminf_{n\to\infty} \inf_{\substack{(F,\Delta)\in\mathcal{F}\times\Theta:\\ \Delta\in\Theta^I_{g,F}}} \frac{\widehat{\sigma}_{M_g,n}(\Delta)}{\sigma^F_{M_g,n}(\Delta)^2} \geq 1,$$

our proposed CS satisfies,

$$\liminf_{n\to\infty} \inf_{\substack{(F,\Delta)\in\mathcal{F}\times\Theta:\\ \Delta\in\Theta^I_{g,F}}} P_F\left(\Delta \in CS^{\Delta}_n(1-\alpha)\right) \geq 1-\alpha$$

And, under conditions such that,

$$\sup_{\substack{(F,\Delta)\in\mathcal{F}\times\Theta:\\ \Delta\in\Theta^I_{g,F}}} \left|\frac{\widehat{\sigma}_{M_g,n}(\Delta)}{\sigma^F_{M_g,n}(\Delta)^2} - 1\right| \xrightarrow{p} 0,$$

our proposed CS satisfies,

$$\lim_{n\to\infty} \sup_{\substack{(F,\Delta)\in\mathcal{F}\times\Theta:\\ \Delta\in\Theta^I_{g,F}}} \left|P_F\left(\Delta \in CS^{\Delta}_n(1-\alpha)\right) - (1-\alpha)\right| = 0.$$

S3 Asymptotic properties of the estimator $\widehat{\theta}$ proposed in Section 3.5.2

The estimator proposed in Section 3.5.2 is analogous to the estimator we described in Section 2.6. Accordingly, we will maintain Assumptions E1, G4 -G5, I5 -I7 and we will properly modify Assumptions E2 and E3 based on the difference in the specifications of equations (15) and (38). Consider the following.

Assumption E2' ***(A modified version of Assumption E2)***

(i) Let M be the integer described in Assumption E1. The following functionals are $M-$times continuously differentiable with respect to z, with bounded derivatives for all $(z,\theta)\in\mathcal{Z}\times\Theta$,

- $f_Z(z)$

  Element-wise:

- $\mu_I^{\Xi_Y}(y|z,\theta)\equiv E[\Xi(y|X,\theta)(D(y)-m_1(y|X,\theta))\,|Z=z]\;\;\forall\; y\in\mathcal{Y}$
- $\mu_{II,1}^{\Xi_Y}(y|z,\theta)\equiv E[\Xi(y|X,\theta)\,|Z=z]\;\;\forall\; y\in\mathcal{Y}$
- $\mu_{II,2}^{\Xi_Y}(y|z,\theta)\equiv E[\Xi(y|X,\theta)\cdot\Xi(y|X,\theta)'\,|Z=z]\;\;\forall\; y\in\mathcal{Y}$
- $\mu_{III,\theta_\ell}^{\Xi_Y}(y|z,\theta)\equiv E\left[\frac{\partial\Xi(y|X,\theta)}{\partial\theta_\ell}\cdot(D(y)-m_1(y|X,\theta))\,\Big|Z=z\right]$ *(for $\ell=1,\ldots,d_\theta$ and $\forall\; y\in\mathcal{Y}$)*
- $\mu_{IV,\theta_\ell}^{\Xi_Y}(y|z,\theta)\equiv E\left[\Xi_Y(y|X,\theta)\cdot\frac{\partial m_1(y|X,\theta)}{\partial\theta_\ell}\,\Big|Z=z\right]$ *(for $\ell=1,\ldots,d_\theta$ and $\forall\; y\in\mathcal{Y}$)*
- $\mu_{V,\theta_\ell}^{\Xi_Y}(y|z,\theta)\equiv E\left[\Xi_Y(y|X,\theta)\cdot\frac{\partial\Xi_Y(y|X,\theta)'}{\partial\theta_\ell}\,\Big|Z=z\right]$ *(for $\ell=1,\ldots,d_\theta$ and $\forall\; y\in\mathcal{Y}$)*
- $\mu_{VI,\theta_\ell}^{\Xi_Y}(y|z,\theta)\equiv E\left[\frac{\partial\Xi_Y(y|X,\theta)}{\partial\theta_\ell}\,\Big|Z=z\right]$ *(for $\ell=1,\ldots,d_\theta$ and $\forall\; y\in\mathcal{Y}$)*

In addition, there exists a $\overline{C}<\infty$ such that each of the functionals described above is bounded above (in a matrix-norm sense) by $\overline{C}$, for all $(z,\theta)\in\mathcal{Z}\times\Theta$ and, letting $\overline{y}\in\mathcal{Y}$ be the potential outcome that satisfies the invertibility restriction in Assumption I6, then $\left\|\mu_{II,2}^{\Xi_Y}(\overline{y}|z,\theta)^{-1}\right\|\leq\overline{C}\;\forall\;(z,\theta)\in\mathcal{Z}\times\Theta$. Finally, there exists a $\underline{C}>0$ such that $f_Z(z)\geq\underline{C}$ for all $z\in\mathcal{Z}$.

(ii) The following functionals are element-wise continuously differentiable with respect to z with bounded first derivative for all $(z,\theta)\in\mathcal{Z}\times\Theta$, $\forall\; y\in\mathcal{Y}$,

- $\Upsilon_{I,\theta_\ell,\theta_j}(y|z,\theta)\equiv E\left[\frac{\partial^2\Xi_Y(y|X,\theta)}{\partial\theta_j\partial\theta_\ell}\cdot(D(y)-m_1(y|X,\theta))\,\Big|Z=z\right]$ *(for $j,\ell=1,\ldots,d_\theta$)*
- $\Upsilon_{II,\theta_\ell,\theta_j}(y|z,\theta)\equiv E\left[\frac{\partial\Xi_Y(y|X,\theta)}{\partial\theta_\ell}\cdot\frac{\partial m_1(y|X,\theta)}{\partial\theta_j}\,\Big|Z=z\right]$ *(for $j,\ell=1,\ldots,d_\theta$)*
- $\Upsilon_{III,\theta_\ell,\theta_j}(y|z,\theta)\equiv E\left[\Xi_Y(y|X,\theta)\cdot\frac{\partial^2 m_1(y|X,\theta)}{\partial\theta_j\partial\theta_\ell}\,\Big|Z=z\right]$ *(for $j,\ell=1,\ldots,d_\theta$)*
- $\Upsilon_{IV,\theta_\ell,\theta_j}(y|z,\theta)\equiv E\left[\frac{\partial^2\Xi_Y(y|X,\theta)}{\partial\theta_j\partial\theta_\ell}\cdot\Xi_Y(y|X,\theta)'\,\Big|Z=z\right]$ *(for $j,\ell=1,\ldots,d_\theta$)*
- $\Upsilon_{V,\theta_\ell,\theta_j}(y|z,\theta)\equiv E\left[\frac{\partial\Xi_Y(y|X,\theta)}{\partial\theta_\ell}\cdot\frac{\partial\Xi_Y(y|X,\theta)'}{\partial\theta_j}\,\Big|Z=z\right]$ *(for $j,\ell=1,\ldots,d_\theta$)*

■

Assumption E3' ***(A modified version of Assumption E3)***

Each of the functions below has the following type of Lipschitz property: $\|g(u,\theta)-g(u,\theta')\| \leq \overline{G}(u)\cdot \|\theta-\theta'\|\ \forall\ u \in Supp(U)$ *(unless noted otherwise), and* $\forall\ \theta,\theta' \in \Theta$*, where* $\overline{G}(\cdot)$ *is a nonnegative function that satisfies* $E[\overline{G}(U)^{2+\delta}] < \infty$ *for some* $\delta > 0$*.*

- *The Lipschitz property holds for all* $x \in Supp(X)$ *and* $y \in \mathcal{Y}$ *for the following functions:* $m_1(y|x,\theta)$, $\Xi_Y(y|x,\theta)$, $\frac{\partial m_1(y|x,\theta)}{\partial\theta_\ell}$, $\frac{\partial \Xi_Y(y|x,\theta)}{\partial\theta_\ell}$ *(for* $\ell = 1,\ldots,d_\theta$*), with* $\overline{G}(X)$ *satisfying* $E[\overline{G}(X)^{2+\delta}] < \infty$ *for some* $\delta > 0$*.*
- *The Lipschitz property holds for all* $z \in \mathcal{Z}$ *and* $y \in \mathcal{Y}$ *for the following functionals (see our definitions in Assumption E2'):* $\mu_I^{\Xi_Y}(y|z,\theta)$, $\mu_{II,\kappa}^{\Xi_Y}(y|z,\theta)$ *(for* $\kappa = 1,2$*),* $\mu_{III,\theta_\ell}^{\Xi_Y}(y|z,\theta)$, $\mu_{IV,\theta_\ell}^{\Xi_Y}(y|z,\theta)$, $\mu_{V,\theta_\ell}^{\Xi_Y}(y|z,\theta)$, $\mu_{VI,\theta_\ell}^{\Xi_Y}(y|z,\theta)$ *(for* $\ell = 1,\ldots,d_\theta$*). In each case,* $\overline{G}(Z)$ *satisfies* $E[\overline{G}(Z)^{2+\delta}] < \infty$ *for some* $\delta > 0$*.*
- *The Lipschitz property holds for all* $z \in \mathcal{Z}$ *and* $y \in \mathcal{Y}$ *for the following functionals, and for every* $\ell, j = 1,\ldots,d_\theta$ *(see our definitions in Assumption E2'):* $\Upsilon_{I,\theta_\ell,\theta_j}(y|z,\theta)$, $\Upsilon_{II,\theta_\ell,\theta_j}(y|z,\theta)$, $\Upsilon_{III,\theta_\ell,\theta_j}(y|z,\theta)$, $\Upsilon_{IV,\theta_\ell,\theta_j}(y|z,\theta)$, $\Upsilon_{V,\theta_\ell,\theta_j}(y|z,\theta)$*. In each case,* $\overline{G}(Z)$ *satisfies* $E[\overline{G}(Z)^{2+\delta}] < \infty$ *for some* $\delta > 0$*.*

■

Analogous arguments leading to the linear representation results in (S1.11) and (S1.27) can be used to show that,

$$
\begin{aligned}
\widehat{\delta}(z,\theta) &= \delta(z,\theta) + \frac{1}{n\cdot h_n^{d_z}}\sum_{i=1}^{n}\psi_n^{\delta}(V_i;z,\theta) + \vartheta_n^{\delta}(z,\theta),\\
\frac{\partial\widehat{\delta}(z,\theta)}{\partial\theta} &= \frac{\partial\delta(z,\theta)}{\partial\theta} + \frac{1}{n\cdot h_n^{d_Z}}\sum_{i=1}^{n}\psi_n^{\nabla\theta\delta}(V_i;z,\theta) + \vartheta_n^{\nabla\theta\delta}(z,\theta),\\
\text{where}\quad & \sup_{\substack{\theta\in\Theta\\ z\in\mathcal{Z}}}\left\|\vartheta_n^{\delta}(z,\theta)\right\| = o_p\left(n^{-1/2}\right),\quad \sup_{\substack{\theta\in\Theta\\ z\in\mathcal{Z}}}\left\|\vartheta_n^{\nabla\theta\delta}(z,\theta)\right\| = o_p\left(n^{-1/2}\right),
\end{aligned}
\tag{S3.1}
$$

and where $E[\psi_n^{\delta}(V;z,\theta)] = 0$, $E[\psi_n^{\nabla\theta\delta}(V;z,\theta)] = 0\ \forall\ (z,\theta) \in \mathcal{Z}\times\Theta$. Next, in what follows, let $(V_1,V_2,V_3,V_4) \sim F_V \otimes F_V \otimes F_V \otimes F_V$ (four randomly drawn observations from our i.i.d sample $(V_i)_{i=1}^n$). Let,

$$
\begin{aligned}
g_\theta^a(y|V_1,V_2,V_3;\theta) &\equiv \frac{\partial\varphi(y|V_3,\theta)}{\partial\theta}\cdot\varphi(y|V_1,\theta)\cdot\mathbb{1}_{\mathcal{Z}}\{U_1\leq U_2\}\cdot\mathbb{1}_{\mathcal{Z}}\{U_3\leq U_2\},\\
g_{\theta,n}^b(y|V_1,V_2,V_3,V_4;\theta) &\equiv \frac{1}{h_n^{d_Z}}\times\Bigg(\Bigg(\psi_n^{\nabla\theta\delta}(V_4;Z_3,\theta)'\Xi(y|X_3,\theta) + \frac{\partial\Xi(y|X_3,\theta)}{\partial\theta}'\psi_n^{\delta}(V_4;Z_3,\theta)\Bigg)\cdot\varphi(y|V_1,\theta)\\
&+ \frac{\partial\varphi(y|V_3,\theta)}{\partial\theta}\cdot\psi_n^{\delta}(V_4;Z_1,\theta)'\Xi(y|X_1,\theta)\Bigg)\cdot\mathbb{1}_{\mathcal{Z}}\{U_1\leq U_2\}\cdot\mathbb{1}_{\mathcal{Z}}\{U_3\leq U_2\}
\end{aligned}
\tag{S3.2}
$$

Since $E[\psi_n^{\delta}(V;z,\theta)] = 0$, $E[\psi_n^{\nabla\theta\delta}(V;z,\theta)] = 0\ \forall\ (z,\theta)\in\mathcal{Z}\times\Theta$, we have

$$
E[g_{\theta,n}^b(y|V_1,V_2,V_3,V_4;\theta)|V_1,V_2,V_3] = 0\ \forall\ \theta\in\Theta,\ y\in\mathcal{Y}
$$

Let

$$\mathcal{V}^a_{\theta,n}(y|\theta) \equiv \frac{1}{n\cdot(n-1)\cdot(n-2)}\sum_{j=1}^{n}\sum_{i\neq j}\sum_{\ell\neq i,j} g^a_\theta(y|V_i,V_j,V_\ell;\theta),$$

$$\mathcal{V}^b_{\theta,n}(y|\theta) \equiv \frac{1}{n\cdot(n-1)\cdot(n-2)\cdot(n-3)}\sum_{j=1}^{n}\sum_{i\neq j}\sum_{\ell\neq i,j}\sum_{k\neq i,j,\ell} g^b_{\theta,n}(y|V_i,V_j,V_\ell,V_k;\theta)$$

Parallel arguments to those leading to the results in equations (S1.59)-(S1.64) now yield,

$$\begin{aligned}&\frac{\partial \widehat{Q}_{Y,\mathcal{Z}}(y|\theta)}{\partial\Delta} = \left(\frac{(n-1)(n-2)}{n^2}\right)\cdot\mathcal{V}^a_{\theta,n}(y|\theta) - \left(\frac{(n-1)(n-2)(n-3)}{n^3}\right)\cdot\mathcal{V}^b_{\theta,n}(y|\theta) + \xi^{\nabla\theta Q}_{I,n}(y|\theta),\\ &\text{where}\quad \sup_{\theta\in\Theta}\left\|\xi^{\nabla\theta Q}_{I,n}(y|\theta)\right\| = o_p\left(n^{-1/2}\right)\end{aligned} \tag{S3.3}$$

From here, the next step is to analyze the U-statistics $\mathcal{V}^a_{\theta,n}(y|\theta)$ and $\mathcal{V}^b_{\theta,n}(y|\theta)$ through their *Hoeffding decompositions* (see Serfling (1980, pages 177-178) or Sherman (1994, equations (6)-(7))). Let us begin with $\mathcal{V}^a_{\theta,n}(y|\theta)$. Let

$$\mathcal{T}(y|\theta) \equiv E\left[g^a_\theta(y|V_1,V_2,V_3;\theta)\right].$$

Note that $\mathcal{T}(y|\theta_0) = 0$ since,

$$\begin{aligned}\mathcal{T}(y|\theta_0) &= E\left[\frac{\partial\varphi(y|V_3,\theta_0)}{\partial\theta}\cdot\varphi(y|V_1,\theta_0)\cdot\mathbb{1}_{\mathcal{Z}}\{U_1\leq U_2\}\cdot\mathbb{1}_{\mathcal{Z}}\{U_3\leq U_2\}\right]\\ &= E\Bigg[\frac{\partial\varphi(y|V_3,\theta_0)}{\partial\theta}\cdot\underbrace{E\left[\varphi(y|V_1,\theta_0)\cdot\mathbb{1}_{\mathcal{Z}}\{U_1\leq U_2\}\,\middle|U_2\right]}_{\equiv T_{\mathcal{Z}}(y|U_2,\theta_0)=0\ \text{ a.e } U_2}\cdot\mathbb{1}_{\mathcal{Z}}\{U_3\leq U_2\}\Bigg]\\ &= E\left[0\cdot\frac{\partial\varphi(y|V_3,\theta_0)}{\partial\theta}\cdot\mathbb{1}_{\mathcal{Z}}\{U_3\leq U_2\}\right]\\ &= 0.\end{aligned}$$

Next, define,

$$\widetilde{g}^a_\theta(y|V_1,V_2,V_3;\theta) \equiv \frac{1}{3!}\sum_p g^a_\theta(y|V_{m_1},V_{m_2},V_{m_3};\theta), \tag{S3.4}$$

where $\sum\limits_p$ denotes the sum over the 3! permutations $\{m_1,m_2,m_3\}$ of $\{1,2,3\}$. By construction, $\widetilde{g}^a_\theta(y|V_1,V_2,V_3;\theta)$ is symmetric in (V_1,V_2,V_3), and $E\left[\widetilde{g}^a_\theta(y|V_1,V_2,V_3;\theta)\right] = \mathcal{T}(y|\theta)$. We can express,

$$\mathcal{V}^a_{\theta,n}(y|\theta) = \binom{n}{3}^{-1}\sum_c \widetilde{g}^a_\theta(y|V_i,V_j,V_\ell;\theta),$$

where $\sum\limits_c$ denotes the sum over the $\binom{n}{3}$ distinct combinations $\{i,j,\ell\}$ from $\{1,\ldots,n\}$. Let

$$\overline{g}^a_\theta(y|V_1;\theta) \equiv 3\cdot\left(E\left[\widetilde{g}^a_\theta(y|V_1,V_2,V_3;\theta)\,\middle|V_1\right] - \mathcal{T}(y|\theta)\right) \tag{S3.5}$$

Note that, by construction, $E[\overline{g}^a_\theta(y|V_1;\theta)] = 0 \; \forall \; \theta$. Also note that evaluated at θ_0, we have,

$$\overline{g}^a_\theta(y|V_1;\theta_0) = E\left[E\left[\frac{\partial\varphi(y|V_3,\theta_0)}{\partial\theta}\cdot \mathbb{1}_{\mathcal{Z}}\{U_3 \leq U_2\}\,\Big|U_2\right]\cdot \mathbb{1}_{\mathcal{Z}}\{U_1 \leq U_2\}\,\Big|U_1\right]\cdot \varphi(y|V_1,\theta_0) \tag{S3.6}$$

The Hoeffding decomposition of $\mathcal{V}^a_{\theta,n}(y|\theta)$ (see Serfling (1980, pages 177-178) is given by,

$$\mathcal{V}^a_{\theta,n}(y|\theta) = \mathcal{T}(y|\theta) + \frac{1}{n}\sum_{i=1}^{n}\overline{g}^a_\theta(y|V_i;\theta) + \mathcal{U}^a_n(y|\theta),$$

where $\mathcal{U}^a_n(y|\theta)$ is a linear combination of *degenerate* U-statistics of orders 2 and 3. Given our smoothness and integrability restrictions, Nolan and Pollard (1987, Lemma 22), Pakes and Pollard (1989, Example 2.10, Lemma 2.13, Lemma 2.14) imply that the class of functions involved is *Euclidean* (see Sherman (1994, Definition 3)). From here, Sherman (1994, Corollary 4) yields,

$$\sup_{\theta\in\Theta}\left|\mathcal{U}^a_n(y|\theta)\right| = O_p\left(\frac{1}{n}\right) = o_p\left(n^{-1/2}\right).$$

Thus,

$$\mathcal{V}^a_{\theta,n}(y|\theta) = \mathcal{T}(y|\theta) + \frac{1}{n}\sum_{i=1}^{n}\overline{g}^a_\theta(y|V_i;\theta) + \mathcal{U}^a_n(y|\theta), \quad \text{where} \quad \sup_{\theta\in\Theta}\left|\mathcal{U}^a_n(y|\theta)\right| = o_p\left(n^{-1/2}\right). \tag{S3.7}$$

And evaluated at θ_0, we have $\mathcal{T}(\theta_0) = 0$, and

$$\begin{aligned} \mathcal{V}^a_{\theta,n}(y|\theta_0) &= \frac{1}{n}\sum_{i=1}^{n}\overline{g}^a_\theta(y|V_i;\theta_0) + o_p\left(n^{-1/2}\right), \quad \text{where} \\ \overline{g}^a_\theta(y|V_1;\theta_0) &= E\left[E\left[\frac{\partial\varphi(y|V_3,\theta_0)}{\partial\theta}\cdot \mathbb{1}_{\mathcal{Z}}\{U_3 \leq U_2\}\,\Big|U_2\right]\cdot \mathbb{1}_{\mathcal{Z}}\{U_1 \leq U_2\}\,\Big|U_1\right]\cdot \varphi(y|V_1,\theta_0) \end{aligned} \tag{S3.8}$$

Under our restrictions, $\sup_{\theta\in\Theta}\left|\mathcal{T}(y|\theta)\right| < \infty$. Thus, from (S3.7), we have $\sup_{\theta\in\Theta}\left|\mathcal{V}^a_{\theta,n}(y|\theta)\right| = O_p(1)$ and,

$$\sup_{\theta\in\Theta}\left|\underbrace{\left(\frac{2-3n}{n^2}\right)}_{=O(\frac{1}{n})}\cdot\mathcal{V}^a_{\theta,n}(y|\theta)\right| = O_p\left(\frac{1}{n}\right).$$

From here, it follows that,

$$\left(\frac{(n-1)(n-2)}{n^2}\right)\cdot \mathcal{V}^a_{\theta,n}(y|\theta) = \mathcal{V}^a_{\theta,n}(y|\theta) + \underbrace{\left(\frac{2-3n}{n^2}\right)\cdot \mathcal{V}^a_{\theta,n}(y|\theta)}_{\equiv\vartheta^a_n(y|\theta)} = \mathcal{T}(y|\theta) + \frac{1}{n}\sum_{i=1}^{n}\overline{g}^a_\theta(y|V_i;\theta) + \vartheta^a_n(y|\theta),$$

$$\text{where} \quad \sup_{\theta\in\Theta}\left|\vartheta^a_n(y|\theta)\right| = O_p\left(\frac{1}{n}\right) = o_p\left(n^{-1/2}\right).$$

In particular, evaluated at θ_0,

$$\left(\frac{(n-1)(n-2)}{n^2}\right)\cdot \mathcal{V}^a_{\theta,n}(y|\theta_0) = \frac{1}{n}\sum_{i=1}^{n}\overline{g}^a_\theta(y|V_i;\theta_0) + o_p\left(n^{-1/2}\right), \quad \text{where}$$
$$\overline{g}^a_\theta(y|V_1;\theta_0) = E\left[E\left[\frac{\partial\varphi(y|V_3,\theta_0)}{\partial\theta}\cdot \mathbb{1}_{\mathcal{Z}}\{U_3 \le U_2\}\,\Big|U_2\right]\cdot \mathbb{1}_{\mathcal{Z}}\{U_1 \le U_2\}\,\Big|U_1\right]\cdot\varphi(y|V_1,\theta_0) \tag{S3.9}$$

Next we analyze the Hoeffding decomposition of $\mathcal{V}^b_{\theta,n}(y|\theta)$, the last remaining term to study in (S3.3). As we pointed out previously,

$$E[g^b_{\theta,n}(y|V_1,V_2,V_3,V_4;\theta)|V_1,V_2,V_3] = 0 \;\forall\; \theta\in\Theta,\, y\in\mathcal{Y}$$

and therefore $E\left[g^b_{\theta,n}(y|V_1,V_2,V_3,V_4;\theta)\right] = 0 \;\forall\; \theta\in\Theta,\, y\in\mathcal{Y}$. Define,

$$\widetilde{g}^b_{\theta,n}(y|V_1,V_2,V_3,V_4;\theta) \equiv \frac{1}{4!}\sum_p g^b_{\theta,n}(V_{m_1},V_{m_2},V_{m_3},V_{m_4};\theta), \tag{S3.10}$$

where $\sum_p$ denotes the sum over the 4! permutations $\{m_1,m_2,m_3,m_4\}$ of $\{1,2,3,4\}$. By construction, $\widetilde{g}^b_{\theta,n}(y|V_1,V_2,V_3,V_4;\theta)$ is symmetric in (V_1,V_2,V_3,V_4), and $E\left[\widetilde{g}^b_{\theta,n}(y|V_1,V_2,V_3,V_4;\theta)\right] = 0 \;\forall\; \theta$. We can express,

$$\mathcal{V}^b_{\theta,n}(y|\theta) = \binom{n}{4}^{-1}\sum_c \widetilde{g}^b_{\theta,n}(y|V_i,V_j,V_\ell,V_k;\theta),$$

where $\sum_c$ denotes the sum over the $\binom{n}{4}$ distinct combinations $\{i,j,\ell,k\}$ from $\{1,\dots,n\}$. Let

$$\overline{g}^b_{\theta,n}(y|V_1;\theta) \equiv 4\cdot E\left[\widetilde{g}^b_{\theta,n}(y|V_1,V_2,V_3,V_4;\theta)\,\Big|V_1\right].$$

By inspection, we can see that,

$$\begin{aligned}
&E\left[\widetilde{g}^b_{\theta,n}(y|V_1,V_2,V_3,V_4;\theta)\,\Big|V_1\right] = \frac{3!}{4!}E\left[g^b_{\theta,n}(y|V_2,V_3,V_4,V_1;\theta)\,\Big|V_1\right]\\
=&\frac{1}{4}\times\Bigg(E\left[\frac{1}{h_n^{d_Z}}\psi_n^{\nabla\delta}(V_1;Z_4,\theta)'\Xi(y|X_4,\theta)\cdot\varphi(y|V_2,\theta)\cdot\mathbb{1}_{\mathcal{Z}}\{U_2\le U_3\}\cdot\mathbb{1}_{\mathcal{Z}}\{U_4\le U_3\}\,\Big|V_1\right]\\
&+E\left[\frac{\partial\Xi(y|X_4,\theta)'}{\partial\theta}\frac{1}{h_n^{d_Z}}\psi_n^{\delta}(V_1;Z_4,\theta)\cdot\varphi(y|V_2,\theta)\cdot\mathbb{1}_{\mathcal{Z}}\{U_2\le U_3\}\cdot\mathbb{1}_{\mathcal{Z}}\{U_4\le U_3\}\,\Big|V_1\right]\\
&+E\left[\frac{\partial\varphi(y|V_4,\theta)}{\partial\theta}\cdot\frac{1}{h_n^{d_Z}}\psi_n^{\delta}(V_1;Z_2,\theta)'\Xi(y|X_2,\theta)\cdot\mathbb{1}_{\mathcal{Z}}\{U_2\le U_3\}\cdot\mathbb{1}_{\mathcal{Z}}\{U_4\le U_3\}\,\Big|V_1\right]\Bigg)
\end{aligned}$$

Therefore,

$$\begin{aligned}\overline{g}^b_{\theta,n}(y|V_1;\theta) = & E\left[\frac{1}{h_n^{d_Z}}\psi_n^{\nabla\delta}(V_1;Z_4,\theta)'\Xi(y|X_4,\theta)\cdot\varphi(y|V_2,\theta)\cdot\mathbb{1}_{\mathcal{Z}}\{U_2\le U_3\}\cdot\mathbb{1}_{\mathcal{Z}}\{U_4\le U_3\}\,\Big|V_1\right] \\ & + E\left[\frac{\partial\Xi(y|X_4,\theta)'}{\partial\theta}\frac{1}{h_n^{d_Z}}\psi_n^{\delta}(V_1;Z_4,\theta)\cdot\varphi(y|V_2,\theta)\cdot\mathbb{1}_{\mathcal{Z}}\{U_2\le U_3\}\cdot\mathbb{1}_{\mathcal{Z}}\{U_4\le U_3\}\,\Big|V_1\right] \\ & + E\left[\frac{\partial\varphi(y|V_4,\theta)}{\partial\theta}\cdot\frac{1}{h_n^{d_Z}}\psi_n^{\delta}(V_1;Z_2,\theta)'\Xi(y|X_2,\theta)\cdot\mathbb{1}_{\mathcal{Z}}\{U_2\le U_3\}\cdot\mathbb{1}_{\mathcal{Z}}\{U_4\le U_3\}\,\Big|V_1\right].\end{aligned} \tag{S3.11}$$

Note that, evaluated at θ_0,

$$\begin{aligned}& E\left[\frac{1}{h_n^{d_Z}}\psi_n^{\nabla\delta}(V_1;Z_4,\theta_0)'\Xi(y|X_4,\theta_0)\cdot\varphi(y|V_2,\theta_0)\cdot\mathbb{1}_{\mathcal{Z}}\{U_2\le U_3\}\cdot\mathbb{1}_{\mathcal{Z}}\{U_4\le U_3\}\,\Big|V_1\right] \\ =& E\left[\frac{1}{h_n^{d_Z}}\psi_n^{\nabla\delta}(V_1;Z_4,\theta_0)'\Xi(y|X_4,\theta_0)\cdot\underbrace{E[\varphi(y|V_2,\theta_0)|U_2]}_{=0}\cdot\mathbb{1}_{\mathcal{Z}}\{U_2\le U_3\}\cdot\mathbb{1}_{\mathcal{Z}}\{U_4\le U_3\}\,\Big|V_1\right]=0, \\ & E\left[\frac{\partial\Xi(y|X_4,\theta_0)'}{\partial\theta}\frac{1}{h_n^{d_Z}}\psi_n^{\delta}(V_1;Z_4,\theta_0)\cdot\varphi(y|V_2,\theta_0)\cdot\mathbb{1}_{\mathcal{Z}}\{U_2\le U_3\}\cdot\mathbb{1}_{\mathcal{Z}}\{U_4\le U_3\}\,\Big|V_1\right] \\ =& E\left[\frac{\partial\Xi(y|X_4,\theta_0)'}{\partial\theta}\frac{1}{h_n^{d_Z}}\psi_n^{\delta}(V_1;Z_4,\theta_0)\cdot\underbrace{E[\varphi(y|V_2,\theta_0)|U_2]}_{=0}\cdot\mathbb{1}_{\mathcal{Z}}\{U_2\le U_3\}\cdot\mathbb{1}_{\mathcal{Z}}\{U_4\le U_3\}\,\Big|V_1\right]=0\end{aligned}$$

Thus, evaluated at θ_0,

$$\begin{aligned}\overline{g}^b_{\theta,n}(y|V_1;\theta_0) &= E\left[\frac{\partial\varphi(y|V_4,\theta)}{\partial\theta}\cdot\frac{1}{h_n^{d_Z}}\psi_n^{\delta}(V_1;Z_2,\theta)'\Xi(y|X_2,\theta)\cdot\mathbb{1}_{\mathcal{Z}}\{U_2\le U_3\}\cdot\mathbb{1}_{\mathcal{Z}}\{U_4\le U_3\}\,\Big|V_1\right] \\ &= E\left[\frac{\partial\varphi(y|V_4,\theta)}{\partial\theta}\cdot E\left[\frac{1}{h_n^{d_Z}}\psi_n^{\delta}(V_1;Z_2,\theta)'\Xi(y|X_2,\theta)\cdot\mathbb{1}_{\mathcal{Z}}\{U_2\le U_3\}\Big|V_1,U_3\right]\cdot\mathbb{1}_{\mathcal{Z}}\{U_4\le U_3\}\,\Big|V_1\right].\end{aligned} \tag{S3.12}$$

The Hoeffding decomposition of $\mathcal{V}^b_{\theta,n}(y|\theta)$ is given by,

$$\mathcal{V}^b_{\theta,n}(y|\theta) = \frac{1}{n}\sum_{i=1}^{n}\overline{g}^b_{\theta,n}(y|V_i;\theta) + \mathcal{U}^b_n(y|\theta),$$

where $\mathcal{U}^b_n(y|\theta)$ is a linear combination of degenerate U-statistics of orders 2, 3 and 4. By the bounded-variation properties of our kernel function described in Assumption E1 and our smoothness and integrability conditions, Nolan and Pollard (1987, Lemma 22), Pakes and Pollard (1989, Example 2.10, Lemma 2.13, Lemma 2.14) imply that the class of functions involved is *Euclidean* (see Sherman (1994, Definition 3)). From here, Sherman (1994, Corollary 4) yields

$$\sup_{\theta\in\Theta}|\mathcal{U}^b_n(y|\theta)| = O_p\left(\frac{1}{n\cdot h_n^{d_Z}}\right) = o_p\left(n^{-1/2}\right),$$

where the last equality follows from our bandwidth convergence restriction $n^{1/2} \cdot h_n^{d_Z} \longrightarrow \infty$. Thus,

$$\mathcal{V}^b_{\theta,n}(y|\theta) = \frac{1}{n}\sum_{i=1}^{n} \overline{g}^b_{\theta,n}(y|V_i;\theta) + \mathcal{U}^b_n(y|\theta), \quad \text{where} \quad \sup_{\theta\in\Theta} |\mathcal{U}^b_n(y|\theta)| = o_p\left(n^{-1/2}\right),$$

Let us focus on $\theta = \theta_0$. Note first that,

$$\left(\frac{(n-1)(n-2)(n-3)}{n^3}\right)\cdot \mathcal{V}^b_{\theta,n}(y|\theta_0) = \mathcal{V}^b_{\theta,n}(y|\theta_0) + \underbrace{\underbrace{\left(\frac{-6n^2+11n-6}{n^3}\right)}_{=O\left(\frac{1}{n}\right)}\cdot \underbrace{\mathcal{V}^b_{\theta,n}(y|\theta_0)}_{=O_p(1)}}_{=O_p\left(\frac{1}{n}\right)=o_p\left(n^{-1/2}\right)}$$

Thus,

$$\left(\frac{(n-1)(n-2)(n-3)}{n^3}\right)\cdot \mathcal{V}^b_{\theta,n}(y|\theta_0) = \frac{1}{n}\sum_{i=1}^{n} \overline{g}^b_{\theta,n}(y|V_i;\theta_0) + o_p(n^{-1/2}) \tag{S3.13}$$

where $\overline{g}^b_{\theta,n}(y|V_1;\theta_0)$ is as described in (S3.12). Combining our results in (S3.3), (S3.9) and (S3.13),

$$\begin{aligned}
\frac{\partial \widehat{Q}_{Y,\mathcal{Z}}(y|\theta_0)}{\partial\theta} &= \frac{1}{n}\sum_{i=1}^{n}\Gamma_{\theta,n}(y|V_i;\theta_0) + o_p\left(n^{-1/2}\right), \quad \text{where} \\
\Gamma_{\theta,n}(y|V_1;\theta_0) &\equiv \overline{g}^a_\theta(y|V_1;\theta_0) - \overline{g}^b_{\theta,n}(y|V_1;\theta_0) \\
&= E\left[E\left[\frac{\partial\varphi(y|V_3,\theta_0)}{\partial\theta}\cdot \mathbb{1}_{\mathcal{Z}}\{U_3 \le U_2\}\,\Big|\,U_2\right]\cdot \mathbb{1}_{\mathcal{Z}}\{U_1 \le U_2\}\,\Big|\,U_1\right]\cdot\varphi(y|V_1,\theta_0) \\
&\quad - E\left[\frac{\partial\varphi(y|V_4,\theta)}{\partial\theta}\cdot E\left[\frac{1}{h_n^{d_Z}}\psi^\delta_n(V_1;Z_2,\theta)'\Xi(y|X_2,\theta)\cdot\mathbb{1}_{\mathcal{Z}}\{U_2\le U_3\}\,\Big|\,V_1,U_3\right]\cdot \mathbb{1}_{\mathcal{Z}}\{U_4\le U_3\}\,\Big|\,V_1\right].
\end{aligned} \tag{S3.14}$$

Note, once again, that $E\left[\Gamma_{\theta,n}(y|V_1;\theta_0)\right] = 0$. From here,

$$\frac{\partial \widehat{Q}_{Y,\mathcal{Z}}(\theta_0)}{\partial\theta} = \frac{1}{n}\sum_{i=1}^{n}\Gamma_{\theta,n}(V_i;\theta_0) + o_p\left(n^{-1/2}\right), \quad \text{where} \quad \Gamma_{\theta,n}(V_i;\theta_0) \equiv \sum_{y\in\mathcal{Y}^*}\Gamma_{\theta,n}(y|V_i;\theta_0). \tag{S3.15}$$

Thus, $E[\Gamma_{\theta,n}(V_1;\theta_0)] = 0$.

S4 An inferential procedure for θ_0 in Section 3.7

As we described in Section 3.7, when we allow for the possibility that players cooperate almost surely, we can construct a CS for θ_0 based on (44). Take any $y \in \mathcal{Y}$ and let $g_y : \mathbb{R}^{d_U} \to \mathbb{R}$ be a real-valued, pre-specified function of U. For a given $u \in \mathbb{R}^{d_U}$, let

$$m_g(y|u,\theta) \equiv E\left[\varphi(y|V,\theta)\cdot g_y(U)\cdot \mathbb{1}_{\mathcal{Z}}\{U \le u\}\right] \quad \text{and} \quad M_g(y|\theta) \equiv E[m_g(y|U,\theta)].$$

Using iterated expectations, (44) implies $M_g(y|\theta_0)\ \forall\ y \in \mathcal{Y}^*$ and a CS for θ_0 can be constructed from here. Let $\Theta_g^I = \left\{\theta \in \Theta : M_g(y|\theta) = 0\ \forall\ y \in \mathcal{Y}^*\right\}$ be our target identified set for θ. Our sample statistic for $M_g(y|\theta)$ is,

$$\widehat{M}_g(y|\theta) = \frac{1}{n}\sum_{j=1}^{n}\widehat{m}_g(y|U_j,\theta) = \frac{1}{n^2}\sum_{j=1}^{n}\sum_{i=1}^{n}\widehat{\varphi}(y|V_i,\theta)\cdot g_y(U_i)\cdot \mathbb{1}_{\mathcal{Z}}\{U_i \le U_j\}$$

Maintain Assumptions E1, G4 -G5 and E2'-E3'. Using the linear representation result in (S3.1), we have

$$\begin{aligned}\widehat{M}_g(y|\theta) &= \frac{1}{n^2}\sum_{j=1}^{n}\sum_{i=1}^{n}\varphi(y|V_i,\theta)\cdot g_y(U_i)\cdot \mathbb{1}_{\mathcal{Z}}\{U_i \le U_j\}\\ &- \frac{1}{n^3}\sum_{j=1}^{n}\sum_{i=1}^{n}\sum_{k=1}^{m}\frac{1}{h_n^{d_Z}}\cdot \psi_n^{\delta}(V_k;Z_i,\theta)'\Xi(y|X_i,\theta)\cdot g_y(U_i)\cdot \mathbb{1}_{\mathcal{Z}}\{U_i \le U_j\}\\ &- \frac{1}{n^2}\sum_{j=1}^{n}\sum_{i=1}^{n}\vartheta_n^{\delta}(Z_i,\theta)'\Xi(y|X_i,\theta)\cdot g_y(U_i)\cdot \mathbb{1}_{\mathcal{Z}}\{U_i \le U_j\},\end{aligned} \tag{S4.1}$$

where, as described in (S3.1), $\sup_{\substack{\theta\in\Theta\\ z\in\mathcal{Z}}}\left\|\vartheta_n^{\delta}(z,\theta)\right\| = o_p\left(n^{-1/2}\right)$. As in previous sections, in what follows we will let $(V_1,V_2,V_3)\sim F_V\otimes F_V\otimes F_V$ (three randomly drawn observations from our i.i.d sample $(V_i)_{i=1}^n$). Let

$$\begin{aligned} q_{M_g}^a(y|V_1,V_2;\theta) &\equiv \varphi(y|V_1,\theta)\cdot g_y(U_1)\cdot \mathbb{1}_{\mathcal{Z}}\{U_1 \le U_2\},\\ \widetilde{q}_{M_g}^a(y|V_1,V_2;\theta) &\equiv \frac{1}{2}\cdot\left(q_{M_g}^a(y|V_1,V_2;\theta) + q_{M_g}^a(y|V_2,V_1;\theta)\right)\end{aligned}$$

Note that $E[q_{M_g}^a(y|V_1,V_2;\theta)] = E[\widetilde{q}_{M_g}^a(y|V_1,V_2;\theta)] = M_g(y|\theta)$. Next, let

$$\overline{q}_{M_g}^a(y|V_1;\theta) \equiv 2\cdot\left(E[\widetilde{q}_{M_g}^a(y|V_1,V_2;\theta)|V_1] - M_g(y|\theta)\right).$$

Note that $E[\overline{q}_{M_g}^a(y|V_1;\theta)] = 0\ \forall\ \theta\in\Theta$, and that

$$\overline{q}_{M_g}^a(y|V_1;\theta) = \varphi(y|V_1,\theta)\cdot g_y(U)\cdot E[\mathbb{1}_{\mathcal{Z}}\{U_1 \le U_2\}|U_1] \quad \forall\ \theta\in\Theta_g^I.$$

Next, let

$$\overline{q}_{M_g,n}^b(y|V_1;\theta) \equiv E\left[\frac{1}{h_n^{d_Z}}\cdot \psi_n^{\delta}(V_1;Z_2,\theta)'\Xi(y|X_2,\theta)\cdot g_y(U_2)\cdot \mathbb{1}\{U_2 \le U_3\}\middle| V_1\right].$$

By iterated expectations, we can see that $E[\overline{q}_{M_g,n}^b(y|V_1;\theta)] = 0\ \forall\ \theta\in\Theta$ since, as described in (S3.1), we have $E[\psi_n^{\delta}(V;z,\theta)] = 0\ \forall\ (z,\theta)\in\mathcal{Z}\times\Theta$. Let

$$\psi_{M_g,n}(y|V_1;\theta) \equiv \overline{q}_{M_g}^a(y|V_1;\theta) - \overline{q}_{M_g,n}^b(y|V_1;\theta).$$

Note that $E[\psi_{M_g,n}(y|V_1;\theta)] = 0\ \forall\ \theta\in\Theta$. Following analogous steps to those in Section S2 of this Supplement, we can show that, under our current assumptions, $\widehat{M}_g(y|\theta)$ satisfies the following linear representation,

$$
\begin{aligned}
&\widehat{M}_g(y|\theta) = M_g(y|\theta) + \frac{1}{n}\sum_{i=1}^{n}\psi_{M_g,n}(y|V_i;\theta) + \varsigma_n^{M_g}(y|\theta),\ \forall\ \theta\in\Theta,\\
&\text{where}\quad \sup_{\theta\in\Theta}\left|\varsigma_n^{M_g}(y|\theta)\right| = o_p\left(n^{-1/2}\right).
\end{aligned}
\tag{S4.2}
$$

And over the identified set Θ_g^I, the result in (S4.2) simplifies to,

$$
\begin{aligned}
&\widehat{M}_g(y|\theta) = \frac{1}{n}\sum_{i=1}^{n}\psi_{M_g,n}(y|V_i;\theta) + \varsigma_n^{M_g}(y|\theta),\ \forall\ \theta\in\Theta_g^I,\\
&\text{where}\quad \sup_{\theta\in\Theta}\left|\varsigma_n^{M_g}(y|\theta)\right| = o_p\left(n^{-1/2}\right).
\end{aligned}
\tag{S4.3}
$$

The results in (S4.2) and (S4.3) are analogous to those in (S2.25) and (S2.26), respectively. Construction of a CS for θ_0 can be done in a variety of ways from (S4.3). For example, consider $M_g(\theta) \equiv \sum_{y\in\mathcal{Y}^*} M_g(y|\theta)$. Then, $M_g(\theta) = 0\ \forall\ \theta \in \Theta_g^I$. We can estimate this functional with $\widehat{M}_g(\theta) = \sum_{y\in\mathcal{Y}^*} \widehat{M}_g(y|\theta)$. From (S4.2),

$$
\begin{aligned}
&\widehat{M}_g(\theta) = M_g(\theta) + \frac{1}{n}\sum_{i=1}^{n}\psi_{M_g,n}(V_i;\theta) + \varsigma_n^{M_g}(\theta),\\
&\text{where}\quad \sup_{\theta\in\Theta}\left|\varsigma_n^{M_g}(\theta)\right| = o_p\left(n^{-1/2}\right),\\
&\psi_{M_g,n}(V_1;\theta) \equiv \sum_{y\in\mathcal{Y}^*}\psi_{M_g,n}(y|V_1;\theta),\ \Longrightarrow\ E[\psi_{M_g,n}(V_1;\theta)] = 0\ \forall\ \theta\in\Theta.
\end{aligned}
\tag{S4.4}
$$

Let $\sigma_{M_g,n}(\theta)^2 \equiv E\left[\psi_{M_g,n}(V_1;\theta)^2\right]$ and consider an estimator $\widehat{\sigma}_{M_g,n}(\theta)^2 \equiv \frac{1}{n}\sum_{i=1}^{n}\widehat{\psi}_{M_g,n}(V_i;\theta)^2$. Similar to our inferential approach in Section 2.7 of this Supplement, a CS for θ_0 with target asymptotic coverage probability $1-\alpha$ can be constructed as,

$$
CS_n^{\theta}(1-\alpha) = \left\{\theta\in\Theta :\ \left|\frac{n^{1/2}\cdot\widehat{M}_g(\theta)}{\widehat{\sigma}_{M_g,n}(\theta)}\right| \leq \Phi^{-1}\left(1-\frac{\alpha}{2}\right)\right\}
\tag{S4.5}
$$

Consider the following restriction, which is a modification of Assumption E4.

Assumption E4' *Suppose the DGP F belongs to a family of distributions $\mathcal{F}$ that satisfy all the restrictions in Assumptions G4 -G5, I6-I7, E1, E2' and E3' and, in particular, suppose that the existence of $2+\delta$ moments in Assumption E3' and the smoothness restrictions in Assumptions E2'-E3' hold uniformly over $\mathcal{F}$ (i.e, the bounds described for each one of those restrictions are common to every $F\in\mathcal{F}$). In addition, suppose that the constant δ described in the existence of $2+\delta$ moment restrictions satisfies $\delta\geq 1$, and that for some $\delta\geq 1$ and $\overline{C}<\infty$, the instrument function g also satisfies $E_F[|g_y(U)|^{2+\delta}]\leq\overline{C}$ for all $F\in\mathcal{F}$.*

For each $F\in\mathcal{F}$ let $\Theta_{g,F}^I \equiv \left\{\Delta\in\Theta : M_{g,F}(\gamma_0,\Delta) = 0\ \forall\ y\in\mathcal{Y}^*\right\}$. Our previous results combined with the same Berry-Esseen bound arguments from Section 2.7 in this Supplement can be used to show that, under conditions such that,

$$\liminf_{n\to\infty} \inf_{\substack{(F,\theta)\in\mathcal{F}\times\Theta:\\ \theta\in\Theta^I_{g,F}}} \frac{\widehat{\sigma}_{M_g,n}(\theta)}{\sigma^F_{M_g,n}(\theta)^2} \geq 1,$$

our proposed CS satisfies,

$$\liminf_{n\to\infty} \inf_{\substack{(F,\theta)\in\mathcal{F}\times\Theta:\\ \Delta\in\Theta^I_{g,F}}} P_F\left(\theta \in CS^\theta_n(1-\alpha)\right) \geq 1-\alpha$$

And, under conditions such that,

$$\sup_{\substack{(F,\theta)\in\mathcal{F}\times\Theta:\\ \theta\in\Theta^I_{g,F}}} \left| \frac{\widehat{\sigma}_{M_g,n}(\theta)}{\sigma^F_{M_g,n}(\theta)^2} - 1 \right| \xrightarrow{p} 0,$$

our proposed CS satisfies,

$$\lim_{n\to\infty} \sup_{\substack{(F,\theta)\in\mathcal{F}\times\Theta:\\ \theta\in\Theta^I_{g,F}}} \left| P_F\left(\theta \in CS^\theta_n(1-\alpha)\right) - (1-\alpha) \right| = 0, \tag{S4.6}$$

so our CS has correct asymptotic coverage probability for θ_0. An alternative CS construction can potentially proceed as follows. Denote,

$$\underset{\sim}{M}_g(\theta) \equiv \left(M_g(y|\theta)\right)_{y\in\mathcal{Y}^*}, \quad \underset{\sim}{\widehat{M}}_g(\theta) \equiv \left(\widehat{M}_g(y|\theta)\right)_{y\in\mathcal{Y}^*}, \quad \underset{\sim}{\psi}_{M_g,n}(V_1;\theta) \equiv \left(\psi_{M_g,n}(y|V_1;\theta)\right)_{y\in\mathcal{Y}^*}$$

Recall that $\dim(\mathcal{Y}^*) = 3$, so each of the above vectors belongs in $\mathbb{R}^3$. From (S4.2),

$$\underset{\sim}{\widehat{M}}_g(\theta) = \underset{\sim}{M}_g(\theta) + \frac{1}{n}\sum_{i=1}^n \underset{\sim}{\psi}_{M_g,n}(V_i;\theta) + \underset{\sim}{\varsigma}_n^{M_g}(\theta), \quad \text{where} \quad \sup_{\theta\in\Theta} \left\| \underset{\sim}{\varsigma}_n^{M_g}(\theta) \right\| = o_p\left(n^{-1/2}\right)$$

Let $\underset{\sim}{\Sigma}_{M_g,n}(\theta) \equiv E[\underset{\sim}{\psi}_{M_g,n}(V_1;\theta)\cdot\underset{\sim}{\psi}_{M_g,n}(V_1;\theta)']$. Then, $\sqrt{n}\cdot\underset{\sim}{\widehat{M}}_g(\theta) \xrightarrow{d} \mathcal{N}(0,\underset{\sim}{\Sigma}_{M_g}(\theta))$ for all $\theta\in\Theta^I_g$, where $\underset{\sim}{\Sigma}_{M_g}(\theta) = \lim_{n\to\infty}\underset{\sim}{\Sigma}_{M_g,n}(\theta)$. Suppose $\underset{\sim}{\Sigma}_{M_g}(\theta)$ is positive definite (i.e, invertible) for all $\theta\in\Theta$ (this can be relaxed to only hold for all $\theta\in\Theta^I_g$). Let

$$\underset{\sim}{\widehat{\Sigma}}_{M_g,n}(\theta) \equiv \frac{1}{n}\sum_{i=1}^n \underset{\sim}{\widehat{\psi}}_{M_g,n}(V_i;\theta)\cdot\underset{\sim}{\widehat{\psi}}_{M_g,n}(V_i;\theta)'$$

A CS for θ_0 with target asymptotic coverage probability $1-\alpha$ can be constructed as,

$$CS^\theta_n(1-\alpha) \equiv \left\{\theta\in\Theta : n\cdot\underset{\sim}{\widehat{\Sigma}}_{M_g,n}(\theta)'\underset{\sim}{\Sigma}_{M_g,n}(\theta)^{-1}\underset{\sim}{\widehat{\Sigma}}_{M_g,n}(\theta) \leq c_{3,1-\alpha}\right\}$$

where $c_{k,1-\alpha}$ is the $1-\alpha$ quantile of the χ^2_k distribution. If we drop the assumption that $\underset{\sim}{\Sigma}_{M_g}(\theta)$

has full rank, we can replace the above construction with a generalized Wald statistic (see Andrews (1987)). A more general construction would be of the form,

$$CS_n^{\theta}(1-\alpha) \equiv \left\{ \theta \in \Theta : n \cdot \widehat{\underset{\sim}{\Sigma}}_{M_g,n}(\theta)' \widehat{\underset{\sim}{\Sigma}}_{M_g,n}(\theta)^{+} \widehat{\underset{\sim}{\Sigma}}_{M_g,n}(\theta) \leq c_{rank(\widehat{\underset{\sim}{\Sigma}}_{M_g,n}(\theta)),1-\alpha} \right\}$$

where A^{+} denotes the Moore-Penrose generalized inverse and $rank(A)$ denotes the rank of A. If, uniformly over $\mathcal{F} \times \Theta$, our estimated variance matrix satisfies[5]

$$\left\| \widehat{\underset{\sim}{\Sigma}}_{M_g,n}(\theta) - \underset{\sim}{\Sigma}_{M_g}(\theta) \right\| \xrightarrow{p} 0 \quad \text{and} \quad Pr\Big(rank(\widehat{\underset{\sim}{\Sigma}}_{M_g,n}(\theta)) = rank(\underset{\sim}{\Sigma}_{M_g}(\theta)) \Big) \longrightarrow 1,$$

and if the type of restrictions described in Assumption E4' are satisfied, then $CS_n^{\theta}(1-\alpha)$ would have the type of asymptotic coverage properties described in (S4.6).

[5]See Andrews (1987, Theorem 1).

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