A nonparametric test for cooperation in discrete games

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Abstract

We propose a nonparametric test for cooperative behavior among players in discrete, static games. Assuming that certain exchangeability conditions hold if we match observable characteristics across all players, we obtain testable implications for cooperative behavior when players maximize an unknown, symmetric joint objective function. Cooperation implies the existence of a class of outcomes $\mathcal{Y}$ such that, conditional on the matching, the probability of observing an outcome $y \in \mathcal{Y}$ must be equal to the probability of observing any permutation of y. We present a nonparametric econometric test and we characterize its asymptotic properties. We apply our test to expansion/entry decisions of Lowe's and Home Depot in the contiguous U.S and we find that, while outcomes are consistent with noncooperative behavior in larger markets, we fail to reject cooperation in smaller markets.

Keywords: Econometrics of games, nonparametric tests, matching, conditional moment restrictions.

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1 Introduction

The ability to test alternative behavioral models nonparametrically is a valuable tool in the econometric analysis of games. Of particular interest is the question of whether the

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data observed was generated by noncooperative behavior or if there is evidence of cooperation. Econometric methods to detect the presence of collusion have received particular attention, for example, in auction models, where the various methods proposed have taken advantage of the specific auction rules and have made precise assumptions on bidders' valuations and the nature of underlying collusive scheme in order to identify colluders. Some examples of this work include McAfee and McMillan (1992), Porter and Zona (1993), Baldwin, Marshall, and Richard (1997), Porter and Zona (1999), Pesendorfer (2000), Bajari and Ye (2003), Aryal and Gabrielli (2013), Marmer, Shneyerov, and Kaplan (2017), and Schurter (2020). Some of the tests proposed rely on parametric assumptions while the ones that are nonparametric produce testable implications in the form of independence or orthogonality conditions. All of them leverage specific features about the auction format studied and the particular type of collusive scheme conjectured.

Testing firm conduct in in the context of structural models of demand and supply has also received attention in the empirical Industrial Organization literature, and collusive behavior has been one of the applications. Tests of firm conduct typically involve some form of comparison of the observed markups against the markups predicted by a particular conduct model. Some examples include Bresnahan (1982), Porter (1983), Sullivan (1985), Bresnahan (1987), Gasmi, Laffont, and Vuong (1992), Genovese and Mullin (1998), Berry and Haile (2014), Bergquist and Dinerstein (2020), Sullivan (2020), and Duarte, Magnolfi, Solvsten, and Sullivan (2021). Most existing tests rely on the availability of valid instruments (see Berry and Haile (2014)), and a number of them become special cases of nonnested econometric specification tests (Vuong (1989), Rivers and Vuong (2002)). Some recent approaches (see Bergquist and Dinerstein (2020)) use data from randomized control trials designed to test alternative conduct models. Finally, testing for cooperation has also been studied in experimental economics (see, e.g, Dal Bó (2005), M. and Normann (2012), Fréchette, Lizzeri, and Vespa (2020)). However, the nature of experimental data, where the researcher controls key aspects of the data generating process, which are assumed to be unknown in our setting, makes this type of work fundamentally different, and not applicable to our problem.

In this paper we propose a nonparametric test for cooperation in discrete, static games. Our test relies on the assumption that certain exchangeability conditions in players' payoff functions hold if we match observable payoff covariates across all players. Combined with the assumption that players maximize an unknown joint objective function that treats all players symmetrically (conditional on the aforementioned matching of payoff covariates across players), cooperation implies the existence of a class of outcomes $\mathcal{Y}$ such that, conditional on matching observable payoff covariates across all players, the probability of observing an outcome $y \in \mathcal{Y}$ must be equal to the probability of observing any permutation of y. Our econometric test is based on the conditional moment-equalities implied by cooperation. We contribute to the literature by focusing on a fairly general class of discrete games, and developing a nonparametric econometric test that does not rely on any type of parametric assumption regarding payoffs, nor does it rely on a specific "collusion" scheme. We also do not rely on the existence of instruments, or on exclusion restrictions in payoff covariates across players. Our setup also allows for the possibility that players cooperate in some instances, while they do not cooperate in others.

The paper proceeds as follows. Section 2 focuses on a binary choice game with multiple players in order to illustrate our approach. Section 3 then extends to more general discrete games. Section 4 describes our econometric test and its asymptotic properties. Section 5 summarizes the results of Monte Carlo experiments. Section 6 includes an empirical illustration where we analyze expansion/entry decisions by Home Depot and Lowe's in geographic markets of the contiguous U.S. Section 7 concludes. Proofs are contained in the appendix, with step-by-step details pertaining to our main econometric result included in the online supplement[1].

2 A binary choice game

2.1 Action space

The game consists of a collection of $P \geq 2$ players, where each player p has a binary action $Y_p \in \{0,1\}$. Following convention, we will use lower case letters to denote a potential action, and upper case letters to denote actual choices made. Similarly, we will use the subscript $-p$ to denote all players except p. Thus, $y_p \in \{0,1\}$ and $y_{-p} \in \{0,1\}^{P-1}$ denote, respectively, a potential action by player p and a potential action profile by all players except p, while Y_p and Y_{-p} denote, respectively, the action chosen by p and the action profile chosen by all players except p. We will adopt the convention of listing the choices of each player within Y_{-p} and y_{-p} in order, meaning $Y_{-p} = (Y_1, Y_2, \ldots, Y_{p-1}, Y_{p+1}, \ldots, Y_P)$ and $y_{-p} = (y_1, y_2, \ldots, y_{p-1}, y_{p+1}, \ldots, y_P)$. Finally, we will let $y \in \{0,1\}^P$ denote a particular action profile by all players in the game and we will let Y denote the action profile chosen by the players in the game, both listed in order. That is, $Y = (Y_1, Y_2, \ldots, Y_P)$ and $y = (y_1, y_2, \ldots, y_P)$.

[1] Available online at http://www.personal.psu.edu/aza12/testing-for-cooperation-supplement.pdf

2.1.1 Notation for unit vectors

We will let $e_r \in \mathbb{R}^P$ denote the P−dimensional vector that consists of zeros everywhere and 1 in the r^{th} position, and we will let $\iota_r \in \mathbb{R}^{P-1}$ denote the $(P-1)$−dimensional vector that consists of zeros everywhere and 1 in the r^{th} position. Finally, we let $\boldsymbol{e} \equiv (1,\ldots,1)' \in \mathbb{R}^P$ denote the vector of all-ones in $\mathbb{R}^P$. All these will be treated as column vectors.

2.2 Characteristics of the game

Each player has a collection of observable characteristics to the econometrician, which we will denote as X_p, and we will group $X_G \equiv (X_1,\ldots,X_p)$. We interpret X_p and X_q as denoting the same set of economic characteristics (e.g, market share, firm size, cost measures, distance to distribution center, etc.) for players p and q, and we will denote $\dim(X_p) \equiv d_x$. We denote all other characteristics of the game as ν, which may include a player-specific characteristics and global variables of the game. The distinction in the roles played by X_G and ν in our model will become clear below. We denote $\mathrm{Supp}(\xi)$ as the support of a random variable ξ.

Assumption 1 ***(A support condition)*** *Let $\mathcal{X} \equiv \cap_{p=1}^{P} Supp(X_p) \subset \mathbb{R}^{d_x}$. We do not require $Supp(X_p) = Supp(X_q)$ but we will maintain that $\mathcal{X} \neq \emptyset$. Take any $x \in \mathcal{X}$ and let $\otimes$ be the Kronecker product operator. We have $\boldsymbol{e} \otimes x = (x,\ldots,x) \in \mathbb{R}^{P\cdot d_x}$ (P copies of x). Let f_{X_G} denote the joint density of X_G. Then $\mathcal{X} = \{x \in \mathbb{R}^{d_x} : f_{X_G}(\boldsymbol{e} \otimes x) > 0\} \neq \emptyset$.* ■

2.2.1 A probability definition

For each $x \in \mathcal{X}$, we will denote

$$Q(y|x) \equiv \Pr(Y = y \mid X_G = \boldsymbol{e} \otimes x) = \Pr(Y = y \mid X_1 = \cdots = X_P = x) \tag{1}$$

This is the probability that the outcome observed is y, conditional on matching all players' observable characteristics to x.

2.3 Payoff functions

The payoff for player p is a function denoted as $u_p(Y_p, Y_{-p}, X_G, \nu)$. We will assume that it satisfies the following condition.

Assumption 2 ***(A stochastic payoff-symmetry condition for a class of action profiles)*** *Let $A \stackrel{d}{=} B$ denote A and B having the same distribution. Let $(\ell_1,\ldots,\ell_P)$ and $(k_1,\ldots,k_P)$ denote* *any two permutations of* $(1,\ldots,P)$*. Let* $\mathcal{S} = \{e_r \text{ for some } r = 1,\ldots,P\}$*. Then, for any action profile* $s \equiv (s_1,\ldots,s_P) \in \mathcal{S}$*,*

$$X_1 = \cdots = X_P \quad \Longrightarrow \quad \begin{pmatrix} u_{\ell_1}(s_1, s_{-1}, X_G, \nu) \\ u_{\ell_2}(s_2, s_{-2}, X_G, \nu) \\ \vdots \\ u_{\ell_P}(s_P, s_{-P}, X_G, \nu) \end{pmatrix} \stackrel{d}{=} \begin{pmatrix} u_{k_1}(s_1, s_{-1}, X_G, \nu) \\ u_{k_2}(s_2, s_{-2}, X_G, \nu) \\ \vdots \\ u_{k_P}(s_P, s_{-P}, X_G, \nu) \end{pmatrix}$$

■

Assumption 2 states that, if we match X_p across all players, an exchangeability condition for payoffs follows for all action profiles of the form $y = e_p$. For illustration, suppose $P = 3$ and consider the action profile $s = e_2 = (0,1,0)$. Take the permutations $(1,3,2)$ and $(2,3,1)$ of $(1,2,3)$. Assumption 2 states that, if $X_1 = X_2 = X_3$, then

$$\begin{pmatrix} u_1(0,(1,0),X_G,\nu) \\ u_3(1,(0,0),X_G,\nu) \\ u_2(0,(0,1),X_G,\nu) \end{pmatrix} \stackrel{d}{=} \begin{pmatrix} u_2(0,(1,0),X_G,\nu) \\ u_3(1,(0,0),X_G,\nu) \\ u_1(0,(0,1),X_G,\nu) \end{pmatrix}$$

And the above holds for all permutations of $(1,2,3)$. Note that, in the context of an entry game, the class of action profiles $\mathcal{S}$ in Assumption 2 is the collection of all outcomes where only one player enters the market.

2.3.1 An example of payoff functions that satisfy Assumption 2

Suppose payoffs have the following structure,

$$u_p(Y_p, Y_{-p}, X_G, \nu) = \phi_p^0(X_G, \nu) + \phi_p^1(X_G, \nu) \cdot \lambda(Y_{-p}) + \phi_p^2(X_G, \nu) \cdot Y_p + \Delta_p(Y_{-p}, X_G, \nu) \cdot Y_p. \quad (2)$$

Payoff functions with this structure can satisfy Assumption 2 under the following conditions. Suppose $\lambda(\cdot)$ is the same function for all players and it is symmetric in all its arguments, with $\lambda(\iota_r) = \lambda(\iota_m) \equiv \overline{\lambda}$ for all $(r,m) \in 1,\ldots,P-1$, and $\lambda(0) = 0$. There are two types of strategic effects in (2). λ captures the effect of other players' actions that is independent of the choice made by player p (for example, a shift in p's residual demand). The function Δ_p captures the effect of other players' actions which depends on the choice made by player p. The remaining components of (2) are non-strategic and they depend only on the choice made by p.

Definition (exchangeability, Feller (1970, p.228)): The random variables $\xi_1,\ldots,\xi_P$ are said to be *exchangeable* if the $P!$ permutations $(\xi_{\ell_1},\ldots,\xi_{\ell_P})$ have the same joint probability distribution. ■

Let $\theta_p^0(X_G,\nu) \equiv \phi_p^0(X_G,\nu)+\phi_p^1(X_G,\nu)\cdot\overline{\lambda}$ and $\theta_p^1(X_G,\nu) \equiv \phi_p^0(X_G,\nu)+\phi_p^2(X_G,\nu)$ and suppose,

$$X_1 = \cdots = X_P \implies \begin{cases} \theta_1^0(X_G,\nu),\ldots,\theta_P^0(X_G,\nu) \text{ are exchangeable} \\ \theta_1^1(X_G,\nu),\ldots,\theta_P^1(X_G,\nu) \text{ are exchangeable} \end{cases} \tag{3}$$

We do not impose any exchangeability on the functions $\Delta_p(Y_{-p},X_G,\nu)$, but we assume that[2] $\Delta_p(0,X_G,\nu)=0$ w.p.1. This would be satisfied, for example, if

$$\Delta_p(Y_{-p},X_G,\nu) = \left(\sum_{q\neq p}\Delta_p^q(X_G,\nu)\cdot Y_q\right),$$

with the signs of each $\Delta_p^q(X_G,\nu)$ being unrestricted and unknown. Then, the payoff functions described in (2) satisfy the exchangeability property in Assumption 2.

2.4 A joint objective function

We model cooperation in the following way.

Assumption 3 ***(Cooperation)***
Conditional on the event $X_1=\cdots=X_P$, players choose Y to maximize a joint objective function $\mathcal{U}(Y,X_G,\nu) \equiv V(u_1(Y,X_G,\nu),\ldots,u_P(Y,X_G,\nu);X_G,\nu)$, for some function $V(u_1,\ldots,u_P;X_G,\nu)$ which depends on the payoffs of each player and (possibly) on X_G and ν as direct arguments as well. We will treat the joint objective function as unknown, except for the following features.

(i) Conditional on the event $X_1=\cdots=X_P$, the joint objective function is symmetric in $u_1,\ldots,u_P$. That is, if $X_1=\cdots=X_P$, then $V(u_1,\ldots,u_p;X_G,\nu)=V(u_{\ell_1},\ldots,u_{\ell_p};X_G,\nu)$ for any permutation $(\ell_1,\ldots,\ell_P)$ of $(1,\ldots,P)$.

(ii) $Pr\Big(V\big(u_1(y,X_G,\nu),\ldots,u_P(y,X_G,\nu);X_G,\nu\big) = V\big(u_1(y',X_G,\nu),\ldots,u_P(y',X_G,\nu);X_G,\nu\big)\Big) = 0$ for any pair $y\neq y'$ in $\{0,1\}^P$. ■

Part (i) of Assumption 3 states that when we match the characteristics X_p across all players, their joint objective function treats the payoffs of all players equally. Part (ii) ensures $\mathcal{U}(y,X_G,\nu)$ has a unique maximizer for almost every (a.e) realization of (X_G,ν), so

$$Pr(Y=y|X_1=\cdots=X_P) = Pr\Big(\mathcal{U}(y,X_G,\nu) > \mathcal{U}(y',X_G,\nu)\ \forall\ y'\neq y\ \Big|X_1=\cdots=X_P\Big) \quad \forall\ y. \tag{4}$$

[2] We only need $\Delta_p(0,X_G,\nu)=C$ w.p.1 for some constant C, not necessarily zero.

Remark 1 *Assumption 3 presupposes cooperation only when $X_1 = \cdots = X_P$. Our results will leave players' behavior unspecified otherwise, allowing, for example, for a situation where firms behave noncooperatively in some markets while they cooperate in others. Naturally, we include the scenario where players always cooperate as a special case.*

2.4.1 Examples of joint objective functions that satisfy the symmetry condition in Assumption 3

A vast class of joint objective functions can satisfy the symmetry restrictions in Assumption 3. Some immediate examples include,

$$
\begin{aligned}
V(u_1, \ldots, u_P; X_G, \nu) &= \sum_{p=1}^{P} \zeta(X_p, \nu) \cdot u_p, \\
V(u_1, \ldots, u_P; X_G, \nu) &= \max\{\zeta(X_1, \nu) \cdot u_1, \ldots, \zeta(X_P, \nu) \cdot u_P\}, \\
V(u_1, \ldots, u_P; X_G, \nu) &= \min\{\zeta(X_1, \nu) \cdot u_1, \ldots, \zeta(X_P, \nu) \cdot u_P\}.
\end{aligned}
$$

For any ν, if we match $X_1 = \cdots = X_P$, we have $\zeta(X_1, \nu) = \cdots = \zeta(X_P, \nu)$, making these objective functions symmetric in $(u_1, \cdots, u_P)$.

2.5 Implications of Assumptions 1, 2 and 3

Exchangeability, symmetry and cooperation under our previous conditions produce the following result in our binary choice game.

Proposition 1 *Suppose Assumptions 1, 2 and 3 hold. Then,*

$$
Q(e_p|x) = Q(e_q|x) \ \forall \ p, q, \ \textit{for a.e } x \in \mathcal{X} \tag{5}
$$

The proof of Proposition is included in the appendix.

Remark 2 *Our test will only rely on assumptions about players' behavior when their payoff covariates are matched. We make no assumptions about their behavior otherwise. Thus, our setup allows for the possibility that players cooperate in some instances, while they do not cooperate in others.*

2.6 Violations of condition (5) when players' true behavior is noncooperative

Let us discuss conditions under which (5) can be violated if players' true behavior is noncooperative while the exchangeability conditions in Assumption 2 are satisfied. Suppose the actions observed are the realization of a complete-information Nash equilibrium[3] (NE). NE outcomes can be characterized as follows. Partition the support of (X_G, ν) into J mutually exclusive regions, $R_1, \ldots, R_J$. The number of equilibria in region R_j (in pure and mixed strategies) is denoted as $\mathcal{E}_j$. Let us list the $\mathcal{E}_j$ NE in region R_j as $\pi_{j_1}, \ldots, \pi_{j_{\mathcal{E}_j}}$, where $\pi_{j_\ell}(y) \equiv \Pr(Y = y \text{ under NE } j_\ell)$. There is also an equilibrium selection mechanism $\mathscr{M}$ which determines which NE is selected in regions of multiplicity. Let $\mathscr{M}_{j_\ell}$ be the indicator function for $\mathscr{M}$ selecting NE j_ℓ. Then,

$$
\begin{aligned}
Pr(Y = y | X_1 = \cdots = X_P) = \sum_{j=1}^{J} \sum_{\ell=1}^{\mathcal{E}_j} \Big\{ & E\Big[\pi_{j_\ell}(y) \,\Big|\, (X_G, \nu) \in R_j \,,\, \mathscr{M}_{j_\ell} = 1 \,,\, X_1 = \cdots = X_P\Big] \\
& \cdot Pr\Big((X_G, \nu) \in R_j \,\Big|\, \mathscr{M}_{j_\ell} = 1 \,,\, X_1 = \cdots = X_P\Big) \cdot Pr\Big(\mathscr{M}_{j_\ell} = 1 \,\Big|\, X_1 = \cdots = X_P\Big) \Big\}
\end{aligned}
\tag{6}
$$

From (6), there are different ways in which (5) can be violated while Assumption 2 holds. For example, the following can be sufficient conditions for such a violation to occur.

(i) For some pair of players $r \neq p$, there exist regions R_j and $R_{j'}$ where only e_r or e_p is a NE outcome, but not both and,

$$
\sum_{\ell=1}^{\mathcal{E}_j} E\Big[\pi_{j_\ell}(e_r) \,\Big|\, (X_G, \nu) \in R_j \,,\, \mathscr{M}_{j_\ell} = 1 \,,\, X_1 = \cdots = X_P\Big] \neq \sum_{\ell=1}^{\mathcal{E}_{j'}} E\Big[\pi_{j'_\ell}(e_p) \,\Big|\, (X_G, \nu) \in R_{j'} \,,\, \mathscr{M}_{j'_\ell} = 1 \,,\, X_1 = \cdots = X_P\Big],
$$

or

$$
Pr\Big((X_G, \nu) \in R_j \,\Big|\, X_1 = \cdots = X_P\Big) \neq Pr\Big((X_G, \nu) \in R_{j'} \,\Big|\, X_1 = \cdots = X_P\Big)
$$

(ii) For some pair of players $r \neq p$, there exists a region R_j where both e_r and e_p are NE outcomes for two different NE, $\boldsymbol{j_\ell}$ and $\boldsymbol{j_m}$, and,

$$
E\Big[\pi_{j_\ell}(e_r) \,\Big|\, (X_G, \nu) \in R_j \,,\, \mathscr{M}_{j_\ell} = 1 \,,\, X_1 = \cdots = X_P\Big] \neq E\Big[\pi_{j_m}(e_p) \,\Big|\, (X_G, \nu) \in R_j \,,\, \mathscr{M}_{j_m} = 1 \,,\, X_1 = \cdots = X_P\Big],
$$

or

$$
Pr\Big(\mathscr{M}_{j_\ell} = 1 \,\Big|\, X_1 = \cdots = X_P\Big) \neq Pr\Big(\mathscr{M}_{j_m} = 1 \,\Big|\, X_1 = \cdots = X_P\Big)
$$

As the above arguments suggest, there are various channels through which (5) can be

[3] Other noncooperative behavioral models can be considered, we focus on Nash equilibrium because it is the most widely assumed.

violated if true behavior is noncooperative. This can occur through differences in the NE mixing probabilities across different equilibrium regions, differences in the likelihood of falling in equilibrium regions (as in case (i) above), or through the properties of the equilibrium selection mechanism (as in case (ii)). All of these scenarios are compatible with Assumption 2. On the other hand, in certain cases it is possible for (6) to be satisfied even if the true behavior is Nash equilibrium. For example, this would happen if the following is true.

(i) There exists a collection of P regions, $R_1, \ldots, R_P$ such that, each e_p is a pure-strategy Nash equilibrium (PSNE) if and only if $(X, \nu) \in R_p$ and if, in addition, these regions are such that $Pr((X, \nu) \in R_p | X_1 = \cdots = X_P) = Pr((X, \nu) \in R_q | X_1 = \cdot = X_P)$ for all p, q.

(ii) If any outcome $y = e_r$ is played with positive probability in a mixed-strategy Nash equilibrium (MSNE), then every outcome $y = e_p$ is also played with positive probability in this MSNE and they are all played with the same probability, so we have $E[\pi(e_p) | X_1 = \cdots = X_p] = E[\pi(e_q) | X_1 = \cdots = X_p]$ for all $p.q$.

2.7 The 2×2 case

In a binary choice game where $Y_p \in \{0, 1\}$ with two players (i.e, a 2×2 game), equation (5) reduces to,

$$Q(1, 0|x) = Q(0, 1|x), \quad \text{a.e } x \in \mathcal{X} \tag{7}$$

In other words, cooperation in a 2×2 game implies $Pr(1, 0 | X_1 = X_2) = Pr(0, 1 | X_1 = X_2)$ a.s. In the online supplement we take a look at 2×2 games in more detail and we illustrate the restrictions implied by our assumptions, as well as the power of our testable implications when the true underlying behavior is noncooperative. There, we show how Nash equilibrium behavior can lead to violations of (7) if players' strategic-interaction effects are asymmetric, or if the equilibrium selection mechanism selects equilibria with different probabilities.

3 A more general discrete game

Assume now that $Y_p \in \mathcal{Y}$ (a discrete, finite set) for each p. We still assume that the action space is the same for all players but it is not restricted to be binary. The joint action space is then $\mathcal{Y}^P$. The following condition generalizes Assumption 2.

Assumption 4 *There exists $\mathcal{S} \subseteq \mathcal{Y}^P$ such that, for any $s \equiv (s_1,\ldots,s_P) \in \mathcal{S}$, the following holds. Let $(\ell_1,\ldots,\ell_P)$ and $(k_1,\ldots,k_P)$ denote any pair of permutations of $(1,\ldots,P)$. Then,*

$$X_1 = \cdots = X_P \quad \Longrightarrow \quad \begin{pmatrix} u_{\ell_1}(s_1, s_{-1}, X_G, \nu) \\ u_{\ell_2}(s_2, s_{-2}, X_G, \nu) \\ \vdots \\ u_{\ell_P}(s_P, s_{-P}, X_G, \nu) \end{pmatrix} \stackrel{d}{=} \begin{pmatrix} u_{k_1}(s_1, s_{-1}, X_G, \nu) \\ u_{k_2}(s_2, s_{-2}, X_G, \nu) \\ \vdots \\ u_{k_P}(s_P, s_{-P}, X_G, \nu) \end{pmatrix} \quad \blacksquare$$

Assumption 2 in our binary choice game is a special case of Assumption 4, with $\mathcal{S} = \{e_p : p = 1,\ldots,P\}$. In (2) we provided an expression for payoff functions that would satisfy Assumption 2. We can generalize this class of payoff functions as follows. Suppose,

$$u_p(Y_p, Y_{-p}, X_G, \nu) = \phi_p^0(X_G, \nu) + \phi_p^1(X_G, \nu) \cdot \lambda(Y_{-p}) + \phi_p^2(X_G, \nu) \cdot m(Y_p) + \Delta_p(X_G, \nu) \cdot \eta(Y_p, Y_{-p}),$$

where the functions $\lambda(\cdot)$, $m(\cdot)$ and $\eta(\cdot)$ are the same across all players and, for any given $y_p \in \mathcal{Y}$, the functions $\lambda(Y_{-p})$ and $\eta(y_p, Y_{-p})$ are symmetric in Y_{-p}. The condition in Assumption 4 would be satisfied for $\mathcal{S} = \mathcal{Y}^P$ (the entire action space) if, $X_1 = \cdots = X_P$ implies: (a) $\gamma_1(X_G, \nu),\ldots,\gamma_P(X_G, \nu)$ are exchangeable, (b) $\theta_1(X_G, \nu),\ldots,\theta_P(X_G, \nu)$ are exchangeable, (c) $\phi_1(X_G, \nu),\ldots,\phi_P(X_G, \nu)$ are exchangeable, and (d) $\Delta_1(X_G, \nu),\ldots,\Delta_P(X_G, \nu)$ are exchangeable.

3.1 Implications of Assumptions 1, 3 and 4

The following Theorem generalizes the binary-choice result in Proposition 1.

Theorem 1 *Suppose Assumptions 1, 3 and 4 hold. Then, for any $y \in \mathcal{S}$ and any permutation y' of y,*

$$Q(y|x) = Q(y'|x) \ \text{ for a.e } x \in \mathcal{X} \tag{8}$$

The proof of Theorem 1 is included in the appendix. This result is a generalization of (5) and it summarizes the testable implications of cooperation in our setting. We describe an econometric test based on (8) next.

4 An econometric test

Assume that X_p can be split into a collection of r continuously distributed and m discrete random variables, where $r + m = d_x \equiv \dim(X_p)$,

$$X_p = \Big(\underbrace{X_p^{c_1}, \ldots, X_p^{c_r}}_{\text{continuous}}, \underbrace{X_p^{d_1}, \ldots, X_p^{d_m}}_{\text{discrete}}\Big) \equiv \Big(X_p^c, X_p^d\Big),$$

where $X_p^c \equiv (X_p^{c_1}, \ldots, X_p^{c_r})$ and $X_p^d \equiv (X_p^{d_1}, \ldots, X_p^{d_m})$. Let $\mathcal{X}$ be as defined in Assumption 1. Then, each $x \in \mathcal{X}$ can be expressed as

$$x \equiv (\underbrace{x^{c_1}, \ldots, x^{c_r}}_{\text{continuous}}, \underbrace{x^{d_1}, \ldots, x^{d_m}}_{\text{discrete}}) \equiv (x^c, x^d).$$

We will let $\mathcal{X}^c$ and $\mathcal{X}^d$ denote the range of values of x^c and x^d within $\mathcal{X}$. Recall that $X_{Gi} \equiv (X_{1i}, \ldots, X_{Pi})$ denotes the collection of all players' observable characteristics. Let $X_G^c \equiv (X_1^c, \ldots, X_P^c)$ and $X_G^d \equiv (X_1^d, \ldots, X_P^d)$ denote the collection of all players' continuous and discrete observable characteristics, respectively.

4.1 A population statistic to test cooperation

Let $\sigma(y)$ denote the collection of all distinct permutations of y. Let $Q(y|x)$ be as defined in (1). Let $f_{X_G}(\cdot)$ denote the density function of X_G. For each $x \in \mathcal{X}$, denote

$$\tau(y, y'|x) \equiv (Q(y|x) - Q(y'|x)) \cdot f_{X_G}(\boldsymbol{e} \otimes x).$$

Recall that every $x \in \mathcal{X}$ can be partitioned as $x \equiv (x^c, x^d)$. Let $\omega(x)$ denote a pre-specified weight function that satisfies $\omega(x) \geq 0 \ \forall \ x$, and $\omega(x) > 0$ only if $x \in \mathcal{X}$. Let $\mathcal{S} \subseteq \mathcal{Y}^P$ be as described in Assumption 4, and let

$$\mathcal{T} \equiv \sum_{y \in \mathcal{S}} \sum_{y' \in \sigma(y)} \sum_{x^d \in \mathcal{X}^d} \int_{x^c \in \mathcal{X}^c} \tau(y, y'|x)^2 \omega(x) dx^c \tag{9}$$

For simplicity, let us normalize our weight function so that $\sum\limits_{x^d \in \mathcal{X}^d} \int\limits_{x^c \in \mathcal{X}^c} \omega(x) dx^c = 1$. By construction, $\mathcal{T} \geq 0$, and $\mathcal{T} = 0$ only if (8) is satisfied. Therefore, we can test for cooperation by testing the null hypothesis $H_0 : \mathcal{T} = 0$.

Assumption E1 ***(Weight function properties)*** *For each x^d, $\omega(x^c, x^d)$ is continuous in x^c. Let $\underline{x}^c$ and $\underline{x}^c$ denote the (element-wise) minimum and maximum values of x^c for which $\omega(x^c, x^d) > 0$ for some x^d, and for each p let $\underline{X}_p^c$ and $\overline{X}_p^c$ denote the (element-wise) minimum and maximum values of the support of the distribution of X_p^c. Then, there exists a constant $D > 0$ such that $\underline{x}^c - \underline{X}_p^c \geq D$ and $\overline{X}_p^c - \overline{x}^c \geq D$ for each p. In other words, the range of values of x^c for which $\omega(x^c, x^d) > 0$ for some x^d, belongs in the interior of Supp(X_p^c), for each p.* ■

Note that Assumption E1 allows for Supp(X_p^c) to be unbounded.

4.2 Constructing a test-statistic

Our test is based on matching $X_1 = \cdots = X_P = x$. While this matching can be done exactly for X_p^d, we do it asymptotically for X_p^c. Let $\kappa(\cdot)$ be a real-valued, univariate kernel function and let h_n be a bandwidth sequence and denote $L \equiv P \cdot r$ (the total number of continuous covariates in X_{Gi}). For a given $x \equiv (x^c, x^d)$, let

$$\mathcal{K}\left(\frac{X_{Gi}^c - \boldsymbol{e} \otimes x^c}{h_n}\right) \equiv \prod_{p=1}^{P} \prod_{\ell=1}^{r} \kappa\left(\frac{X_{pi}^{c_\ell} - x^{c_\ell}}{h_n}\right), \quad \mathbb{1}\left\{X_{Gi}^d = \boldsymbol{e} \otimes x^d\right\} \equiv \prod_{p=1}^{P} \prod_{s=1}^{m} \mathbb{1}\left\{X_{pi}^{d_s} = x^{d_s}\right\}, \tag{10}$$

$$\Gamma(X_{Gi}, x, h_n) \equiv \frac{1}{h_n^L} \mathcal{K}\left(\frac{X_{Gi}^c - \boldsymbol{e} \otimes x^c}{h_n}\right) \cdot \mathbb{1}\left\{X_{Gi}^d = \boldsymbol{e} \otimes x^d\right\}.$$

Let $S(Y_i, y, y') \equiv \mathbb{1}\{Y_i = y\} - \mathbb{1}\{Y_i = y'\}$, and

$$\widehat{\tau}(y, y'|x)^2 = \binom{n}{2}^{-1} \sum_{i<j} S(Y_i, y, y') \cdot S(Y_j, y, y') \cdot \Gamma(X_{Gi}, x, h_n) \cdot \Gamma(X_{Gj}, x, h_n).$$

This is a U-statistic of order two. Then,

$$\widehat{\mathcal{T}} = \sum_{y \in \mathcal{S}} \sum_{y' \in \sigma(y)} \sum_{x^d \in \mathcal{X}^d} \int_{x^c \in \mathcal{X}^c} \widehat{\tau}(y, y'|x)^2 \omega(x) dx^c. \tag{11}$$

Assumption E2 ***(Smoothness conditions with respect to X_G^c)***

For each p, and each $x_p \equiv (x_p^c, x_p^d) \in Supp(X_p)$, express $x_p^c \equiv (x_p^{c_1}, \ldots, x_p^{c_r})$ and $x_p^d \equiv (x_p^{d_1}, \ldots, x_p^{d_m})$, and group $x_G^c \equiv (x_1^c, \ldots, x_P^c)$, and $x_G^d \equiv (x_1^d, \ldots, x_P^d)$. There exists $\mathcal{X}_G^* \subseteq Supp(X_G)$ that satisfies the following.

(i) $\boldsymbol{e} \otimes x \in \mathcal{X}_G^*$ for each $x \in \mathcal{X}$.

(ii) Let $\mu(y|X_G) \equiv Pr(Y = y|X_G)$, and note from our definition in (1) *that $Q(y|x) = \mu(y|\boldsymbol{e}\otimes x)$ for each $x \in \mathcal{X}$. Let $f_{X_G^c|X_G^d}$ denote the conditional density of X_G^c given X_G^d. There exists a constant $\overline{D} < \infty$ and an integer M such that, for each $x_G \equiv (x_G^c, x_G^d) \in \mathcal{X}_G^*$, each integer $1 \leq j \leq M$, and any collection $\{\{j_p^\ell\}_{p=1}^P\}_{\ell=1}^r$ such that $\sum_{p=1}^P \sum_{\ell=1}^r j_p^\ell = j$, we have $f_{X_G^c|X_G^d}(x_G^c|x_G^d) \leq \overline{D}$, and*

$$\left| \frac{\partial^j f_{X^c|X^d}(x_G^c|x_G^d)}{(\partial x_1^{c_1})^{j_1^1} \cdots (\partial x_1^{c_r})^{j_1^r} \cdots\cdots (\partial x_P^{c_1})^{j_P^1} \cdots (\partial x_P^{c_r})^{j_P^r}} \right| \leq \overline{D}, \quad \left| \frac{\partial^j \mu(v|x_G^c, x_G^d)}{(\partial x_1^{c_1})^{j_1^1} \cdots (\partial x_1^{c_r})^{j_1^r} \cdots\cdots (\partial x_P^{c_1})^{j_P^1} \cdots (\partial x_P^{c_r})^{j_P^r}} \right| \leq \overline{D},$$

where the last holds for any $v \in \sigma(y) : y \in \mathcal{S}$, with $\mathcal{S}$ as defined in Assumption 4. ■

Assumption E3 ***(Kernels and bandwidths)***
Let M be the integer described in Assumption E2. The following conditions hold,

(i) The kernel is constructed as described in (10)*. The kernel $\kappa(\cdot)$ is bias-reducing of order M with support of the form $[-S, S]$ ($\kappa(v) = 0\ \forall\ v \notin (-S, S)$, with $\int_{-S}^S \kappa(v)dv = 1$, $\int_{-S}^S v^j \kappa(v)dv = 0$ for $j = 1, \ldots, M-1$ and $\int_{-S}^S |v|^M \kappa(v)dv < \infty$) and symmetric around zero (i.e, $\kappa(v) = \kappa(-v)$ for all v). In addition, $|\kappa(\cdot)| \leq \overline{\kappa}$ for a constant $\overline{\kappa} < \infty$.*

(ii) The bandwidth sequence satisfies $n \cdot h_n^{2L-r} \longrightarrow \infty$, and $n \cdot h_n^{M+\frac{r}{2}} \longrightarrow 0$. ■

Part (ii) of Assumption E3 requires $M + \frac{r}{2} > 2L - r$. Since $L \equiv P \cdot r$, the smallest value of M (the number of higher-order derivatives of the functionals described in Assumption E2) consistent with Assumption E3 is $M = \lceil r \cdot (2P - 3/2) \rceil$, where $\lceil x \rceil \equiv \text{ceiling}(x)$. For example, if $P = 2$ (a two-player game), the smallest value of M consistent with Assumption E3 is $M = \lceil 5r/2 \rceil$. Recall that M also determines the order of the kernel used. If our bandwidth is of the form $h_n \propto n^{-\alpha_h}$, part (ii) of Assumption E3 requires $\frac{1}{M+r/2} < \alpha_h < \frac{1}{r(2P-1)}$.

4.3 Asymptotic properties of $\widehat{\mathcal{T}}$

Group $U_i \equiv (Y_i, X_{Gi})$, denote $S(Y_i, Y_j, y, y') \equiv S(Y_i, y, y') \cdot S(Y_j, y, y')$,

$$\begin{aligned}
\overline{S}(Y_i, Y_j) &\equiv \sum_{y \in \mathcal{S}} \sum_{y' \in \sigma(y)} S(Y_i, Y_j, y, y'), \\
\varphi(X_{Gi}, X_{Gj}, h_n) &\equiv \sum_{x^d \in \mathcal{X}^d} \int_{x^c \in \mathcal{X}^c} \Gamma(X_{Gi}, x, h_n) \cdot \Gamma(X_{Gj}, x, h_n) \omega(x) dx^c, \qquad (12) \\
H_n(U_i, U_j) &\equiv \overline{S}(Y_i, Y_j) \cdot \varphi(X_{Gi}, X_{Gj}, h_n).
\end{aligned}$$

We have

$$\widehat{\mathcal{T}} = \binom{n}{2}^{-1} \sum_{i<j} H_n(U_i, U_j), \tag{13}$$

so, our test-statistic is a U-statistic of order 2. From here, applying the results in Hall (1986), we obtain the following result.

Theorem 2 *Suppose Assumptions E1, E2 and E3 hold and that each X_p includes at least one continuously distributed covariate (i.e, $r \geq 1$). Then,*

(i) If players cooperate under the conditions described in Section 3.1, then

$$\frac{n \cdot \widehat{\mathcal{T}}}{\sqrt{2 \cdot E\left[H_n(U_i, U_j)^2\right]}} \xrightarrow{d} \mathcal{N}(0,1).$$

(ii) If players do not cooperate and (8) *is violated (and therefore $\mathcal{T} > 0$), we have*

$$\lim_{n\to\infty} Pr\left(\frac{n \cdot \widehat{\mathcal{T}}}{\sqrt{2 \cdot E\left[H_n(U_i, U_j)^2\right]}} > c\right) = 1 \quad \forall\ c$$

The main steps of the proof of Theorem 2 are described in the appendix. All the step-by-step details and derivations are included in the online supplement. The proof is based on the conditions of Hall (1986, Theorem 1), which result in asymptotic normality for degenerate U-statistics whose *kernel functions* (not to be confused with the kernel used to construct our nonparametric estimators) change with n, as is our case –under the null hypothesis of cooperation– due to the presence of the bandwidth sequence h_n. While limiting distributions for degenerate U-statistics of order two, whose kernel functions are fixed, are a linear combination of independent, centered χ_1^2 distributions (see Gregory (1977), Neuhaus (1977), Serfling (1980, Section 5.5.2)), using Margingale theory, Hall (1986, Theorem 1) shows sufficient conditions to derive a central limit theorem. This CLT has been used to construct consistent specification tests (Zheng (1996), Fan and Li (1996)), and it has been generalized for degenerate U-statistics of higher order by Fan and Li (1996). The bandwidth convergence conditions in Assumption E3 are designed to satisfy the conditions in Hall (1986, Theorem 1).

Using the result in Theorem 2, we can use the following rejection rule for testing cooperation. Let

$$\widehat{t} \equiv \frac{n \cdot \widehat{\mathcal{T}}}{\sqrt{2 \cdot \widehat{E}\left[H_n(U_i, U_j)^2\right]}}, \quad \text{where} \quad \widehat{E}\left[H_n(U_i, U_j)^2\right] = \binom{n}{2}^{-1} \sum_{i<j} H_n(U_i, U_j)^2.$$

Choose a target significance level α and let $z_{1-\alpha}$ denote the Standard Normal $(1-\alpha)^{th}$ quantile. Our rejection rule is the following,

Reject the null hypothesis of cooperation if and only if $\widehat{t} > z_{1-\alpha}$

By part (ii) of Theorem 2, our proposed test is consistent (it rejects cooperation if the conditional moment restrictions are violated with nonzero probability), and by part (i), it achieves the target significance level asymptotically.

5 Monte Carlo experiments

The details of our Monte Carlo experiments are included in the online supplement. We summarize them here. Our experiments revolve around a 2×2 game,

	$Y_2 = 1$	$Y_2 = 0$
$Y_1 = 1$	$X_1^1 + X_1^2 + \Delta_1 + \varepsilon_1$, $X_2^1 + X_2^2 + \Delta_2 + \varepsilon_2$	$X_1^1 + X_1^2 + \varepsilon_1$, 0
$Y_1 = 0$	0 , $X_2^1 + X_2^2 + \varepsilon_2$	0 , 0

In all our experiments, $(X_p^1, X_p^2, \varepsilon_p)$ are iid $\mathcal{N}(0,1)$. The strategic interaction parameters (Δ_1, Δ_2) are constant, with $\Delta_p < 0$ (strategic substitutes). First, we generate data first assuming non-cooperative, pure-strategy Nash-equilibrium (PSNE) behavior. We use two DGPs. In the first DGP, we set $\Delta_1 = -2$ and $\Delta_2 = -1$ and the PSNE selection mechanism chooses both coexisting Nash equilibria, $\{(1,0),(0,1)\}$ in the multiple NE region with equal probability. Our goal here is to evaluate the power of our test when the strategic effects are different across players. In the second DGP, we set $\Delta_1 = \Delta_2 = -2$ and assume that the selection mechanism chooses the coexisting PSNE with different probabilities. Our goal is to evaluate the power of our test when the strategic effects are equal but the selection mechanism does not choose the coexisting NE with uniform probability.

For both DGPs we also generate data assuming cooperation, with three alternative joint objective functions: $V(u_1, u_2) = u_1 + u_2$, $V(u_1, u_2) = \max\{u_1, u_2\}$ and $V(u_1, u_2) = \min\{u_1, u_2\}$. In all cases where cooperation is the true model, we find that, while our rejection rates are above our target significance levels in small samples, as the sample size increases (approaching $n = 1000$ in our experiments), the rejection rates are much closer to the asymptotic size predictions in Theorem 2. We also find that our test has power to reject cooperation when the true behavior is noncooperative, and that this power can be derived from an asymmetry in players' strategic-interaction effects or from the properties of the equilibrium selection mechanism. The kernels and bandwidths employed are described in the online supplement, but they have the same structure as those we use in our empirical illustration, described in Section 6.2.1. As we describe there, we use bandwidths of the form $h_n = c_h \cdot \widehat{\sigma}(X) \cdot n^{-\alpha_h}$, where $\widehat{\sigma}(X)$ is the sample standard deviation of X (we use covariate-specific bandwidths) and $\alpha_h > 0$ is our bandwidth convergence rate, designed to satisfy the restrictions in Assumption E3. Tables 1 and 2 summarize our Monte Carlo results when $c_h = 1.25$, the value that showed the best balance between size and power in our experiments. In the online supplement, we repeat our experiments for a range of values $c_h \in [1, 2]$ and we find that our results are robust.

Table 1: Monte Carlo experiment results. Rejection rates for the null hypothesis of cooperation when $\Delta_1 = -2$, $\Delta_2 = -1$ and PSNE selection mechanism is uniform

	Cooperative behavior			**PSNE behavior**
Sample size	$V(u_1, u_2) = u_1 + u_2$	$V(u_1, u_2) = \max\{u_1, u_2\}$	$V(u_1, u_2) = \min\{u_1, u_2\}$	$P_{\mathcal{M}}(1, 0) = 0.50$
$n = 250$	8.2%	8.1%	6.7%	36.8%
$n = 500$	6.2%	6.2%	7.3%	58.3%
$n = 1000$	5.6%	5.5%	4.6%	82.4%

• $P_{\mathcal{M}}(y) \equiv Pr$ (mechanism $\mathcal{M}$ will choose PSNE y), with $P_{\mathcal{M}}(1, 0) + P_{\mathcal{M}}(0, 1) = 1$.
• 1000 simulations in each case.

Table 2: Monte Carlo experiment results. Rejection rates for the null hypothesis of cooperation when $\Delta_1 = -2$, $\Delta_2 = -2$ and PSNE selection mechanism is non-uniform

	Cooperative behavior			**PSNE behavior**	
Sample size	$V(u_1, u_2) = u_1 + u_2$	$V(u_1, u_2) = \max\{u_1, u_2\}$	$V(u_1, u_2) = \min\{u_1, u_2\}$	$P_{\mathcal{M}}(1, 0) = 0.25$	$P_{\mathcal{M}}(1, 0) = 0.10$
$n = 250$	8.1%	8.1%	6.7%	30.8%	62.3%
$n = 500$	6.2%	6.2%	7.3%	51.9%	87.0%
$n = 1000$	5.5%	5.5%	4.6%	74.4%	98.9%

• $P_{\mathcal{M}}(y) \equiv Pr$ (mechanism $\mathcal{M}$ will choose PSNE y), with $P_{\mathcal{M}}(1, 0) + P_{\mathcal{M}}(0, 1) = 1$.
• 1000 simulations in each case.

6 Empirical illustration

We apply our methodology to analyze expansion and entry decisions by Lowe's and Home Depot into geographic markets in the continental United States between 2008 and 2022. We model expansion (entry) as a binary choice, with

$$Y_{pi} = \mathbb{1}\{\text{ Player } p \text{ expands its presence in market } i \text{ between 2008 and 2022}\}$$

We model player $p = 1$ as Lowe's and $p = 2$ as Home Depot, and we define a market i as a core-based statistical area (CBSA) in the contiguous United States (the lower 48 states in North America, including the District of Columbia). We say that $Y_{pi} = 1$ if and only if the number of stores of player p in market i increased between 2008 and 2022. Note that expansion implies entry in markets where player p had no presence in 2008. Our sample consists of $n = 954$ markets. Our matching covariates include three elements. X^1_{pi} ≡*Number of stores percapita of player p within 100 miles of market i in 2008.* X^2_{pi} ≡*Number of stores percapita of player p's opponent within 100 miles of market i in 2008.* X^3_{pi} ≡*Distance between market i and the nearest regional distribution center of player p in 2008.* Thus, we have $X_{pi} \equiv (X^1_{pi}, X^2_{pi}, X^3_{pi})$, and we treat each one of them as continuous random variables. Thus, we have $r = 3$ continuously distributed covariates.

Table 3: Some summary statistics of expansion decisions between 2008-2022

Proportion of markets where <u>at least one</u> firm expanded	**Proportion of markets where <u>both</u> firms expanded**	**Proportion of markets where <u>only one</u> firm expanded**	**Proportion of markets where <u>Lowe's</u> expanded**	**Proportion of markets where <u>Home Depot</u> expanded**
21.9%	3.2%	18.7%	16.7%	8.5%

The summary statistics in Table 3 indicate that approximately one-fifth of all markets observed an expansion by at least one of the two players, with the vast majority of these cases corresponding to an expansion by only one firm. We also observe that the proportion of markets where Lowe's expanded was almost twice as large as that of Home Depot. Our test for cooperation focuses on a comparison between the proportion of markets where we observe $Y = (1, 0)$ and the proportion of markets where we observe $Y = (0, 1)$. As we will discuss next, once we analyze markets by size (population), we observe patterns that are consistent with cooperation under the conditions of this paper.

6.1 Observed data patterns consistent with cooperation

As we pointed out in previous sections, cooperation in a 2×2 game under Assumptions 1, 2 and 3 implies $Pr(1,0|X_1 = X_2) = Pr(0,1|X_1 = X_2)$, a.s (see equation 7). As Table 4 suggests, a preliminary inspection of the data reveals patterns consistent with this condition in smaller markets (measured by population), and in markets where there were no stores in 2008.

Table 4: Proportion of markets where only one firm expanded between 2008-2022

	All markets	**Markets below the 85th percentile in size**	**Markets below the 70th percentile in size**	**Markets below the 50th percentile in size**	**Markets with no stores in 2008**
Proportion of markets where only Lowe's expanded (i.e, $Y = (1,0)$)	13.4%	8.8%	5.2%	4.2%	6.5%
Proportion of markets where only Home Depot expanded (i.e, $Y = (0,1)$)	5.2%	4.9%	4.5%	4.2%	6.5%

• Market size refers to population in 2008.

6.2 Results of our test for cooperation

6.2.1 Choice of testing range and tuning parameters

Our choice for testing range and tuning parameters (bandwidth and kernel) are guided by the Monte Carlo experiment findings, included in the online supplement. We describe them next. Let $Z_{(\tau)}$ denote the τ^{th} quantile of the r.v Z. For $\ell = 1,2,3$, let $\underline{x}^{\ell} \equiv X^{\ell}_{1,(0.01)} \vee X^{\ell}_{2,(0.01)}$ and $\overline{x}^{\ell} \equiv X^{\ell}_{1,(0.99)} \wedge X^{\ell}_{2,(0.99)}$. Our testing range is $\mathcal{X} \equiv [\underline{x}^1, \overline{x}^1] \times [\underline{x}^2, \overline{x}^2] \times [\underline{x}^3, \overline{x}^3]$. Our weight function $\omega(\cdot)$ is the uniform distribution over $\mathcal{X}$. Next, we choose a bandwidth of the form $h_n = c_h \cdot \widehat{\sigma}(X) \cdot n^{-\alpha_h}$, where $\widehat{\sigma}(X)$ is the sample standard deviation of

X (we use covariate-specific bandwidths). As we discussed in the paragraph following Assumption E3, we must have $\frac{1}{M+r/2} < \alpha_h < \frac{1}{r(2P-1)}$, and the smallest value of M is $\lceil \frac{5r}{2} \rceil$. Since $P = 2$ and $r = 3$, we set $M = 8$ and $\alpha_h = 0.11$. These are also the guidelines we used in our Monte Carlo experiments. As shown in the online supplement, we found that $c_h = 1.25$ provided the best finite-sample results. Consequently, we implemented our test with $h_n = 1.25 \cdot n^{-0.11}$. The online supplement repeats our test for a range of values of $c_h \in [1, 1.75]$. As the results there show, the findings we present below remained qualitatively unchanged. Lastly, we employed a bias-reducing kernel of order $M = 8$ of the form, $\kappa(\psi) = \left(c_1 \cdot (S^2 - \psi^2)^2 + c_2 \cdot (S^2 - \psi^2)^4 + c_3 \cdot (S2 - \psi^2)^6 + c_4 \cdot (S^2 - \psi^2)^8\right) \cdot \mathbb{1}\{|\psi| \leq S\}$. The kernel has support $[-S, S]$, with $S = 10$. The coefficients $c_1, \ldots, c_4$ were chosen to satisfy the conditions of a bias-reducing kernel of order $M = 8$.

6.2.2 Results

The results of our test are included in Table 5. Our findings suggest that, while cooperation in expansion/entry decisions can be rejected in larger markets, this type of behavior cannot be rejected in smaller markets. To be precise, we fail to reject cooperation in markets below the 70th percentile in population size, as well as in markets that did not have any stores in 2008. On the other hand, when we consider all markets or, for example, markets whose size are above the 85th percentile, we reject cooperation at a significance level < 1%. As we show in the online supplement, these findings are robust to alternative bandwidth choices.

Table 5: Test results for cooperation in expansion decisions.

All markets	**Markets below the 85^{th} percentile in size**	**Markets below the 70^{th} percentile in size**	**Markets below the 50^{th} percentile in size**	**Markets with no stores in 2008**
21.642*** (0.000)	4.294*** (0.000)	−0.912 (0.819)	−0.773 (0.780)	−0.543 (0.706)

• Results show the value of our test-statistic, with p-value in parenthesis.
(∗∗∗) Cooperation rejected at < 1% significance level.

7 Concluding remarks

The assumptions made about players' behavior are fundamental for the econometric analysis of games. Since most existing work presupposes noncooperative behavior, having the ability to test for evidence of cooperation is extremely important. In this paper we proposed a nonparametric test for a fairly general class of discrete games. Our procedure is based on some symmetry and exchangeability conditions that are assumed to hold when we match observable payoff characteristics across players. Under our assumptions, cooperation implies a class of conditional moment-equalities for certain permutations of outcomes. We proposed an econometric test that is consistent (it rejects cooperation if the conditional moment restrictions are violated with nonzero probability) and asymptotically normal under the null hypothesis of cooperation. We also presented evidence that our test has good power properties when players' true behavior is noncooperative. We applied our test to analyze expansion/entry decisions of Lowe's and Home Depot in the contiguous U.S and our results found that, while outcomes are consistent with noncooperative behavior in larger markets, we failed to reject cooperation in smaller markets.

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Appendix A: Proofs of Results

A.1 Proposition 1

Take any $y \equiv (y_1, \ldots, y_P) \in \{0,1\}^P$. For any permutation $(k_1, \ldots, k_P)$ of $(1, \ldots, P)$ where we assign action y_{k_p} to player p and payoffs are, $u_1(y_{k_1}, y_{-k_1}, X_G, \nu)$, ... , $u_P(y_{k_P}, y_{-k_P}, X_G, \nu)$, there exists a permutation $(\ell_1, \ldots, \ell_P)$ of $(1, \ldots, P)$ where player ℓ_p plays action y_p and the resulting payoffs are, $u_{\ell_1}(y_1, y_{-1}, X_G, \nu)$, ... , $u_{\ell_P}(y_P, y_{-P}, X_G, \nu)$. From here we have the following result. Take any $y \equiv (y_1, \ldots, y_P) \in \{0,1\}^P$ and let $(k_1, \ldots, k_P)$ be any permutation of $(1, \ldots, P)$. Let $\sigma(y) \equiv (y_{k_1}, \ldots, y_{k_P})$ be the resulting permutation of y. By the symmetry properties of the joint objective function in Assumption 3, there exists a permutation $(\ell_1, \ldots, \ell_P)$ such that

$$X_1 = \cdots = X_P \quad \Longrightarrow \quad \begin{aligned} \mathcal{U}(\sigma(y), X_G, \nu) &= V\Big(u_1(y_{k_1}, y_{-k_1}, X_G, \nu), \ldots, u_P(y_{k_p}, y_{-k_p}, X, \nu); X_G, \nu\Big) \\ &= V\Big(u_{\ell_1}(y_1, y_{-1}, X_G, \nu), \ldots, u_{\ell_P}(y_P, y_{-P}, X, \nu); X_G, \nu\Big). \end{aligned}$$

Thus, if we let $\sigma_1(y),\ldots,\sigma_{R_y}(y)$ be the collection of all *distinct* permutations of y, there exist a corresponding collection of permutations $(\ell_1^1,\ldots,\ell_P^1);(\ell_1^2,\ldots,\ell_P^2);\cdots;(\ell_1^{R_y},\ldots,\ell_P^{R_y})$ of $(1,\ldots,P)$ such that, if $X_1=\cdots=X_P$, then we have,

$$\begin{pmatrix} \mathcal{U}(\sigma_1(y),X_G,\nu) \\ \mathcal{U}(\sigma_2(y),X_G,\nu) \\ \vdots \\ \mathcal{U}(\sigma_{R_y}(y),X_G,\nu) \end{pmatrix} = \begin{pmatrix} V\Big(u_{\ell_1^1}(y_1,y_{-1},X_G,\nu),\ldots,u_{\ell_P^1}(y_P,y_{-P},X_G,\nu);X_G,\nu\Big) \\ V\Big(u_{\ell_1^2}(y_1,y_{-1},X_G,\nu),\ldots,u_{\ell_P^2}(y_P,y_{-P},X_G,\nu);X_G,\nu\Big) \\ \vdots \\ V\Big(u_{\ell_1^{R_y}}(y_1,y_{-1},X_G,\nu),\ldots,u_{\ell_P^{R_y}}(y_P,y_{-P};X_G,\nu),X_G,\nu\Big) \end{pmatrix}$$

From here and the exchangeability condition in Assumption 2, we obtain the following result. For any pair of permutations $(m_1,\ldots,m_P)$ and $(m_1',\ldots,m_P')$ of $(1,\ldots,P)$,

$$X_1=\cdots=X_P \quad \Longrightarrow \quad \begin{pmatrix} \mathcal{U}(e_{m_1},X_G,\nu) \\ \vdots \\ \mathcal{U}(e_{m_P},X_G,\nu) \end{pmatrix} \stackrel{d}{=} \begin{pmatrix} \mathcal{U}(e_{m_1'},X_G,\nu) \\ \vdots \\ \mathcal{U}(e_{m_P'},X_G,\nu) \end{pmatrix} \tag{A-1}$$

Let $s_1,\ldots,s_Q$ denote the collection of all action profiles in $\{0,1\}^P$ such that $s_\ell \neq e_r$ for any r (note that $Q=2^P-P$). From our previous result, if Assumptions 2 and 3 are satisfied, then for any pair of permutations $(m_1,\ldots,m_P)$ and $(m_1',\ldots,m_P')$ of $(1,\ldots,P)$,

$$X_1=\cdots=X_P \quad \Longrightarrow \quad \begin{pmatrix} \mathcal{U}(e_{m_1},X_G,\nu) \\ \vdots \\ \mathcal{U}(e_{m_P},X_G,\nu) \\ \mathcal{U}(s_1,X_G,\nu) \\ \vdots \\ \mathcal{U}(s_Q,X_G,\nu) \end{pmatrix} \stackrel{d}{=} \begin{pmatrix} \mathcal{U}(e_{m_1'},X_G,\nu) \\ \vdots \\ \mathcal{U}(e_{m_P'},X_G,\nu) \\ \mathcal{U}(s_1,X_G,\nu) \\ \vdots \\ \mathcal{U}(s_Q,X_G,\nu) \end{pmatrix} \tag{A-2}$$

Thus, for every p,q, we have $Pr\Big(\mathcal{U}(e_p,X_G,\nu)>\mathcal{U}(y,X_G,\nu)\ \forall\ y\neq e_p \mid X_1=\cdots=X_P\Big)$ $=Pr\Big(\mathcal{U}(e_q,X_G,\nu)>\mathcal{U}(y,X_G,\nu)\ \forall\ y\neq e_q \mid X_1=\cdots=X_P\Big)$. Using the definition in (1), we have $Q(e_p|x)=Q(e_q|x)\ \forall\ p,q,$, for a.e $x\in\mathcal{X}$. This proves the result in Proposition 1. ■

A.2 Theorem 1

Take any $y\in\mathcal{S}$ and let $\sigma_1(y),\ldots,\sigma_{R_y}(y)$ denote the collection of all *distinct* permutations of y. Using the same arguments leading to equation (A-1), we have that, if Assumptions 3 and 4 hold, then for any pair of permutations $(m_1, \ldots, m_{R_y})$ and $(m'_1, \ldots, m'_{R_y})$ of $(1, \ldots, R_y)$,

$$X_1 = \cdots = X_P \quad \Longrightarrow \quad \begin{pmatrix} \mathcal{U}(\sigma_{m_1}(y), X_G, \nu) \\ \vdots \\ \mathcal{U}(\sigma_{m_{R_y}}(y), X_G, \nu) \end{pmatrix} \stackrel{d}{=} \begin{pmatrix} \mathcal{U}(\sigma_{m'_1}(y), X_G, \nu) \\ \vdots \\ \mathcal{U}(\sigma_{m'_{R_y}}(y), X_G, \nu) \end{pmatrix} \tag{A-3}$$

This is a generalization of (A-1), which we obtained for the binary choice game. From (A-3), we have that if we take any $y \in \mathcal{S}$, and if we let $s_1, \ldots, s_{Q_y}$ denote all the action profiles in $\mathcal{Y}^P$ that are not a permutation of y, if Assumptions 3 and 4 hold, then for any pair of permutations $(m_1, \ldots, m_{R_y})$ and $(m'_1, \ldots, m'_{R_y})$ of $(1, \ldots, R_y)$,

$$X_1 = \cdots = X_P \quad \Longrightarrow \quad \begin{pmatrix} \mathcal{U}(\sigma_{m_1}(y), X_G, \nu) \\ \vdots \\ \mathcal{U}(\sigma_{m_{R_y}}(y), X_G, \nu) \\ \mathcal{U}(s_1, X_G, \nu) \\ \vdots \\ \mathcal{U}(s_{Q_y}, X_G, \nu) \end{pmatrix} \stackrel{d}{=} \begin{pmatrix} \mathcal{U}(\sigma_{m'_1}(y), X_G, \nu) \\ \vdots \\ \mathcal{U}(\sigma_{m'_{R_y}}(y), X_G, \nu) \\ \mathcal{U}(s_1, X_G, \nu) \\ \vdots \\ \mathcal{U}(s_{Q_y}, X_G, \nu) \end{pmatrix}$$

And from here we have that, for any $y \in \mathcal{S}$, any permutation y' of y, and for a.e $x \in \mathcal{X}$,

$$\begin{aligned} &Pr\left(\mathcal{U}(y, X_G, \nu) > \mathcal{U}(t, X_G, \nu) \ \forall \ t \neq y \mid X_1 = \cdots = X_P = x\right) \\ =&Pr\left(\mathcal{U}(y', X_G, \nu) > \mathcal{U}(t, X_G, \nu) \ \forall \ t \neq y' \mid X_1 = \cdots = X_P = x\right). \end{aligned}$$

Using the definition in (1), it follows that for any $y \in \mathcal{S}$ and any permutation y' of y,

$$Q(y|x) = Q(y'|x) \ \text{ for a.e } x \in \mathcal{X}$$

This proves the result in Theorem 1. ■

A.3 Theorem 2

Here we describe the main steps of the proof of Theorem 2. All the step-by-step details and derivations are included in the online supplement. In what follows, recall that M is the integer described in Assumptions E2 and E3, r is the number of continuously distributed observable payoff shifters for each individual player and $L \equiv P \cdot r$ is the total number of continuously distributed payoff shifters combined for all players in the game. Let $H_n(U_i, U_j)$ be as defined in equation (12). As we showed in equation (13), our test-statistic $\widehat{\mathcal{T}}$ can be expressed as $\widehat{\mathcal{T}} = \binom{n}{2}^{-1} \sum\limits_{i<j} H_n(U_i, U_j)$, a U-statistic of order 2. Let $m_n(U_i) \equiv E[H_n(U_i, U_j)|U_i]$, $b_n(U_i) \equiv m_n(U_i) - E[m_n(U_i)]$, and $G_n(U_i, U_j) \equiv H_n(U_i, U_j) - E[H_n(U_i, U_j)] - b_n(U_i) - b_n(U_j)$. Under the assumptions of Theorem 2, the Hoeffding decomposition (see Serfling (1980, pages 177-178)) of the U-statistic $\binom{n}{2}^{-1} \sum_{i<j} H_n(U_i, U_j)$ yields the following representation for $\widehat{\mathcal{T}}$

$$\widehat{\mathcal{T}} = \mathcal{T} + \frac{2}{n} \sum_{i=1}^{n} b_n(U_i) + \binom{n}{2}^{-1} \sum_{i<j} G_n(U_i, U_j) + O\left(h_n^M\right). \tag{A-4}$$

And, under the null hypothesis of cooperation, (A-4) becomes,

$$\widehat{\mathcal{T}} = \binom{n}{2}^{-1} \sum_{i<j} G_n(U_i, U_j) + O_p\left(h_n^{M+r-L}\right). \tag{A-4'}$$

By construction, $\binom{n}{2}^{-1} \sum_{i<j} G_n(U_i, U_j)$ is a *degenerate* U-statistic of order 2. To determine its asymptotic distribution, we verify the conditions in Theorem 1 of Hall (1986), which result in asymptotic normality of degenerate U-statistics of order two. Let (i, j, k) denote three distinct observations from our iid sample. Let $\widetilde{G}_n(U_j, U_k) \equiv E[G_n(U_i, U_j) \cdot G_n(U_i, U_k)|U_j, U_k]$. Suppose,

$$\frac{E[\widetilde{G}_n(U_j, U_k)^2] + n^{-1} E[G_n(U_i, U_j)^4]}{\left(E[G_n(U_i, U_j)^2]\right)^2} \longrightarrow 0, \tag{A-5}$$

as $n \to \infty$. Theorem 1 in Hall (1986) shows that, in this case, $\sum_{i<j} G_n(U_i, U_j)$ is asymptotically normally distributed with zero mean and variance given by $\frac{1}{2} n^2 \cdot E[G_n(U_i, U_j)^2]$. The bulk of the proof is devoted to showing that (A-5) is satisfied under our assumptions. Let $\widetilde{H}_n(U_j, U_k) \equiv E[H_n(U_i, U_j) \cdot H_n(U_i, U_k)|U_k, U_k]$. Our first series of steps is to show that, under our assumptions, we have $E[\widetilde{H}_n(U_j, U_k)^2] = O\left(\frac{1}{h_n^{4L-3r}}\right)$, $E[H_n(U_i, U_j)^2] = O\left(\frac{1}{h_n^{2L-r}}\right)$, $E[H_n(U_i, U_j)^3] = O\left(\frac{1}{h_n^{4L-2r}}\right)$, and $E[H_n(U_i, U_j)^4] = O\left(\frac{1}{h_n^{6L-3r}}\right)$. In particular, we show that $E\left[H_n(U_i, U_j)^2\right] = \frac{1}{h_n^{2L-r}} \cdot \left(\sigma_n^2 + o(1)\right)$, where $\sigma_n^2 > 0$ is described in the online supplement and satisfies $\sigma_n^2 \to \sigma^2 > 0$. From here, the next series of steps show that $E[\widetilde{G}_n(U_j, U_k)^2] = E[\widetilde{H}_n(U_j, U_k)^2] + O\left(h_n^{M+3\cdot(r-L)}\right)$, $E\left[G_n(U_i, U_j)^4\right] = E\left[H_n(U_i, U_j)^4\right] + O\left(h_n^{M+3r-5L}\right)$, $E\left[G_n(U_i, U_j)^2\right] =$

$E\left[H_n(U_i,U_j)^2\right]+O\left(h_n^{M+r-L}\right)$, and from here,

$$\frac{E\left[\widetilde{G}_n(U_j,U_k)^2\right]+n^{-1}E\left[G_n(U_i,U_j)^4\right]}{\left(E\left[G_n(U_i,U_j)^2\right]\right)^2}=\frac{O(h_n^r)+O\left(\frac{1}{n\cdot h_n^{2L-r}}\right)}{\left(\sigma_n^2+o(1)\right)^2}\longrightarrow 0,$$

where the last result follows from our bandwidth-convergence restrictions. Thus, (A-5) is satisfied and, going back to (A-4'), we have that, under the null hypothesis of cooperation,

$$\frac{n\cdot\widehat{\mathcal{T}}}{\sqrt{2\cdot E\left[H_n(U_i,U_j)^2\right]}}=n\cdot\left(\frac{\binom{n}{2}^{-1}\sum_{i<j}G_n(U_i,U_j)}{\sqrt{2\cdot E\left[G_n(U_i,U_j)^2\right]+o(1)}}\right)+\underbrace{O_p\left(n\cdot h_n^{M+\frac{r}{2}}\right)}_{=o_p(1)}\xrightarrow{d}\mathcal{N}(0,1). \quad \text{(A-6)}$$

To establish the asymptotic behavior of $\widehat{\mathcal{T}}$ under the alternative hypothesis of no-cooperation, where we have $\mathcal{T}>0$. We go back to (A-4), and we have,

$$\frac{n\cdot\widehat{\mathcal{T}}}{\sqrt{2\cdot E\left[H_n(U_i,U_j)^2\right]}}=\frac{n\cdot h_n^{\frac{2L-r}{2}}\cdot\mathcal{T}}{\sqrt{2\cdot\left(\sigma_n^2+o(1)\right)}}+n^{\frac{1}{2}}\cdot h_n^{\frac{L}{2}}\cdot\vartheta_n,\quad \text{where}\quad |\vartheta_n|=O_p(1),$$

Under the bandwidth convergence restrictions in Assumption E3 we have $n^{\frac{1}{2}}\cdot h_n^{\frac{L}{2}}\longrightarrow\infty$ and $n^{\frac{1}{2}}\cdot h_n^{\frac{L-r}{2}}\longrightarrow\infty$. Take any sequence $c_n>0$ such that, $c_n\longrightarrow+\infty$, and $\frac{c_n}{n^{\frac{1}{2}}\cdot h_n^{\frac{L}{2}}}\longrightarrow 0$. Going back to (A-6), under the alternative hypothesis of no cooperation,

$$Pr\left(\frac{n\cdot\widehat{\mathcal{T}}}{\sqrt{2\cdot E\left[H_n(U_i,U_j)^2\right]}}>c_n\right)=Pr\left(\vartheta_n>\frac{c_n}{n^{\frac{1}{2}}\cdot h_n^{\frac{L}{2}}}-\frac{n^{\frac{1}{2}}\cdot h_n^{\frac{L-r}{2}}\cdot\mathcal{T}}{\sqrt{2\cdot\left(\sigma_n^2+o(1)\right)}}\right)\longrightarrow 1,$$

where the last result follows from the fact that $\vartheta_n=O_p(1)$ and $\frac{c_n}{n^{\frac{1}{2}}\cdot h_n^{\frac{L}{2}}}-\frac{n^{\frac{1}{2}}\cdot h_n^{\frac{L-r}{2}}\cdot\mathcal{T}}{\sqrt{2\cdot\left(\sigma_n^2+o(1)\right)}}\longrightarrow-\infty$ (since $\mathcal{T}>0$). Thus, under the alternative hypothesis of no cooperation,

$$\frac{n\cdot\widehat{\mathcal{T}}}{\sqrt{2\cdot E\left[H_n(U_i,U_j)^2\right]}}\longrightarrow+\infty\quad \text{with probability approaching one (w.p.a.1)} \quad \text{(A-7)}$$

The results in (A-6) and (A-7) prove Theorem 2. ■