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A simple test for moment inequality models with an application to English auctions☆

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ABSTRACT

Testable predictions of many economic models involve inequality comparisons between transformations of nonparametric functionals. We introduce an econometric test for these types of restrictions based on one-sided L_p-statistics that adapt asymptotically to the contact sets without having to directly estimate them. Monte Carlo experiments show that our test is less conservative than procedures based on least-favorable configurations and has power comparable to other contact-set based procedures. As an application, we test for interdependence of bidders' valuations in ascending auctions. Using USFS timber auction data we reject the Independent Private Values model in favor of a model of correlated private values.

1. Introduction

Testable implications of economic models often involve restrictions on moments identified in the data. In this paper we study models whose restrictions involve inequalities between nonlinear transformations of conditional moments, and develop a computationally simple econometric methodology to test such restrictions. Our main motivating example is testable implications concerning the interdependence of bidders' valuations in ascending auctions but our method can be applied to many other settings.

There has been a recent growth in the literature on testing and inferential methods involving some form of conditional moment inequalities. Recent contributions include Ghosal et al. (2000), Barrett and Donald (2003), Hall and Yatchew (2005), Lee et al. (2009), Andrews and Shi (2013, 2011), Chernozhukov et al. (2013), Lee et al. (2013, 2014), Ponomareva (2010), Kim (2009), Menzel (2014), Armstrong (2015, 2014), Chetverikov (2012). Our approach is more closely comparable to that of Lee et al. (2014), as both rely on one-sided L_p-statistics meant to exploit the properties of the contact sets (the regions where the inequalities are binding). Having a test whose properties rely on the contact sets improves upon conservative approaches based on least-favorable configurations (such as Lee et al. (2013)). The difference between our approach and Lee et al. (2014) is that our test will asymptotically adapt to the properties of the contact sets while the procedure in Lee et al. (2014) relies on a direct estimate of such sets. As our analysis will show, the asymptotic power properties of both methods are comparable, an assertion supported by our Monte Carlo experiments. We view our test as a complement to the approach in Lee et al. (2014): the two use very different ways to construct tests based on the properties of the contact sets, and each outperforms the other in certain settings. As our Monte Carlo experiments will suggest, both of these tests in turn complement other existing procedures, outperforming them in many, but not all, settings.

We apply our approach to test the Independent Private Values (IPV) assumption widely used in empirical studies of auctions. Whether the valuations of bidders in an auction are independent or correlated has significant policy implications.[1] Furthermore, in

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[1] If private values are independently and symmetrically distributed, a classic result in auction theory is that the optimal selling mechanism takes the form of

auction models both non-parametric identification and the choice of an empirical strategy depend on whether or not values are independent (see, e.g., Athey and Haile (2007)). When values are correlated, policy implications drawn from an IPV-based empirical approach can be misleading; thus, if the tests presented in this paper lead to rejection of the IPV model, an empirical strategy that allows for correlation is necessary.[2] We derive testable implications of both IPV and correlated private values models of bidding in English auctions. Even though the power analysis of our econometric test in these models is limited because it is impractical to evaluate our procedure against all possible alternative auction models of interest, we find that it has good power properties in the context of two standard models of endogenous entry in auctions (those of Levin and Smith (1994) and Samuelson (1985)). We apply our tests to data from USFS timber auctions, which has been widely studied in the literature under the assumption of IPV. Even after we condition on a rich collection of auction characteristics we find clear evidence to reject independence of values in favor of correlation.

The rest of the paper proceeds as follows. Section 2 describes the structure of our general setup. Section 3 describes our econometric testing procedure and its asymptotic properties. There we also compare it to the existing literature. Section 4 derives testable implications of IPV and positively-correlated private values under standard models of bidding in English auctions, and shows how they fit within our general econometric setup. A series of Monte Carlo experiments are described in Section 5, where we analyze the performance of our procedure in auction models as well as its comparison to other existing tests. Section 6 applies our test to data on USFS timber auctions. Our results there lead us to reject independence of bidders' valuations, an important finding in terms of identification and policy implications. Section 7 concludes. A condensed version of our econometric proofs is included in Appendices A and B that describes the results of our Monte Carlo experiments. Step-by-step econometric proofs, as well as proofs of our auction results in Section 4 and additional material can be found in the online Supplementary Appendix, at: http://www.personal.psu.edu/aza12/testing_auctions_supplement.pdf.

2. General setup

Our setup is motivated by our English auctions application but it encompasses many other econometric models as special cases.

2.1. Variables and index parameter

The variables observed in the data can be classified into the following categories:

(i) **Outcome variables:** Denoted by $Y \in \mathbb{R}^{d_y}$, these are quantities under the control of the decision-makers in the model, such as bids in an auction. Economic theory will characterize how these decisions are made.

(ii) **Conditioning variable of interest:** Denoted by $N \in \mathbb{R}$, this refers generally to a variable whose relationship with Y is predicted by theory and which we will be testing.

(iii) **Controls:** Denoted by $X \in \mathbb{R}^{d_x}$, these are other observable characteristics of the environment on which we will be conditioning.

At a high level, we will be testing the relationship between N and Y, while holding X fixed. The variable N could be allowed to be multidimensional, but we focus on the case $N \in \mathbb{R}$ because it corresponds to the main economic application we study here, where N will denote the number of bidders in an auction and X will denote all other observable details of the auction.

(iv) **Index parameter:** Denoted by $z \in \mathcal{Z} \subset \mathbb{R}^{d_z}$. This index will be present when the theoretical predictions to be tested must hold over a range of values. For example, a test of first-order stochastic dominance can be thought of as a test that the relationship $F_1(z) \le F_2(z)$ holds for each value of z in some range. We will focus on the case where $\mathcal{Z}$ is a compact, connected set.

Some of our examples do not involve an index z. In this case we will normalize $z \equiv 1$ and $\mathcal{Z} \equiv \{1\}$. If $\mathcal{Z}$ were discrete then z would be absorbed by our index of transformations 'q' which is described below.

2.2. Structural functions

Our next component is a *known*, vector-valued function of the variables above. This function will be denoted by $S(y, x, z, n) \in \mathbb{R}^{d_s}$, and its expected value over y, conditional on (x, z, n), by

$$s(x, z, n) = \int S(y, x, z, n) dF_{Y|X,N}(y|x, n) = E_{Y|X,N}\left[S(Y, x, z, n) \,\middle|\, X = x, N = n\right].$$

(Throughout the paper, F_ξ refers to the marginal distribution of a random variable ξ, and $F_{\xi|\eta}$ to the conditional distribution of ξ given η, with f_ξ and $f_{\xi|\eta}$ the respective densities.)

2.3. Transformations

For each pair $(n, n') \in \mathrm{Supp}(N)^2$, the model produces a finite collection of $Q_{n,n'}$ *known* real-valued transformations $\{m^q\}$. Each of these transformations depends on the pair (n, n') in question and on the conditional moments $s(\cdot, n)$ and $s(\cdot, n')$. We will abbreviate

$$R^q\left(X, z; n, n'\right) = m^q\left(s(X, z, n), s(X, z, n'); n, n'\right) \in \mathbb{R}.$$

The models we consider have predictions of the type

$$\forall n, n' \in \mathrm{Supp}(N)^2, \forall z \in \mathcal{Z}, Pr\left(R^q\left(X, z; n, n'\right) \le 0\right) = 1 \quad \text{for } q = 1, \ldots, Q_{n,n'}. \tag{1}$$

2.4. Examples

While motivated by our English auctions application, our setup encompasses many other examples as special cases.

Example 1: First order stochastic dominance

Suppose Y is real-valued, and that the economic model predicts the first-order stochastic dominance relation $F_{Y|X,N}(\cdot|x, n) \succsim_{FOSD} F_{Y|X,N}(\cdot|x, n')$ for a.e x whenever $n > n'$. Thus, the model predicts

$$n > n' \Longrightarrow F_{Y|X,N}(z|x, n) \le F_{Y|X,N}(z|x, n') \quad \forall z, \text{ a.e } x.$$

This can be written as an instance of our general setup by letting $S(y, x, z, n) = \mathbb{1}\{y \le z\}$, so that $s(x, z, n) = F_{Y|X,N}(z|x, n)$, and using the single transformation m for each n, n'

$$m\left(s(x, z, n); s(x, z, n'); n, n'\right) = \left(s(x, z, n) - s(x, z, n')\right) \cdot \mathbb{1}\left\{n > n'\right\}.$$

a standard auction in which the only relevant design parameter is the reserve price. IPV also implies revenue-equivalence of the two most prevalent auction formats, first-price and English auctions; optimality of a reserve price strictly higher than the seller's valuation; and invariance of that optimal reserve price to the number of bidders present. All of these results can break down when bidder values are correlated.

[2] Identification and estimation in auctions with correlated values have been studied by Li et al. (2002), Krasnokutskaya (2011), and Hu et al. (2013) in the case of first-price auctions, and Aradillas-López et al. (2013) in the case of ascending auctions.

Example 2: Second order stochastic dominance

Next, consider testing whether $F_{Y|X,N}(\cdot|x,n) \succsim_{SOSD} F_{Y|X,N}(\cdot|x,n')$ whenever $n > n'$, where SOSD denotes second-order stochastic dominance. This requires that

$$n > n' \implies \int_{-\infty}^{z} F_{Y|X,N}(v|x,n)dv \le \int_{-\infty}^{z} F_{Y|X,N}(v|x,n')dv \quad \forall z, \text{ a.e } x.$$

This too is an instance of our general setup, with $S(y,x,z,n) = \max\{z-y,0\}$. To see why, note that

$$\int_{-\infty}^{z} \mathbb{1}\{Y \le v\}\, dv = \max\{z-Y,0\},$$

and so for a given (x,z,n),

$$\begin{aligned} s(x,z,n) &= E_{Y|X,N}[S(Y,x,z,n)|X=x,N=n] \\ &= E_{Y|X,N}\left[\max\{z-Y,0\}\,\middle|X=x,N=n\right] \\ &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{z}\mathbb{1}\{y\le v\}\,dv\right) f_{Y|X,N}(y|x,n)dy \\ &= \int_{-\infty}^{z}\left(\int_{-\infty}^{\infty}\mathbb{1}\{y\le v\} f_{Y|X,N}(y|x,n)dy\right)dv \\ &= \int_{-\infty}^{z} F_{Y|X,N}(v|x,n)dv. \end{aligned}$$

Once again, we would have $Q_{n,n'} = 1$ and

$$m\left(s(x,z,n); s(x,z,n'); n, n'\right) = \left(s(x,z,n) - s(x,z,n')\right) \cdot \mathbb{1}\left\{n > n'\right\}.$$

Example 3: Covariance restrictions

Third, let Y_1 and Y_2 denote two real-valued outcome variables, and X a vector of controls. There exist economic models that yield restrictions of the general form

$$Cov\left(\mathbb{1}\{Y_1 \le z\}, Y_2\middle|X=x\right) \ge 0 \quad \forall z \in \mathcal{Z}. \tag{2}$$

One example is the "positive correlation" test proposed by Chiappori and Salanie (2000) and Chiappori et al. (2006) for moral hazard and adverse selection in insurance markets. In this case $Y_2 = -A$, where A is the binary indicator variable that equals 1 if the agent incurs an accident (requiring them to exercise their insurance contract). Incomplete-information static games are another example of models that produce restrictions of the general form (2), as is shown by de Paula and Tang (2012) for binary action games, and generalized to an ordered set of actions by Aradillas-López and Gandhi (forthcoming). To see why (2) fits within our general setup, note that it can be re-expressed as the restriction

$$\begin{aligned} &E\left[\mathbb{1}\{Y_1 \le z\}\middle|X=x\right]\cdot E\left[Y_2\middle|X=x\right] \\ &\quad - E\left[\mathbb{1}\{Y_1\le z\}\cdot Y_2\middle|X=x\right] \le 0 \quad \forall z, \text{ a.e } x. \end{aligned}$$

In this case there is no variable playing the role of N, so this model fits the special case (1′) discussed below. The structural functions are

$$S(Y,z) = \left(\mathbb{1}\{Y_1\le z\}, Y_2, \mathbb{1}\{Y_1 \le z\}\cdot Y_2\right),$$

and

$$\begin{aligned} s(x,z) &= E\left[S(Y,z)\middle|X=x\right] \\ &= \Big(E\left[\mathbb{1}\{Y_1\le z\}\middle|X=x\right], E\left[Y_2\middle|X=x\right], \\ &\qquad E\left[\mathbb{1}\{Y_1\le z\}\cdot Y_2\middle|X=x\right]\Big) \\ &\equiv \left(s^1(x,z), s^2(x), s^3(x,z)\right). \end{aligned}$$

And the transformation m is then given by

$$\begin{aligned} m(s(x,z)) &= s^1(x,z)\cdot s^2(x) - s^3(x,z) \\ &= E\left[\mathbb{1}\{Y_1\le z\}\middle|X=x\right]\cdot E\left[Y_2\middle|X=x\right] \\ &\quad - E\left[\mathbb{1}\{Y_1\le z\}\cdot Y_2\middle|X=x\right]. \end{aligned}$$

Example 4: Conditional moment inequality models

A special case of our general framework is when there is no N variable and no index z, so $S(y,x,z) = S(y,x)$. In this case, the vector-valued transformations are of the form $S(y,x) \in \mathbb{R}^{d_s}$ and

$$s(x) = \int S(y,x)dF_{Y|X}(y|x) = E_{Y|X}\left[S(Y,x)\middle|X=x\right].$$

In this case the model consists of $q = 1,\ldots,Q$ transformations $\{m^q\}$. The arguments of each transformation m^q are now simply $s(x)$, with

$$R^q(x) = m^q(s(x)).$$

These models predict

$$Pr\left(R^q(X) \le 0\right) = 1, \quad q = 1,\ldots,Q. \tag{1′}$$

3. An econometric test for our general model

Testing of conditional moment inequalities has received increasing attention in the recent past. Some examples include Andrews and Shi (2011, 2013), Armstrong (2015, 2014), Chernozhukov et al. (2013), Lee et al. (2013, 2014). Among the existing work, our approach is more closely related to Lee et al. (2014) since they are both based on one-sided L_p tests driven by the properties of *contact sets*, improving upon conservative methods based on least-favorable configurations such as Lee et al. (2013). The distinction is that, while our test adapts asymptotically to the properties of these sets, the method in Lee et al. (2014) is based on a direct estimate of such sets. Our results will suggest that both methods complement each other, and that they also complement alternative approaches based on instrument-functions (like Andrews and Shi (2013)) and sup-norm tests (like Chernozhukov et al. (2013)).

3.1. Expressing (1) as an unconditional mean-zero condition

We develop a test for the general model described in (1). Our procedure also covers the special case described in (1′). For simplicity (and because it corresponds to our auction models where N denotes number of bidders) we will assume that N has discrete support, even though our approach can be extended to the continuous$-N$ case. X will be allowed to include discrete and/or continuously distributed covariates. For simplicity we will assume that the support of X does not vary with N. Let $\mathcal{P}$ denote a *pre-specified* measure function with Lebesgue density and support concentrated on $\mathcal{Z}$. We will use $\mathcal{P}$ as our weight function for the index z in our statistic. Since $\mathcal{Z}$ is compact,[3] without loss of generality we can normalize $\int_{z\in\mathcal{Z}} d\mathcal{P}(z) = 1$. Define

$$\mathcal{T}^q_{n,n'}(z) = E_X\left[\max\left\{R^q(X,z;n,n'),0\right\}\right], \quad \text{and}$$

$$\mathcal{T}^q_{n,n'} = \int_{z\in\mathcal{Z}} \mathcal{T}^q_{n,n'}(z)d\mathcal{P}(z)$$

[3] Naturally we can also have $\int_{z\in\mathcal{Z}} d\mathcal{P}(z) = 1$ if $\mathcal{Z}$ is unbounded but it would preclude, e.g., giving each $z \in \mathcal{Z}$ a uniform weight.

and note that this expectation is nonnegative, and it is zero *if and only if* (1) holds for this (n, n') and q. Nonnegativity allows us to combine all restrictions into

$$\mathcal{T} = \sum_{n,n' \in \text{Supp}(N)} \sum_{q=1}^{Q_{n,n'}} \mathcal{T}_{n,n'}^q.$$

We then have $\mathcal{T} \geq 0$, with $\mathcal{T} = 0$ if and only if (1) holds.

3.2. Choosing a testing range

We separate X into its continuous X^c and discrete X^d components, and assume X^c has an absolutely continuous distribution with respect to the Lebesgue measure. Our test will rely on nonparametric estimators which must satisfy certain asymptotic properties *uniformly* over the range of values considered for (x, n, z). For this reason we will specify a *testing range* $\mathcal{W} \equiv \mathcal{X} \times \mathcal{N} \times \mathcal{Z}$ for (x, n, z), where $\mathcal{N} \subseteq \text{Supp}(N)$ and $\mathcal{X} \subset \text{Supp}(X)$ are pre-specified sets such that $x \equiv (x^c, x^d) \in \mathcal{X} \implies x^c \in \text{int}(\text{Supp}(X^c))$, and $f_{X,N}(x, n) \geq \underline{f} > 0\ \forall (x, n) \in \mathcal{X} \times \mathcal{N}$, where $f_{X,N}$ is the joint density of X and N. We will focus on the expectation $\mathcal{T}_{n,n'}^q$ taken *conditional over this range*. Let $\mathbb{I}_{\mathcal{X}}$ denote a trimming function with the property $\mathbb{I}_{\mathcal{X}}(x) \geq 0$, and $\mathbb{I}_{\mathcal{X}}(x) > 0 \iff x \in \mathcal{X}$. Since we will concentrate on the testing range, hereafter we will re-define

$$\begin{aligned}
\mathcal{T}_{n,n'}^q(z) &= E_X\left[\max\left\{R^q(X, z; n, n'), 0\right\} \cdot \mathbb{I}_{\mathcal{X}}(X)\right], \\
\mathcal{T}_{n,n'}^q &= \int \mathcal{T}_{n,n'}^q(z) d\mathcal{P}(z), \quad \text{and} \quad \mathcal{T} = \sum_{n,n' \in \mathcal{N}} \sum_{q=1}^{Q_{n,n'}} \mathcal{T}_{n,n'}^q.
\end{aligned} \tag{3}$$

3.3. Nonparametric estimators

We group $U \equiv (Y, X, N)$ and maintain the assumption that we observe a sample $\{U_i : 1 \leq i \leq L\}$ with $U_i \sim F$. We employ kernel-based estimators constructed using a kernel $K : \mathbb{R}^r \longrightarrow \mathbb{R}$ (recall that $r \equiv \dim(X^c)$) and a bandwidth sequence $h_L \longrightarrow 0$. For a given $x \equiv (x^c, x^d)$ and $h > 0$ denote

$$\mathcal{H}(X_i - x; h) = K\left(\frac{X_i^c - x^c}{h}\right) \cdot \mathbb{1}\left\{X_i^d - x^d = 0\right\}.$$

For a given (x, z, n) we use

$$\begin{aligned}
\widehat{f}_{X,N}(x, n) &= \frac{1}{L \cdot h_L^r} \sum_{i=1}^{L} \mathcal{H}(X_i - x; h_L) \cdot \mathbb{1}\{N_i = n\}, \\
\widehat{s}(x, z, n) &= \frac{1}{L \cdot h_L^r} \sum_{i=1}^{L} S(Y_i, x, z, n) \cdot \mathcal{H}(X_i - x; h_L) \\
&\quad \cdot \mathbb{1}\{N_i = n\} \Big/ \widehat{f}_{X,N}(x, n), \\
\widehat{R}^q(x, z; n, n') &= m^q\left(\widehat{s}(x, z, n), \widehat{s}(x, z, n'); n, n'\right).
\end{aligned}$$

As we described above, our target testing range depends on $f_{X,N}$. We estimate $\mathcal{T}_{n,n'}^q$ and $\mathcal{T}$ as

$$\begin{aligned}
\widehat{\mathcal{T}}_{n,n'}^q(z) &= \frac{1}{L} \sum_{i=1}^{L} \widehat{R}^q(X_i, z; n, n') \\
&\quad \cdot \mathbb{1}\left\{\widehat{R}^q(X_i, z; n, n') \geq -b_L\right\} \cdot \mathbb{I}_{\mathcal{X}}(X_i),
\end{aligned} \tag{4}$$

$$\widehat{\mathcal{T}}_{n,n'}^q = \int \widehat{\mathcal{T}}_{n,n'}^q(z) \mathcal{P}(z), \quad \text{and} \quad \widehat{\mathcal{T}} = \sum_{n,n' \in \mathcal{N}} \sum_{q=1}^{Q_{n,n'}} \widehat{\mathcal{T}}_{n,n'}^q.$$

$b_L \longrightarrow 0$ is a nonnegative bandwidth sequence, whose inclusion will allow us to deal with the "kink" at zero in the function $\max\{v, 0\}$ and obtain (under assumptions described below) asymptotically pivotal properties for $\widehat{\mathcal{T}}_{n,n'}^q$ and $\widehat{\mathcal{T}}$. Its role is analogous, e.g., to the bandwidth sequences β_n in Jun et al. (2010) and τ_n in Kim (2009). Both of these papers also involve testing and inference with moment inequalities.[4]

3.4. Asymptotic properties

We characterize the asymptotic distribution of $\widehat{\mathcal{T}}$ under four types of assumptions: (i) Smoothness conditions. (ii) A special regularity assumption. (iii) Kernels and bandwidth convergence conditions. (iv) Manageability properties of the empirical processes involved.

Assumption 3.1 (*Smoothness Conditions*). (i) For each $(n, n', z) \in \mathcal{N}^2 \times \mathcal{Z}$ and a.e $(x_1, x_2) \in \mathcal{X}^2$, the functionals

$$E_{Y|X,N}\left[S(Y, x_1, z, n') \middle| X = x_2, N = n\right] \quad \text{and} \quad f_{X,N}(x_2, n)$$

are M times differentiable with respect to x_2^c, with bounded derivatives. Below, we will describe how large M needs to be.
(ii) $-\infty < \underline{s} \leq \|s(x, z, n)\| \leq \overline{s} < \infty \forall (x, z, n) \in \mathcal{W}$. Let $\frac{\partial m^q(s_1, s_2; n, n')}{\partial s_1}$ and $\frac{\partial m^q(s_1, s_2; n, n')}{\partial s_2}$ denote the partial derivatives of m^q with respect to its first and second arguments, respectively. Denote the Jacobian and Hessian

$$\frac{\partial m^q(s_1, s_2; n, n')}{\partial s} = \left(\frac{\partial m^q\left(s_1, s_2; n, n'\right)}{\partial s_1} \quad \frac{\partial m^q\left(s_1, s_2; n, n'\right)}{\partial s_2}\right)',$$

$$\frac{\partial^2 m^q(s_1, s_2; n, n')}{\partial s \partial s'} = \begin{pmatrix} \dfrac{\partial^2 m^q\left(s_1, s_2; n, n'\right)}{\partial s_1 \partial s_1} & \dfrac{\partial^2 m^q\left(s_1, s_2; n, n'\right)}{\partial s_1 \partial s_2} \\ \dfrac{\partial^2 m^q\left(s_1, s_2; n, n'\right)}{\partial s_2 \partial s_1} & \dfrac{\partial^2 m^q\left(s_1, s_2; n, n'\right)}{\partial s_2 \partial s_2} \end{pmatrix}.$$

Then

$$\sup_{\substack{(n,n') \in \mathcal{N} \\ s_1, s_2 : \max\{\|s_1\|, \|s_2\|\} \leq D}} \max\left\{\left\|\frac{\partial m^q(s_1, s_2; n, n')}{\partial s}\right\|, \left\|\frac{\partial^2 m^q(s_1, s_2; n, n')}{\partial s \partial s'}\right\|\right\} \leq D \quad \text{for some } D < \infty.$$

For a given $(x, z, n, n') \in \mathcal{W}$ we will denote from now on,

$$\begin{aligned}
\nabla_s m^q\left(x, z; n, n'\right) &= \frac{\partial m^q(s(x, z, n), s(x, z, n'); n, n')}{\partial s}, \\
\nabla_{ss'} m^q\left(x, z; n, n'\right) &= \frac{\partial^2 m^q(s(x, z, n), s(x, z, n'); n, n')}{\partial s \partial s'}.
\end{aligned}$$

Our testing range $\mathcal{W}$ is chosen such that $\nabla_s m^q\left(x, z; n, n'\right)$ and $\nabla_{ss'} m^q\left(x, z; n, n'\right)$ exist for all (x, z, n, n') in $\mathcal{W}$. Furthermore, the previous conditions imply that

[4] All the results that follow can hold if we let the bandwidth sequence b_L depend on (z, n, n') and q, thus generalizing (4) to

$$\widehat{\mathcal{T}}_{n,n'}^q(X_i) = \int_{z \in \mathcal{Z}} \widehat{R}^q\left(X_i, z; n, n'\right) \cdot \mathbb{1}\left\{\widehat{R}^q\left(X_i, z; n, n'\right) \geq -b_L^q(z, n, n')\right\} d\mathcal{P}(z). \tag{4'}$$

As long as each of the bandwidth functions $b_L^q(z, n, n')$ satisfy the conditions to be described in Assumption 3.3, all our asymptotic results would follow through. We focus on the expression given in (4) for expositional purposes.

$$\sup_{(x,z,n,n')\in\mathcal{W}} \left\| \nabla_s m^q\left(x, z; n, n'\right)\right\| \le D,$$
$$\sup_{(x,z,n,n')\in\mathcal{W}} \left\| \nabla_{ss'} m^q\left(x, z; n, n'\right)\right\| \le D.$$

Contact sets. Contact sets refer to regions in $\mathcal{W}$ where the inequalities in (1) are binding. For a given (n, n') we let $CO^q(n, n') = \left\{(x, z) \in \mathcal{W}: R^q\left(x, z; n, n'\right) = 0\right\}$ denote the contact set for $R^q(\cdot; n, n')$. The contact set has measure zero if $E_X\left[\int_{z\in\mathcal{Z}} \mathbb{1}\left\{(X, z) \in CO^q(n, n')\right\} d\mathcal{P}(z)\right] = 0$. We must allow for contact sets to have positive measure. While we allow for this, we assume that $R^q\left(X, z; n, n'\right)$ has a finite density in an open neighborhood to the left of zero. More precisely we impose the following condition.

Assumption 3.2 (*A Regularity Condition*). There exist constants $\overline{b} > 0$ and $\overline{A} < \infty$ such that, for any $0 < b \le \overline{b}$ and each $(z, n, n') \in \mathcal{Z} \times \mathcal{N}^2$ and each $q = 1, \ldots, Q_{n,n'}$,

$$Pr\left(-b \le R^q\left(X, z; n, n'\right) < 0 \,\middle|\, X \in \mathcal{X}\right) \le b \cdot \overline{A}.$$

The smoothness conditions in Assumption 3.1 can lead to $\sqrt{L}$-consistency of $\widehat{\mathcal{T}}$ if we can make the bias in our nonparametric estimators disappear at a fast enough rate. We describe conditions under which this can be achieved in the following assumption.

Assumption 3.3 (*Kernels and Bandwidths*). Let M be as described in Assumption 3.1. We use a kernel K of order M with bounded support. The kernel is a function of bounded variation, symmetric around zero, and satisfies $\sup_{v\in\mathbb{R}^r}\left|K(v)\right| \le \overline{K} < \infty$. The bandwidth sequences b_L and h_L satisfy

$$L^{1/2} \cdot h_L^r \cdot b_L \longrightarrow \infty$$

and for a small enough $\epsilon_1 > 0$,

$$\frac{b_L \cdot L^{\epsilon_1}}{\sqrt{h_L^r}} \longrightarrow 0 \quad \text{and} \quad L^{1/2+\epsilon_1} \cdot b_L^2 \longrightarrow 0.$$

In addition, M is large enough that

$$L^{1/2+\epsilon_1} \cdot h_L^M \longrightarrow 0.$$

The latter can be understood as an undersmoothing condition. If our bandwidths are of the form $h_L \propto L^{-\alpha_h}$ and $b_L \propto L^{-\alpha_b}$ it is not hard to verify that the smallest value for which Assumption 3.3 can hold is $M = 2r + 1$. This is the smallest number of bounded derivatives that the functionals in Assumption 3.1 must possess.

Assumption 3.4 (*Empirical Process Conditions*). Let

$$\overline{S}(y) = \sup_{(x,z,n)\in\mathcal{X}\times\mathcal{Z}\times\mathcal{N}} \left\|S(y, x, z, n)\right\|.$$

Then $E\left[\exp\left\{\overline{S}(Y)^2 \cdot \epsilon\right\}\right] \le C$ for some $\epsilon > 0$ and $C < \infty$ (i.e., $\overline{S}(Y)^2$ possesses a moment generating function). For each $\ell = 1, \ldots, d_s$ and each $n \in \mathcal{N}$, the class of functions

$$\mathcal{F}^\ell = \left\{f: f(y) = S^\ell(y, x, z, n) \ \text{ for some } (x, z, n) \in \mathcal{X} \times \mathcal{Z} \times \mathcal{N}\right\}$$

is *Euclidean* (see Definition 2.7 in Pakes and Pollard (1989)) with respect to the envelope $\overline{S}(\cdot)$.

Using the results in Pakes and Pollard (1989) (in particular, Lemmas 2.4 and 2.14), the Euclidean property is satisfied[5] by our auction models in Section 4, as well as the examples briefly outlined in Section 2.4.

3.4.1. A linear representation result

Let $\varepsilon\left(y, x, z, n\right) = S(y, x, z, n) - s(x, z, n)$. The functionals that follow will have three sets of arguments: $z \in \mathcal{Z}$, $(n, n') \in \mathcal{N}^2$, $x_1 \in \mathcal{X}$ and $u_2 \equiv (y_2, x_2, n_2) \in \text{Supp}(Y) \times \mathcal{X} \times \mathcal{N}$. Let

$$\widetilde{\varepsilon}(u_2, x_1, z; n, n') = \left(\frac{\varepsilon\left(y_2, x_1, z, n\right)' \cdot \mathbb{1}\left\{n_2 = n\right\}}{f_{X,N}(x_1, n)}, \frac{\varepsilon\left(y_2, x_1, z, n'\right)' \cdot \mathbb{1}\left\{n_2 = n'\right\}}{f_{X,N}(x_1, n')}\right)'.$$

Let $\nabla_s m^q\left(x, z; n, n'\right)$ and $\nabla_{ss'} m^q\left(x, z; n, n'\right)$ be as defined in 3.1. We will denote

$$\begin{aligned}
&\varphi^q\left(u_2, x_1; n, n'\right) = \int_{z\in\mathcal{Z}} \left[\nabla m^q\left(x_1, z; n, n'\right)' \widetilde{\varepsilon}(u_2, x_1, z; n, n')\right] \\
&\quad \cdot \mathbb{1}\left\{R^q(x_1, z; n, n') \ge 0\right\} d\mathcal{P}(z), \\
&f^q(x_1, u_2, n, n'; h) = \varphi^q\left(u_2, x_1; n, n'\right) \cdot \mathbb{I}_{\mathcal{X}}(x_1) \\
&\quad \cdot \frac{1}{h^r}\mathcal{H}(x_2 - x_1; h), \\
&\Delta^q(u_2, n, n'; h) = E_X\left[f^q(X, u_2, n, n'; h)\right], \\
&\Delta^q(u_2, n, n'; h_L) \equiv \Delta_L^q(u_2, n, n').
\end{aligned} \tag{5}$$

The following functional describes the "influence function" in the linear representation of $\widehat{\mathcal{T}}^q_{n,n'}$ under our previous assumptions. Let

$$\begin{aligned}
\mathcal{D}^q_{n,n'}(X) &= \int_{z\in\mathcal{Z}} \max\left\{R^q(X, z; n, n'), 0\right\} d\mathcal{P}(z), \\
\lambda_L^q(U_i; n, n') &= \Big(\mathcal{D}^q_{n,n'}(X_i) \cdot \mathbb{I}_{\mathcal{X}}(X_i) - \underbrace{E_X\left[\mathcal{D}^q_{n,n'}(X) \cdot \mathbb{I}_{\mathcal{X}}(X)\right]}_{=\mathcal{T}^q_{n,n'}}\Big) \\
&\quad + \left(\Delta_L^q(U_i, n, n') - E_U\left[\Delta_L^q(U, n, n')\right]\right).
\end{aligned}$$

Note that $E_U\left[\lambda_L^q(U; n, n')\right] = 0$. Under our assumptions, $\widehat{\mathcal{T}}^q_{n,n'} - \mathcal{T}^q_{n,n'}$ will possess an asymptotic linear representation, with $\lambda_L^q(U_i; n, n')$ as the influence function. The next result follows from an Mth-order approximation.

Remark 1. Fix $(n, n') \in \mathcal{N}$ and $u_2 \equiv (y_2, x_2, n_2) \in \text{Supp}(Y) \times \mathcal{X} \times \mathcal{N}$ and define

$$\Xi^q\left(u_2, n, n'\right) = \varphi^q(u_2, x_2; n, n') \cdot \mathbb{I}_{\mathcal{X}}(x_2) \cdot f_X(x_2).$$

Suppose $\varphi^q(u_2, x_1; n, n')$ and $\mathbb{I}_{\mathcal{X}}(x_1)$ are M times differentiable with respect to x_1^c with bounded derivatives for all $u_2 \in \text{Supp}(Y) \times \mathcal{X} \times \mathcal{N}$ and $x_1 \in \mathcal{X}$. Then,

$$\Delta^q\left(u_2, n, n'; h\right) = \Xi^q\left(u_2, n, n'\right) + o\left(h^M\right).$$

[5] Euclidean classes of functions are rich and diverse (Pakes and Pollard (1989), include a list of classes of functions that satisfy this property). They range from smooth (e.g., Lipschitz-continuous) to non-smooth (e.g., functions with "kinks" which are not differentiable but possess directional derivatives) functions, to discontinuous ones (e.g., indicator functions over so-called VC or "polynomial" classes of sets). This includes all the examples studied in this paper and many more cases of interest. The Euclidean property can fail if the class of functions is too rich (for example, if they consist of indicator functions over classes of sets that are too rich, like the class of all Borel sets, as is shown in Andrews (1994)). Ideally, applying our results to classes of functions not yet known to be Euclidean (e.g., classes not covered in Pakes and Pollard (1989)) should be preceded by a formal analysis of this property. In general this would be a very challenging problem as it would involve a detailed characterization of the covering numbers (or packing numbers) of said class.

Note that the above conditions are an extension of the smoothness properties in Assumption 3.1 to the functional φ^q. Combined with our bandwidth convergence restrictions this would yield

$$\begin{aligned}\lambda_L^q(U_i;n,n') =& \left(\mathcal{D}^q_{n,n'}(X_i)\cdot\mathbb{I}_{\mathcal{X}}(X_i) - E_X\left[\mathcal{D}^q_{n,n'}(X)\cdot\mathbb{I}_{\mathcal{X}}(X)\right]\right)\\ &+\left(\Xi^q\left(U_i,n,n'\right) - E_U\left[\Xi^q\left(U,n,n'\right)\right]\right)\\ &+o\left(L^{-1/2-\epsilon}\right)\quad\text{for some } \epsilon>0.\end{aligned}$$

Thus, up to a term of order $o\left(L^{-1/2}\right)$, the influence function $\lambda_L^q(U_i;n,n')$ would be the same for any h_L that satisfies our bandwidth convergence restrictions. This can be a useful result for bandwidth selection (see our discussion in Section 3.7.2 on a proposed Jackknifed-bandwidth approach).

Theorem 1. *If Assumptions 3.1–3.4 hold, then for each $n,n'\in\mathcal{N}$ and $q=1,\ldots,Q_{n,n'}$,*

$$\widehat{\mathcal{T}}^q_{n,n'} = \mathcal{T}^q_{n,n'} + \frac{1}{L}\sum_{i=1}^{L}\lambda_L^q\left(U_i;n,n'\right)+\xi_L^q(n,n'),$$

where

(i) $E\left[\lambda_L^q\left(U_i;n,n'\right)\right]=0$.
(ii) *If $P_X\left(R^q(X,z;n,n')<0\middle|X\in\mathcal{X}\right)=1$ for a.e $z\in\mathcal{Z}$ (i.e., if the contact set has measure zero), then $\lambda_L^q\left(U_i;n,n'\right)=0$ w.p.1.*
(iii) $\left|\xi_L^q(n,n')\right| = O_p\left(L^{-1/2-\epsilon}\right)$ *for some $\epsilon>0$.*

Proof. In the Appendix. □

3.4.2. Properties of $\boldsymbol{\lambda_L^q}$ and its relation to the contact sets

By construction we have $E\left[\lambda_L^q\left(U_i;n,n'\right)\right]=0$. Also as Remark 1 points out, under appropriate smoothness conditions $\lambda_L^q\left(U_i;n,n'\right)$ would be independent of h_L up to a negligible $o\left(L^{-1/2}\right)$ term (in which case we can omit the subscript L in λ^q). However, the key property of λ_L^q is its relationship with the contact sets $CO^q(n,n')$. By inspection of (5) it follows immediately that *if the contact sets have measure zero* (i.e., if $R^q\left(x,z;n,n'\right)<0$ for a.e $(x,z)\in\mathcal{W}$), then $\varphi^q(u_2,x_1;n,n')=0$ a.e. and therefore $\lambda_L(U_i;n,n')=0$ w.p.1. Summarizing,

(i) $E\left[\lambda_L^q\left(U_i;n,n'\right)\right]=0$.
(ii) If $P_X\left(R^q(X,z;n,n')<0\middle|X\in\mathcal{X}\right)=1$ for a.e $z\in\mathcal{Z}$ (i.e., if the contact set has measure zero), then $\lambda_L^q\left(U_i;n,n'\right)=0$ w.p.1.

Notice that we can have $\lambda_L^q\left(U_i;n,n'\right)=0$ w.p.1. even if the contact set has positive measure. This would occur, for example, if for a.e $u_2\in\text{Supp}(Y)\times\mathcal{X}\times\mathcal{N}$,

$$\nabla_s m^q\left(x_1,z;n,n'\right)'\widetilde{\varepsilon}(u_2,x_1,z;n,n')=0$$
$$\text{for a.e } (x_1,z)\in CO^q(n,n').$$

This case could be seen as an anomaly, but we will explicitly rule it out through the following assumption. While we believe this assumption can be relaxed under certain conditions, the asymptotic properties of our procedure, like the fact that it adapts asymptotically to the contact sets, will rely upon it.

Assumption 3.5. As defined previously, let $\varepsilon\left(y,x,z,n\right)=S(y,x,z,n)-s(x,z,n)$. Let us stack

$$\varepsilon(y,x,z;n,n')=\left(\varepsilon\left(y,x,z,n\right),\varepsilon\left(y,x,z,n'\right)\right).$$

Let $X_1\sim F_X$, $Y_2\sim F_Y$ with $(X_1,Y_2)\sim F_X\otimes F_Y$ (i.e., $X_1\perp Y_2$). Then for a.e $(z,n,n')\in\mathcal{W}$ and any $\mathcal{C}\subseteq\mathcal{X}$ such that $P_{X_1}\left[X_1\in\mathcal{C}\right]>0$,

$$P_{X_1,Y_2}\left[\nabla_s m^q\left(X_1,z;n,n\right)'\varepsilon(Y_2,X_1,z;n,n')\neq 0\middle| X_1\in\mathcal{C}\right]>0.$$

Assumption 3.5 would suffice to ensure that, if H_0 is satisfied, $\lambda_L^q\left(U_i;n,n'\right)=0$ w.p.1. if and only if $CO^q(n,n')$ has measure zero. Ruling out anomalous cases will help us in the exposition that follows. Let

$$\lambda_L\left(U_i\right)=\sum_{n,n'\in\mathcal{N}}\sum_{q=1}^{Q_{n,n'}}\lambda_L^q\left(U_i;n,n'\right).$$

By Theorem 1,

(i) $E\left[\lambda_L\left(U_i\right)\right]=0$.
(ii) If $P_X\left(R^q(X,z;n,n')<0\middle|X\in\mathcal{X}\right)=1$ for a.e $z\in\mathcal{Z}$ and each (n,n') and q (i.e., if every contact set has measure zero), then $\lambda_L\left(U_i\right)=0$ w.p.1.

By the representation result in Theorem 1, we have

$$\widehat{\mathcal{T}}=\mathcal{T}+\frac{1}{L}\sum_{i=1}^{L}\lambda_L\left(U_i\right)+\xi_L,$$
$$\text{where } \xi_L=O_p\left(L^{-1/2-\epsilon}\right)\text{ for some }\epsilon>0. \tag{6}$$

Note that the asymptotic properties of $\widehat{\mathcal{T}}$ adapt to the contact sets. This is captured by

$$\sigma_L^2=\text{Var}\left(\lambda_L\left(U_i\right)\right).$$

If H_0 is satisfied but every contact set $CO^q(n,n')$ has measure zero we will have $\sigma_L^2=0$. Otherwise (by Assumption 3.5) it will be positive. Our test will adapt to the contact sets through the properties of the influence function λ_L, and σ_L^2 will be the relevant measure of *slackness* in the inequalities (1).

3.5. A test statistic

The linear representation in (6) and the specific properties of the influence function $\lambda_L(U_i)$ are the foundation of our test. Let $\kappa_L\longrightarrow 0$ be a nonnegative sequence such that $L^\epsilon\cdot\kappa_L\longrightarrow\infty$ for any $\epsilon>0$ (e.g., $\kappa_L\propto\log(L)^{-1}$). Define

$$t_L=\frac{\sqrt{L}\cdot\widehat{\mathcal{T}}}{\max\left\{\sigma_L,\kappa_L\right\}}.$$

We can characterize the asymptotic properties of t_L in three relevant cases.

(i) If (1) is violated with positive probability over our testing range, then

$$t_L=\underbrace{\frac{\sqrt{L}\cdot\mathcal{T}}{\max\left\{\sigma_L,\kappa_L\right\}}}_{\to+\infty}+\underbrace{\frac{1}{\sqrt{L}}\sum_{i=1}^{L}\frac{\lambda_L(U_i)}{\max\left\{\sigma_L,\kappa_L\right\}}}_{=O_p(1)}+\underbrace{O_p\left(\frac{L^{-\epsilon}}{\kappa_L}\right)}_{=o_p(1)}.$$

(ii) If the restrictions in (1) are satisfied as strict inequalities w.p.1 over our testing range, then

$$t_L=O_p\left(\frac{L^{-\epsilon}}{\kappa_L}\right)=o_p(1).$$

In this case, for any $c>0$ we have $\lim_{L\to\infty}Pr\left(t_L\leq c\right)=1$.

(iii) If the restrictions in (1) are satisfied w.p.1 over our testing range but at least one of them holds with equality with positive probability,

$$t_L=\frac{1}{\sqrt{L}}\sum_{i=1}^{L}\frac{\lambda_L(U_i)}{\max\left\{\sigma_L,\kappa_L\right\}}+\underbrace{O_p\left(\frac{L^{-\epsilon}}{\kappa_L}\right)}_{=o_p(1)}.$$

In this case, for any $c > 0$ we have $\lim_{L\to\infty} Pr\,(t_L \le c) \ge \Phi(c)$, where Φ is the Standard Normal distribution.

Take any $\alpha \in (0, 1)$ and let $c_{1-\alpha}$ be the Standard Normal critical value that satisfies $\Phi\,(c_{1-\alpha}) = 1 - \alpha$. From the above results we have

(i) $\lim\limits_{L\to\infty} Pr\,(t_L \le c_{1-\alpha}) \ge 1 - \alpha$
if (1) is satisfied w.p.1 over our testing range,
(ii) $\lim\limits_{L\to\infty} Pr\,(t_L \le c_{1-\alpha}) = 0$ otherwise.

σ_L^2 is unknown but is nonparametrically identified and can be estimated as

$$\widehat{\sigma}_L^2 = \frac{1}{L}\sum_{i=1}^{L}\widehat{\lambda}_L^2(U_i).$$

The estimator $\widehat{\lambda}_L(U_i)$ for the influence function is described in the Appendix. Under the conditions leading to Theorem 1 we will have $\left|\widehat{\sigma}_L^2 - \sigma_L^2\right| = o_p(1)$. Let

$$\widehat{t}_L = \frac{\sqrt{L}\cdot\widehat{\mathcal{T}}}{\max\{\widehat{\sigma}_L, \kappa_L\}}.$$

For a target size $\alpha \in (0, 1)$, consider the rejection rule

$$\text{“}\textit{Reject } (1) \textit{ if } \widehat{t}_L > c_{1-\alpha}\text{”}. \tag{7}$$

This decision rule would have the following properties:

$$\lim_{L\to\infty} Pr\Big(\ (1) \text{ is rejected when it is true}\Big) \le \alpha,$$

$$\lim_{L\to\infty} Pr\Big(\ (1) \text{ is rejected when it is violated over our testing range}\Big) = 1.$$

3.6. A study of uniform asymptotic properties

Here we outline the uniform properties of our approach under certain assumptions about the family of distributions that produced our data. We will denote this space as $\mathcal{F}$ and for each $F \in \mathcal{F}$ we will index the various functionals involved in our test by F. Thus, we will refer to $\mathcal{T}(F)$, $\lambda_L(U, F)$, $\sigma_L(F)$, and $R^q\left(X, z; n, n', F\right) = m^q\left(s(X, z, n, F), s(X, z, n', F); n, n'\right)$.

Assumption 3.6. Let $\mathcal{F}$ be a family of distributions for U which has common support and satisfies $P_F\,[X \in \mathcal{X}] \ge \underline{p} > 0$. In addition,

(i) Every $F \in \mathcal{F}$ satisfies Assumptions 3.1, 3.2, 3.4 and 3.5.
(ii) Let

$$\mathcal{F}_W^* = \Big\{F \in \mathcal{F}: P_F\left(R^q\left(X, z, n, n', F\right) < 0\middle|X \in \mathcal{X}\right) \\ = 1 \text{ a.e } z \in \mathcal{Z}, \forall n, n' \in \mathcal{N},\ q = 1, \ldots, Q_{n,n'}\Big\}$$

$\mathcal{F}_W^*$ is therefore the subset of distributions in $\mathcal{F}$ where the inequalities are satisfied but the contact sets have F-measure zero. There exist $b > 0$ and $\epsilon > 0$ such that

$$\lim_{L\to\infty} E_F\left[\frac{\lambda_L(U, F)^{2+\epsilon}}{\sigma_L^{2+\epsilon}(F)}\right] \le b \quad \forall F \in \mathcal{F}\setminus\mathcal{F}_W^*.$$

Part (i) is meant to ensure that the linear representation in Theorem 1 is valid for each $F \in \mathcal{F}$. Part (ii) is an integrability condition which, intuitively, bounds the information about $\mathcal{T}(F)$ (and $\lambda_L(U, F)$) that is contained in the tails of F. Since one sample of observations yields little information about the tails of the distribution, allowing $\mathcal{T}(F)$ to be sufficiently sensitive to the tails of F could bring about poor (uniform) size properties for our procedure.

Assumption 3.6(ii) is analogous to the integrability condition in Romano (2004, Section 4.2). By Theorem 1, for each $F \in \mathcal{F}_W^*$ we have $\lambda_L(U, F) = 0F$-a.s (and therefore $\sigma_L(F) = 0$). Assumption 3.6(ii) ensures that, for any sequence of distributions $\{F_L\} \in \mathcal{F}$ and any nonnegative sequence $\delta_L \to 0$,

$$\lim_{L\to\infty} E_{F_L}\left[\frac{\lambda_L(U, F_L)^{2+\epsilon}}{\max\left\{\sigma_L^{2+\epsilon}(F_L), \delta_L\right\}}\right] \le b.$$

Assumption 3.6(ii) is also a stronger version of Assumption 3.5. Let us generalize our setting and assume we have a triangular array $\{U_i : 1 \le i \le L, L \ge 1\}$ that is row-wise iid with distribution F_L where $\{F_L\} \in \mathcal{F}$. To study the properties of $\widehat{t}_L$, let us focus for now on $t_L = \frac{\sqrt{L}\cdot\widehat{\mathcal{T}}}{\max\{\sigma_L(F_L),\kappa_L\}}$, the (unfeasible) statistic normalized by the true standard deviation $\sigma_L(F_L)$. The properties of t_L will be preserved for $\widehat{t}_L$ if $\mathcal{F}$ is equipped with conditions that ensure $\left|\max\{\widehat{\sigma}_L(F_L), \kappa_L\} - \max\{\sigma_L(F_L), \kappa_L\}\right| \xrightarrow{p} 0$. Let

$$\widetilde{\mathcal{F}}_W = \Big\{F \in \mathcal{F}: P_F\left(R^q\left(X, z, n, n', F\right) \le 0\middle|X \in \mathcal{X}\right) \\ = 1 \text{ a.e } z \in \mathcal{Z}, \forall n, n' \in \mathcal{N},\ q = 1, \ldots, Q_{n,n'}\Big\}.$$

$\widetilde{\mathcal{F}}_W$ is the subset of distributions such that the inequalities in (1) are satisfied. Note that $\mathcal{F}_W^* \subseteq \widetilde{\mathcal{F}}_W$. Under the conditions in Assumption 3.6,

$$\lim_{L\to\infty} P_{F_L}\,(t_L > c_{1-\alpha}) \le \alpha \quad \forall\,\{F_L\} \in \widetilde{\mathcal{F}}_W.$$

The proof combines the results in Theorem 1 and the verification of the Lindeberg condition, which would follow from Assumption 3.6 (as in Romano (2004, Lemma 1)). Furthermore if $\widetilde{\mathcal{F}}_W \setminus \mathcal{F}_W^* \ne \emptyset$ (i.e., if there are distributions in $\mathcal{F}$ that produce contact sets with positive measure), then there exists a sequence[6] $\{F_L\} \in \widetilde{\mathcal{F}}_W$ such that

$$\lim_{L\to\infty} P_{F_L}\,(t_L > c_{1-\alpha}) = \alpha.$$

Combining the above results, we would obtain

$$\lim_{L\to\infty}\sup_{F\in\widetilde{\mathcal{F}}_W} P_F\,(t_L > c_{1-\alpha}) \le \alpha, \quad \text{with} \\ \lim_{L\to\infty}\sup_{F\in\widetilde{\mathcal{F}}_W} P_F\,(t_L > c_{1-\alpha}) = \alpha \quad \text{if } \widetilde{\mathcal{F}}_W \setminus \mathcal{F}_W^* \ne \emptyset. \tag{8}$$

(8) is relevant for the size properties of our approach.

Moving on to the issue of power, there will be two objects of interest. For any sequence of distributions $\{F_L\} \in \mathcal{F}$ let

$$\delta_1 = \lim_{L\to\infty}\left[\frac{\max\{\sigma_L(F_L), \kappa_L\}}{\sigma_L(F_L)}\right] \quad \text{and}$$

$$\delta_2 = \lim_{L\to\infty}\left[\frac{\sqrt{L}\mathcal{T}(F_L)\sigma_L(F_L)}{\max\{\sigma_L(F_L), \kappa_L\}^2}\right],$$

where δ_1 and δ_2 are allowed to be ∞ (note that $\delta_1 \ge 1$ and $\delta_2 \ge 0$). Under our previous conditions,[7]

$$\lim_{L\to\infty} P_{F_L}\,(t_L > c_{1-\alpha}) = 1 - \Phi\left(\delta_1\cdot(c_{1-\alpha} - \delta_2)\right). \tag{9}$$

This expression helps us characterize the power properties of our approach. Since $\delta_1 \ge 1$, we will have asymptotic power of 1 for any sequence $\{F_L\}$ such that $\delta_2 = \infty$. Consider a sequence such that

[6] Simply choose a sequence $\{F_L\} \in \widetilde{\mathcal{F}}_W \setminus \mathcal{F}_W^*$.

[7] Recall that $\mathcal{T}(F_L) = 0$ for any $\{F_L\} \in \widetilde{\mathcal{F}}_W$ and therefore $\delta_2 = 0$ for any such sequence. Our result in (8) follows as a special case of (9) since $\delta_1 \ge 1$.

$\mathcal{T}(F_L) > 0$ and $\lim_{L\to} \mathcal{T}(F_L) = 0$ and denote $\lim_{L\to\infty} \sigma_L(F_L) = \sigma_0$. Whether or not our test has nontrivial local power against such a sequence will depend on the properties of the corresponding contact sets. Let $\mathcal{M}_0(CO)$ denote the limiting measure of the contact sets. There are two relevant cases.

- $\mathcal{M}_0(CO) > 0$: By Assumption 3.6 (specifically the fact that Assumption 3.5 is satisfied by each $F \in \mathcal{F}$) we must have $\sigma_0 > 0$ and $\delta_1 = 1$. Let $\lim_{L\to\infty} L \cdot \mathcal{T}(F_L) = d$. In this case we will have,
$$\delta_2 = \begin{cases} \infty & \text{if } d = \infty, \\ \dfrac{d}{\sigma_0} \geq \alpha & \text{if } d < \infty. \end{cases}$$
It follows that, if $d \neq 0$, our test will *always* have nontrivial local power as long as the limiting contact set has positive measure.
- $\mathcal{M}_0(CO) = 0$: These are the cases where our test can have trivial local power. We now must have $\sigma_0 = 0$. Suppose for example that $\mathcal{T}(F_L) \propto L^{-\alpha_1}$ and $\sigma_L(F_L) \propto L^{-\alpha_2}$. Then $\delta_1 = 1$ and $\delta_2 = \lim_{L\to\infty} L^{1/2+\alpha_2-\alpha_1}$. In this case our test will have nontrivial power if and only if $1/2 + \alpha_2 - \alpha_1 \leq 0$. This rules out, e.g., the case $\alpha_1 = 1/2$.

To summarize, our test will have trivial local power only if the limiting contact sets have measure zero. Nontrivial local power can be achieved in cases where the limiting contact sets have measure zero but this requires very specific conditions on the relative rates at which $\mathcal{T}(F_L)$ and $\sigma_L(F_L)$ converge to zero.

Next we need to take into account the fact that σ_L is estimated in our test. If $\lambda_L(U_i)$ were directly observed, Assumption 3.6(ii) would suffice to ensure that an appropriate LLN for triangular arrays holds (see Romano (2004, Lemma 2)) and we would have

$$\frac{\max\left\{\sum_{i=1}^{L} \lambda_L^2(U_i)/L, \kappa_L\right\}}{\max\left\{\sigma_L^2(F_L), \kappa_L\right\}} \longrightarrow 1.$$

However, $\lambda_L(U_i)$ is not observed and is replaced with a nonparametric estimator in the construction of our test. Thus, to ensure that the size and power properties described in (8) and (9) are preserved for our testing procedure it would suffice to endow $\mathcal{F}$ with conditions that guarantee that

$$\frac{\max\left\{\sum_{i=1}^{L} \widehat{\lambda}_L^2(U_i)/L, \kappa_L\right\}}{\max\left\{\sigma_L^2(F_L), \kappa_L\right\}} \longrightarrow 1.$$

Analyzing each one of the pieces that goes into the construction of $\widehat{\lambda}_L^2(U_i)$ illuminates the type of additional conditions on $\mathcal{F}$ that would suffice to ensure that[8]

$$\sup_{u\in \mathrm{Supp}(U)\cap \mathcal{W}\times\mathcal{N}} \left|\widehat{\lambda}_L(u, F_L) - \lambda_L(u, F_L)\right| \xrightarrow{p} 0$$

for sequences $\{F_L\}$ in $\mathcal{F}$. From here and Assumption 3.6, the size and power properties described in previously would be satisfied by our actual test.

The measure of contact sets also determines the local power in the test proposed in Lee et al. (2014), suggesting that the two procedures might be comparable in terms of their power; this conjecture is strongly supported by our Monte Carlo experiments. The difference is that our test asymptotically adapts to the contact sets while theirs relies on a direct estimate of such sets. Our conclusion – supported by the evidence from these experiments – will be that the two procedures complement each other, and that they complement other existing methods.

3.7. Choice of tuning parameters

A general theory for bandwidth selection in our problem is beyond the scope of this paper, but here we present some guidelines for choosing our tuning parameters. We will propose a Jackknife-based method that combines information from several bandwidth choices, each of which satisfies our general restrictions and therefore the linear representation in Theorem 1. Our Monte Carlo experiments suggest that this approach to the bandwidth selection performs well in our context.

3.7.1. Tuning parameters and scale invariance

Our inequalities will be unchanged by any proportional rescaling of the functional $R^q(X, z; n, n')$. For this reason it is desirable to set b_L and κ_L in a way that helps make our test scale-invariant. This can be done, for example, in the following way. Let

$$\begin{aligned} R(X, z) &= \sum_{q=1}^{Q_{n,n'}} \sum_{n,n'\in\mathcal{N}} R^q(X, z; n, n'), \\ \Omega^2 &= E_X\left[\int_{z\in\mathcal{Z}} R(X, z)^2 d\mathcal{P}(z)\right] \\ &\quad - \left(E_X\left[\int_{z\in\mathcal{Z}} R(X, z) d\mathcal{P}(z)\right]\right)^2. \end{aligned}$$

Next, let $\lambda_L^*(U_i)$ denote the expression that would follow if we replace $\mathbb{1}\left\{R^q(X, z; n, n') \geq 0\right\}$ with 1 everywhere in the definition of the influence function $\lambda_L(U_i)$,[9] and let

$$\Sigma_L^2 = \mathrm{Var}\left(\lambda_L(U_i)\right).$$

We use bandwidths of the form

$$\begin{aligned} &b_L = c_b \cdot \widehat{\Omega} \cdot L^{-\alpha_b}, \qquad \kappa_L = c_\kappa \cdot \widehat{\Sigma} \cdot \log(L)^{-1}, \\ &h_L = c_h \cdot \widehat{\sigma}(X) \cdot L^{-\alpha_h}. \end{aligned} \tag{10}$$

The bandwidth convergence restrictions in Assumption 3.3 can be satisfied if we set

$$\alpha_h = \frac{1}{4\cdot r} - \epsilon_h, \qquad \alpha_b = \frac{1}{4} + \epsilon_b,$$
$$\text{where } 0 < \epsilon_h \leq \frac{1}{4\cdot r\cdot(2\cdot r+1)}, \ 0 < \epsilon_b < \epsilon_h.$$

3.7.2. A jackknifed-bandwidth approach

We can simply fix the coefficients c_h, c_b and c_κ and use the corresponding bandwidths. However, we propose an approach that combines different bandwidths that result from different choices of these coefficients. We can refer to this as a *Jackknifed-bandwidth* approach, a terminology used, for example, in Honoré and Powell (2005) in semiparametric estimation settings.

[8] This is stronger than we need since we only require $\frac{\max\left\{\sum_{i=1}^{L} \widehat{\lambda}_L^2(U_i)/L, \kappa_L\right\}}{\max\left\{\sigma_L^2(F_L), \kappa_L\right\}} \longrightarrow 1$, and not necessarily $\frac{\sum_{i=1}^{L} \widehat{\lambda}_L^2(U_i)/L}{\sigma_L^2(F_L)} \longrightarrow 1$.

[9] So $\lambda_L^*(U_i)$ is the expression of the influence function if the contact set includes the entire testing range.

The idea is the following.[10] We take a collection $\left(c_h^j, c_b^j\right)_{j=1}^J$ with corresponding bandwidth sequences $h_L^j = c_{hL}^j \cdot \widehat{\sigma}(X) \cdot L^{-\alpha_h}$ and $b_L^j = c_b^j \cdot \widehat{\Omega} \cdot L^{-\alpha_b}$. Under the smoothness conditions[11] in Remark 1, up to a term $o\left(L^{-1/2}\right)$, each one of these bandwidths will have exactly the same influence function, $\lambda(U_i)$ (whose exact expression is given in Remark 1). Let $\sigma^2 = \mathrm{Var}(\lambda(U))$ and let $\widehat{t}_L^j$ denote the test-statistic constructed with (h_L^j, b_L^j). Now let $\left(w^j\right)_{j=1}^J$ denote a set of weights such that $\sum_{j=1}^J w^j = 1$. A Jackknifed-bandwidth test-statistic would be given by $\ddot{t}_L = \sum_{j=1}^J w^j \cdot \widehat{t}_L^j$. By the linear representation result in Theorem 1 and our construction, we have

$$\begin{aligned}
\ddot{t}_L &= \sum_{j=1}^J w_j \cdot \frac{\sqrt{L}\mathcal{T}}{\max\left\{\widehat{\sigma}^j, \kappa_L\right\}} \\
&\quad + \sum_{j=1}^J w_j \cdot \left(\frac{1}{\sqrt{L}} \sum_{i=1}^L \frac{\lambda(U_i)}{\max\left\{\widehat{\sigma}^j, \kappa_L\right\}}\right) + o_p(1) \\
&= \sum_{j=1}^J w_j \cdot \frac{\sqrt{L}\mathcal{T}}{\max\{\sigma, \kappa_L\} + o_p(1)} \\
&\quad + \sum_{j=1}^J w_j \cdot \left(\frac{1}{\sqrt{L}} \sum_{i=1}^L \frac{\lambda(U_i)}{\max\{\sigma, \kappa_L\}} + \underbrace{o_p\left(\frac{1}{L^{1/2} \cdot \kappa_L^2}\right)}_{=o_p(1)}\right) \\
&\quad + o_p(1) \\
&= \sum_{j=1}^J w_j \cdot \frac{\sqrt{L}\mathcal{T}}{\max\{\sigma, \kappa_L\} + o_p(1)} \\
&\quad + \frac{1}{\sqrt{L}} \sum_{i=1}^L \frac{\lambda(U_i)}{\max\{\sigma, \kappa_L\}} + o_p(1),
\end{aligned} \tag{11}$$

where the last line follows from $\sum_{j=1}^J w^j = 1$. From (11), using $\ddot{t}_L$ in our rejection rule will satisfy the properties described in Section 3. It has the potential advantage of combining the information from different bandwidth choices instead of relying on a single bandwidth choice. Furthermore, the weights w^j used in its construction can be designed to boost power (e.g., by weighting the individual statistics according to their value). In our Monte Carlo experiments we find that this Jackknifed-bandwidth approach can perform well even if we use uniform weights $w^j = 1/J$.

3.8. A comparison to existing methods

Among existing methods that can be used to test conditional moment inequalities ours should be directly compared to Lee et al. (2013) (hereafter LSW1) and Lee et al. (2014) (hereafter LSW2). Like ours, those tests use nonparametric estimators and one-sided L_p-statistics. Like us, LSW1 use critical values from the Standard Normal distribution but, unlike us, their approach is based on the *least favorable configuration* (the case where the inequalities are binding *everywhere*). This strategy renders their approach conservative. To overcome this, LSW2 propose a procedure that focuses on the contact sets. The key difference is that LSW2 rely on a direct estimate of the contact sets, while our method adapts asymptotically to the properties of contact sets. Using our notation the contact set estimates proposed in LSW2 would be of the form $\widehat{CO}(\widehat{c}_L) = \left\{(x, z) \in \mathcal{X} \times \mathcal{Z}\colon \left|\widehat{R}(x, z)\right| \le \widehat{c}_L(x, z)\right\}$.
Lee et al. (2014) study the problem of choosing $\widehat{c}_L(\cdot)$ and describe the properties of their bootstrap procedure. Qualitatively, the power properties of LSW2 are comparable to ours. Like our test, LSW2 can have trivial local power if the limiting contact sets have Lebesgue measure zero (see Lee et al. (2014, Section 6.2)). Like our method, the procedure in LSW2 requires the choice of tuning parameters. Implementing their bootstrap test requires choosing three different constants in addition to the bandwidth used in the kernel-based nonparametric estimators. In our case it requires choosing the constants related to b_L and κ_L.

Our experiments in Section 5.2 support the notion that our approach improves upon LSW1 and it complements LSW2 by providing a different way of using the information in the contact sets to give a non-conservative test. We find that the performance of our method is entirely comparable to LSW2 (better in some instances, equivalent in others and slightly outperformed in others). Furthermore, we also find that our test can perform as well or better than other non-L_p tests, including methods based on sup-norm statistics[12] like Chernozhukov et al. (2013) (hereafter CLR), or tests based on spaces of instrument functions instead of nonparametric estimators like Andrews and Shi (2013) (hereafter AS). Overall we conclude that our method complements LSW2, CLR and AS as econometric tools in problems involving tests of functional inequalities.

4. Testing models of English auctions

In this section we show that certain standard assumptions made in the empirical modeling of English auctions fit within our testing framework.

4.1. General setup

Each auction in the data is characterized by a set of observable (to the researcher) covariates describing that particular auction, X; a number of bidders, N; and a vector of bids, $\boldsymbol{B} = (B_1, \ldots, B_N)$. The joint distribution of the observables $(X, N, \boldsymbol{B})$ is thus nonparametrically identified by the data. We will maintain the following assumption throughout:

Assumption 4.1. Bidders have private values, and the joint distribution of these private values is symmetric.

Thus, we will assume that bidders have private values $\boldsymbol{V} = (V_1, \ldots, V_N)$; let $F(\cdot|x, n)$ denote the joint distribution of these valuations, conditional on $X = x$ and $N = n$.[13] Symmetry imposes the additional restriction that $F(v_1, v_2, \ldots, v_n|x, n) = F(v_{\sigma(1)}, v_{\sigma(2)}, \ldots, v_{\sigma(n)}|x, n)$ for $\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ any permutation. An independent private values (IPV) model would impose the additional restriction that $F(v_1, \ldots, v_n|x, n) = \prod_{i=1}^n F_V(v_i|x, n)$ for some univariate distribution F_V (which may or may not depend on n).

Our primary focus is to understand and test the implications of the IPV assumption in English (ascending) auctions. Unlike the case of first-price auctions (see Section 5.5 in Athey and Haile (2007)), bids in ascending auctions may be correlated even if valuations

[10] What follows can be extended to include the sequence κ_L as well. For simplicity we focus on h_L and b_L.

[11] If the smoothness conditions in Remark 1 are not assumed to hold, we can replace each c_h^j with a sequence $c_{h,L}^j$ where each $c_{h,L}^j$ has the same limit.

[12] For instance, we find that a method such as ours or LSW2 can outperform sup-norm approaches such as CLR in cases where the nonparametric functions are flat near the contact sets.

[13] This is slight abuse of notation, as the domain of $F : \Re_+^n \to [0, 1]$ depends on n, but the meaning should be clear.

are not.[14] Thus, we cannot test the IPV model in ascending auctions simply by testing for conditional covariance among bids. However, in this section we show it is still possible to derive non-parametric testable implications of ascending auctions using the properties of order statistics.

Fixing $N = n$, let $V_{1:n} \le V_{2:n} \le \cdots \le V_{n:n}$ denote the order statistics of the random vector of valuations $\boldsymbol{V}$, and $F_{k:n}(\cdot \mid x)$ the distribution of $V_{k:n}$ conditional on the realization $(X, N) = (x, n)$. Similarly, let $B_{1:n} \le \cdots \le B_{n:n}$ denote the order statistics of the random vector of bids $\boldsymbol{B}$, and $G_{k:n}(\cdot \mid x)$ the distribution of $B_{k:n}$ given $(X, N) = (x, n)$. Since bids are observed, $G_{k:n}(\cdot \mid x)$ are identified from the data.

For $k \le n$, define a function $\psi_{k:n} : [0, 1] \to [0, 1]$ by

$$\psi_{k:n}(s) = \frac{n!}{(n-k)!(k-1)!} \int_0^s t^{k-1}(1-t)^{n-k} dt.$$

For $t \in (0, 1)$, the integrand is positive, so $\psi_{k:n}$ is strictly increasing everywhere and therefore invertible. Athey and Haile (2002) observe that if n independent random variables are drawn from a distribution $H(\cdot)$, the distribution of the kth-lowest is $\psi_{k:n}(H(\cdot))$. Under an IPV model, then, for any k and n, $F_V(v|x, n) = \psi_{k:n}^{-1}(F_{k:n}(v|x))$.

4.2. Testing IPV with fixed N

In an open-outcry English auction, a bidder responds to his opponents' bids; a bidder's valuation therefore does not uniquely determine his bid. Haile and Tamer (2003) address this by imposing weak assumptions about bidder behavior. They assume bidders never bid higher than their valuations, which implies $B_{k:n} \le V_{k:n}$; and they assume that bidders never allow the auction to end at a price they could profitably beat, which implies that for $k < n$, $V_{k:n} \le B_{n:n} + \Delta$, where Δ is the minimum bid increment at the end of the auction. In order to create an IPV test that will have power even in such an unstructured model of bidding, we must place some structure on how we might expect independence to be violated, that is, what we see as the alternative hypothesis to IPV. We do this in a theoretically general and non-parametric way:

Assumption 4.2. For each n and x, the joint distribution $F(\cdot|x, n)$ is such that for any v and i, the probability $\Pr(V_i < v | X = x, N = n, \|\{j \ne i : V_j < v\}\| = k)$ is nondecreasing in k.

This formulation of correlated private values was introduced by Aradillas-López et al. (2013). There, we show (Lemma 1) that Assumption 4.2 holds under all the standard models of symmetric, correlated private values: specifically, affiliated private values, conditionally-independent private values, and IPV with unobserved heterogeneity. We will say that Assumption 4.2 holds strictly at (x, n, v) if $\Pr(V_i < v | X = x, N = n, \|\{j \ne i : V_j < v\}\| = k)$ is not the same for all k.

Proposition 1. *Under IPV, for any (x, n, v) and any $k \le n-2$,*

$$\psi_{k:n}^{-1}(F_{k:n}(v|x)) = \psi_{n-1:n}^{-1}(F_{n-1:n}(v|x)).$$

On the other hand, if values are not independent, then

$$\psi_{k:n}^{-1}(F_{k:n}(v|x)) < \psi_{n-1:n}^{-1}(F_{n-1:n}(v|x))$$

at any (x, n, v) where Assumption 4.2 holds strictly.

Under Haile-and-Tamer bidding, $B_{k:n} \le V_{k:n}$, and therefore $G_{k:n}(v|x) \ge F_{k:n}(v|x)$. Similarly, $V_{n-1:n} \le B_{n:n} + \Delta$, and therefore $F_{n-1:n}(v|x) \ge G_{n:n}^{\Delta}(v|x)$, where $G_{n:n}^{\Delta}(\cdot|x)$ is the distribution (given $X = x$) of $B_{n:n} + \Delta$. Under Haile-and-Tamer bidding, then, we can base a test of IPV on the relationship

$$\psi_{k:n}^{-1}(G_{k:n}(v|x)) \ge \psi_{n-1:n}^{-1}\left(G_{n:n}^{\Delta}(v|x)\right) \tag{12}$$

knowing that this must hold under IPV, but should fail when values are not independent, Assumption 4.2 holds strictly, and there is "sufficiently little slack" in the Haile-and-Tamer bounds.

In fact, (12) was noted by Haile and Tamer (2003, Remark 2), who point out that it could be used as a test of the IPV model. The second part of Proposition 1, however, is a new result, and shows that this test can have power against the standard alternative hypotheses to independence.

We can represent (12) as an instance of (1), and in particular, as a case of (1′). The decision variables are $Y = (B_{1:N}, \ldots, B_{N-1:n}, B_{N:N} + \Delta)$. The index variable Z could be any real-valued random variable with the property that Support$(V) \subseteq$ Support(Z).[15] For a given $w \equiv (x, z)$, y and n, the structural function S is the vector-valued indicator $S(y, w, n) = \mathbb{1}\{y \le z\}$, so

$$\begin{aligned} s(w, n) &= E_{Y|X,N}\left[S(Y, w, n) \middle| X = x, N = n\right] \\ &= \left(G_{1:n}(z|x), \ldots, G_{n-1:n}(z|x), G_{n:n}^{\Delta}(z|x)\right). \end{aligned}$$

For each n, the model involves $Q_n = n - 2$ transformations $\{m^q\}$, with

$$m^q(s(w, n); n) = \psi_{n-1:n}^{-1}\left(G_{n:n}^{\Delta}(z|x)\right) - \psi_{q:n}^{-1}\left(G_{q:n}(z|x)\right), \quad q = 1, \ldots, n-2.$$

As noted above, the power of the test depends on how close $G_{n:n}^{\Delta}$ is to $F_{n-1:n}$ and $G_{k:n}$ to $F_{k:n}$. If Δ is small and the top two bids are close together in most auctions, then the first inequality will not have much slack: since $G_{n:n}^{\Delta} \le F_{n-1:n} \le G_{n-1:n}$, if $G_{n-1:n}$ and $G_{n:n}^{\Delta}$ are close together, $F_{n-1:n}$ must be close to $G_{n:n}^{\Delta}$. Thus, the real concern is whether $G_{k:n}$ is close to $F_{k:n}$ for $k \le n-2$—that is, whether losing bidders other than the second-highest bid close to their valuations. Song (2004) considers the possibility that the "top two losers" bid close to their values, even if the others do not, implying $G_{n-2:n} \approx F_{n-2:n}$; this would be enough for our test to have power. Unfortunately, there is no easy way to check this in the data; and if only the highest losing bidder approaches his value, a test based on (12) may have little power.

As a result, we consider another approach to testing the IPV model, which relies only on transaction prices (or the winning and highest losing bids) but requires variation in the number of bidders.

4.3. Testing IPV using variation in N

Exploiting variation in N requires an assumption about the nature of this variation. We will assume that variation in the number of bidders is independent of the realization of their valuations. To formalize this condition, let $F_m^n(\cdot \mid x)$ denote the joint distribution of m bidders drawn at random from an auction with n bidders, conditional on $X = x$.[16]

14 For example, in the absence of jump bidding, the winning bid will always be very close to the second-highest, even when the highest valuation is much higher than the second-highest.

15 This is to ensure the inequality is tested everywhere; in practice, however, for econometric/regularity reasons, we will apply our test on a strict subset of Support(V).

16 Since $F(\cdot \mid x, n)$ is symmetric, $F_m^n(v_1, \ldots, v_m \mid x) = F(v_1, \ldots, v_m, \infty, \infty, \ldots, \infty \mid x, n)$.

Definition. Values are **independent of** N if $F_m^n(\cdot \mid x) = F_m^{n'}(\cdot \mid x)$ for all (x, n, n', m).

Under the IPV model, this simply means that the marginal distribution $F_V(\cdot \mid x, n)$ does not depend on n. This assumption has been used in Haile et al. (2003), Guerre et al. (2009), Gillen (2009), and Aradillas-López et al. (2013), and has been termed an "exclusion restriction" since N is excluded from the distribution $F_V(\cdot \mid x)$.

This test, and the subsequent ones, are based only on the distribution of the second-highest valuation $V_{N-1:N}$ as N changes. In many applications, including in Aradillas-López et al. (2013), this valuation is assumed to be equal to the transaction price $B_{N:N}$—as it would be in a "button auction". If bidders only increase their bids by the minimum amount toward the end of the auction, this should be true to within a bid increment under the Haile-and-Tamer bidding assumptions. Here, we present the test under this stronger assumption, as this is what we use in our application; we show in our online Supplementary Appendix how to modify the tests to be based on the weaker Haile-and-Tamer assumptions.

Proposition 2. *Assume $B_{n:n} = V_{n-1:n}$ and values are independent of N.*

(a) *Under IPV, for any (x, n, n', v),*

$$\psi_{n-1:n}^{-1}\left(G_{n:n}(v|x)\right) = \psi_{n'-1:n'}^{-1}\left(G_{n':n'}(v|x)\right). \tag{13}$$

(b) *Under Assumption 4.2 (nonnegatively correlated private values), for any (x, n, n', v),*

$$n > n' \longrightarrow \psi_{n-1:n}^{-1}\left(G_{n:n}(v|x)\right) \geq \psi_{n'-1:n'}^{-1}\left(G_{n':n'}(v|x)\right) \tag{14}$$

and (14) holds strictly wherever Assumption 4.2 holds strictly at (x, n, v).

(13) was proposed by Athey and Haile (2002) as a possible basis for a test of the IPV model. (See also the discussion in Athey and Haile, 2007.) The drawback is that a rejection of (13) could follow from a violation either of IPV or of the exclusion restriction. (14), on the other hand, is a new result, and contributes to our testing strategy in two ways. First, it ensures that (13) has power against all the standard models of positively-correlated values when the exclusion restriction holds. More importantly, it provides a testable implication of the exclusion restriction itself which does not depend on independence of values. Studying the power properties of our test against every possible alternative auction model of interest is impractical for reasons of space. In our online Supplementary Appendix, we show that (14) has power as a test of the exclusion restriction. Specifically, we show fairly general conditions under which a correlated private values model, combined with either of the two standard models of endogenous entry in auctions (those of Levin and Smith (1994) and Samuelson (1985)), would lead to a violation of (14). (This is also illustrated in a numerical example in Section 4.6.) Thus, if the data violates (13) but satisfies (14), this supports the hypothesis that the failure of (13) is caused by a violation of IPV rather than a violation of the exclusion restriction. If this is indeed the case – values are correlated, but independent of the number of bidders – then both upper and lower bounds are identified for the seller's expected profit and optimal reserve price, using the approach laid out in Aradillas-López et al. (2013).[17]

To gain intuition for Proposition 2, consider what happens to the distribution of transaction prices as N increases. As N increases, transaction prices get stochastically higher (the distribution shifts to the right), since the price is set by the second-highest of a bigger group. (Pinkse and Tan, 2005 refer to this as the *sampling effect*.) If values are IPV and F_V does not vary with n, Proposition 2 says that this must happen at a particular "speed"—that is, for each v, $F_{n-1:n}(v)$ must fall exactly fast enough so that $\psi_{n-1:n}^{-1}(F_{n-1:n}(v))$ remains constant.

Relative to that benchmark, correlation of values *slows down* the sampling effect—if values are correlated, then each incremental bidder has less impact on transaction price, as bidder values are more prone to be close together. So if values are correlated but independent of N, $F_{n-1:n}(v)$ falls more slowly than under IPV, and $\psi_{n-1:n}^{-1}(F_{n-1:n}(v))$ therefore increases with n.

On the other hand, violations of the exclusion restriction would likely be due to a positive relationship between valuations and N—that is, endogenous participation favoring auctions for more-valuable prizes. This would augment the sampling effect, causing $F_{n-1:n}(v)$ to fall more quickly than under IPV; provided this effect was stronger than the slowing-down due to correlation, it would result in $\psi_{n-1:n}^{-1}(F_{n-1:n}(v))$ decreasing with n. As noted above, we have shown that even in the presence of correlation, the test of (14) has power against a fairly wide class of "typical" violations of the exclusion restriction.

The restriction in (14) is an instance of (1) where the decision variable is $Y = B_{N:N}$ and (as before) the index variable Z could be any real-valued random variable with the property that $\text{Support}(V) \subseteq \text{Support}(Z)$. The structural function is $S(y, w, n) = S(y, z) = \mathbb{1}\{y \leq z\}$ and

$$s(w, n) = E_{Y|X,N}\left[S(Y, w, n)\middle|X = x, N = n\right] = G_{n:n}(z|x).$$

For each n, n' the model involves a single transformation (i.e., $Q_{n,n'} = 1$) given by

$$m\left(s(w, n); s\left(w, n'\right); n, n'\right) = \left(\psi_{n'-1:n'}^{-1}\left(G_{n':n'}(z|x)\right) - \psi_{n-1:n}^{-1}\left(G_{n:n}(z|x)\right)\right) \cdot \mathbb{1}\left\{n > n'\right\}.$$

The restriction in (13) involves an equality between transformations of conditional moments. Notice however that we can frame it as the combination of two inequalities: (14) along with its reverse inequality. If either fails, we would reject (13).

4.4. Testing IPV when the exclusion restriction fails

When the exclusion restriction is rejected, of course, (13) no longer offers a test of IPV. Without any restriction on how $F_V(\cdot|x, n)$ can vary with n, the IPV model is just-identified from transaction price data, and therefore not testable. However, a natural restriction would be a general positive relationship between the number of bidders and their valuations. This can be formalized as the following condition[18]:

Assumption 4.3. If valuations are IPV but the distribution $F_V(\cdot \mid x, n)$ depends on n, then it does so in such a way that (for any x) $n > n'$ implies $F_V(\cdot \mid x, n) \succsim_{FOSD} F_V(\cdot \mid x, n')$.

Proposition 3. *Assume $B_{n:n} = V_{n-1:n}$. Under IPV and Assumption 4.3,*

$$n > n' \longrightarrow \psi_{n-1:n}^{-1}\left(G_{n:n}(v|x)\right) \leq \psi_{n'-1:n'}^{-1}\left(G_{n':n'}(v|x)\right) \tag{15}$$

for all (x, n, n', v).

[17] In that paper, we also show that the same upper bound on profit, and a weaker upper bound on the optimal reserve price, still hold if the exclusion restriction is violated.

[18] In Aradillas-López et al. (2013), we generalize this notion of valuations being "stochastically increasing in N" to settings with correlated values, and show conditions under which it follows from three different models of endogenous entry.

Table 1
Overview of observable implications.

Eq.	Test of	N	Bidding assumptions
(12)	IPV	Fixed	Haile-and-Tamer bidding
(13)	IPV $\wedge V \perp N$	Variable	Transaction price $= V_{n-1:n}$
(14)	$V \perp N$	Variable	Transaction price $= V_{n-1:n}$
(15)	IPV	Variable	Transaction price $= V_{n-1:n}$

Rejecting (14) or (15) would also reject (13).

Observe that Proposition 3 gives the opposite conclusion as part (b) of Proposition 2; that is, relative to the benchmark of (13), violations of the exclusion restriction work in the opposite direction as correlation among values. When both the exclusion restriction and independence fail, (15) need not always have power as a test of IPV. Nevertheless, a rejection would serve as evidence against IPV.

For testing, (15) can be framed as an instance of (1). The decision variable is $Y = B_{N:N}$, and (as before) the index variable Z could be any real-valued random variable with the property that $\text{Support}(V) \subseteq \text{Support}(Z)$. The structural transformation is $S(y, w, n) = S(y, z) = \mathbb{1}\{y \leq z\}$ and

$$s(w, n) = E_{Y|X,N}\left[S(Y, w, n)\middle|X = x, N = n\right] = G_{n:n}(z|x).$$

For each n, n' the model involves a single transformation (i.e., $Q_{n,n'} = 1$) given by

$$m\left(s(w, n); s\left(w, n'\right); n, n'\right) = \left(\psi^{-1}_{n-1:n}\left(G_{n:n}(z|x)\right) - \psi^{-1}_{n'-1:n'}\left(G_{n':n'}(z|x)\right)\right) \cdot \mathbb{1}\left\{n > n'\right\}.$$

4.5. Summarizing the results from auction models

Table 1 summarizes the various tests we have derived above:

The tests based on (13)–(15) all have analogs based on the Haile-and-Tamer bidding assumptions; these are given in our online Supplementary Appendix.

4.6. Illustration of Propositions 2 and 3

Eqs. (13)–(15) are claims that $\psi^{-1}_{n-1:n}(G_{n:n}(v|x))$ is constant, increasing, or decreasing in n, respectively. To illustrate how $\psi^{-1}_{n-1:n}(G_{n:n}(v|x))$ behaves for various types of data-generating processes, we graph in Fig. 1 its value (as a function of v) for different values of N under four versions of a parametric example. For the example, there are no observable covariates X; bidder values are *i.i.d.* draws from a log-normal distribution, $\log(V_i) \sim N(\mu, \sigma^2)$, with $\sigma^2 = 0.5$ throughout but μ potentially variable. The four cases are as follows:

1. Values are IPV and independent of N: specifically, $\mu = 2.25$ for every N.
2. Values are independent of N, but correlated with each other via conditional independence: regardless of N, $\mu = 2.0$ with probability $\frac{1}{2}$ and 2.5 with probability $\frac{1}{2}$. (Variation in μ induces correlation among values.)
3. Values are IPV, but the distribution varies with N: specifically, $\mu = 2 + 0.05N$.
4. Values are correlated with each other, and with N. $\mu = 2.5$ or 1.5 with probabilities $\frac{1}{3}$ and $\frac{2}{3}$ respectively, and N is determined endogenously via equilibrium play of the entry game described in Samuelson (1985). There are 12 potential bidders, each of whom learns μ and his own valuation before deciding whether to pay a cost of 10 to participate in the auction. Bidders play a symmetric, cutoff-strategy equilibrium, with the cutoff value varying with μ[19]; this induces a positive relationship between N and μ, and therefore between N and valuations.

Fig. 1 shows plots of $\psi^{-1}_{n-1:n}(G_{n:n}(v))$ against v for various values of n for each scenario. For DGP1, we would fail to reject (13), and conclude (correctly) that the data was consistent with both IPV and the exclusion restriction. For DGP2, we would reject (13) but fail to reject (14), (correctly) rejecting IPV but not the exclusion restriction. For DGP3, we would reject both (13) and (14), but fail to reject (15), concluding (correctly) that the data was consistent with an IPV model violating the exclusion restriction. Finally, for DGP4, we would reject all three tests, concluding (correctly) that both IPV and the exclusion restriction failed.[20]

5. Monte Carlo experiments

This section has two objectives. First, we study the finite-sample performance of our test in auction models. Second, we compare our procedure against alternative approaches in the existing literature. In all cases our goal is to study the sensitivity of our results to various choices of the tuning parameters (bandwidths) involved.

5.1. Auction designs

First, we applied the tests of (14) and (15) on simulated auction data (our main application). The data is based on modifications of the last three DGPs from Section 4.6 to include a single auction-specific covariate X. In all cases, the maximum number of bidders was 12, and valuations satisfy $\log(V_i) \sim \mathcal{N}\left(\mu, \sigma^2\right)$. We fixed $\sigma^2 = 0.5$, let $X \sim \mathcal{N}(0, 1)$, and generated μ in ways analogous to DGPs 2, 3 and 4 above in the following way:

(A) Values are independent of N conditional on X but are correlated with each other (even conditional on X) in the following way. Let $\varepsilon \sim \mathcal{N}(0, 1)$ such that $\varepsilon \perp X$. If $X + \varepsilon > 0$ then $\mu = 2.0$, otherwise $\mu = 2.5$.

(B) Values are IPV conditional on X, but the distribution of values varies with N in the following way. If $X < 0$ then $\mu = 1.7 + .05 \cdot N$. Otherwise $\mu = 2.3 + .05 \cdot N$. Note that on average we have $\mu = 2 + .05 \cdot N$.

(C) Let $\varepsilon \sim \mathcal{N}(0, 1)$ with $\varepsilon \perp X$ and let $c_{\frac{1}{3}}$ denote the $\frac{1}{3}$rd quantile from the Standard Normal distribution. Then $\mu = 2.5$ if $\frac{X+\varepsilon}{\sqrt{2}} < c_{\frac{1}{3}}$ and $\mu = 1.5$ otherwise. Everything else is as described in DGP4 above, with N being determined endogenously (given μ) by the equilibrium outcome of an entry game.

By design, DGP(A) satisfies (14) almost surely as a *strict* inequality, while it violates (15). The reverse is true for DGP(B). Both (14) and (15) are violated with positive probability in DGP(C), but each of these inequalities is satisfied over some range of x and n. Table 2 summarizes the predicted asymptotic behavior of our econometric test for each one of these designs.

[19] When $\mu = 2.5$, the entry cutoff is 30.57, which is exceeded by 9.7% of bidders; when $\mu = 1.5$, the cutoff is 15.54, which is exceeded by 3.9% of bidders. By Bayes' Law, then, $\Pr(\mu = 2.5|N = n)$ is increasing in n.

[20] When neither IPV nor the exclusion restriction holds, it is not necessarily the case that (14) and (15) will both be violated: a similar example based on a different entry model (that of Levin and Smith, 1994) leads to distributions satisfying (15) everywhere.

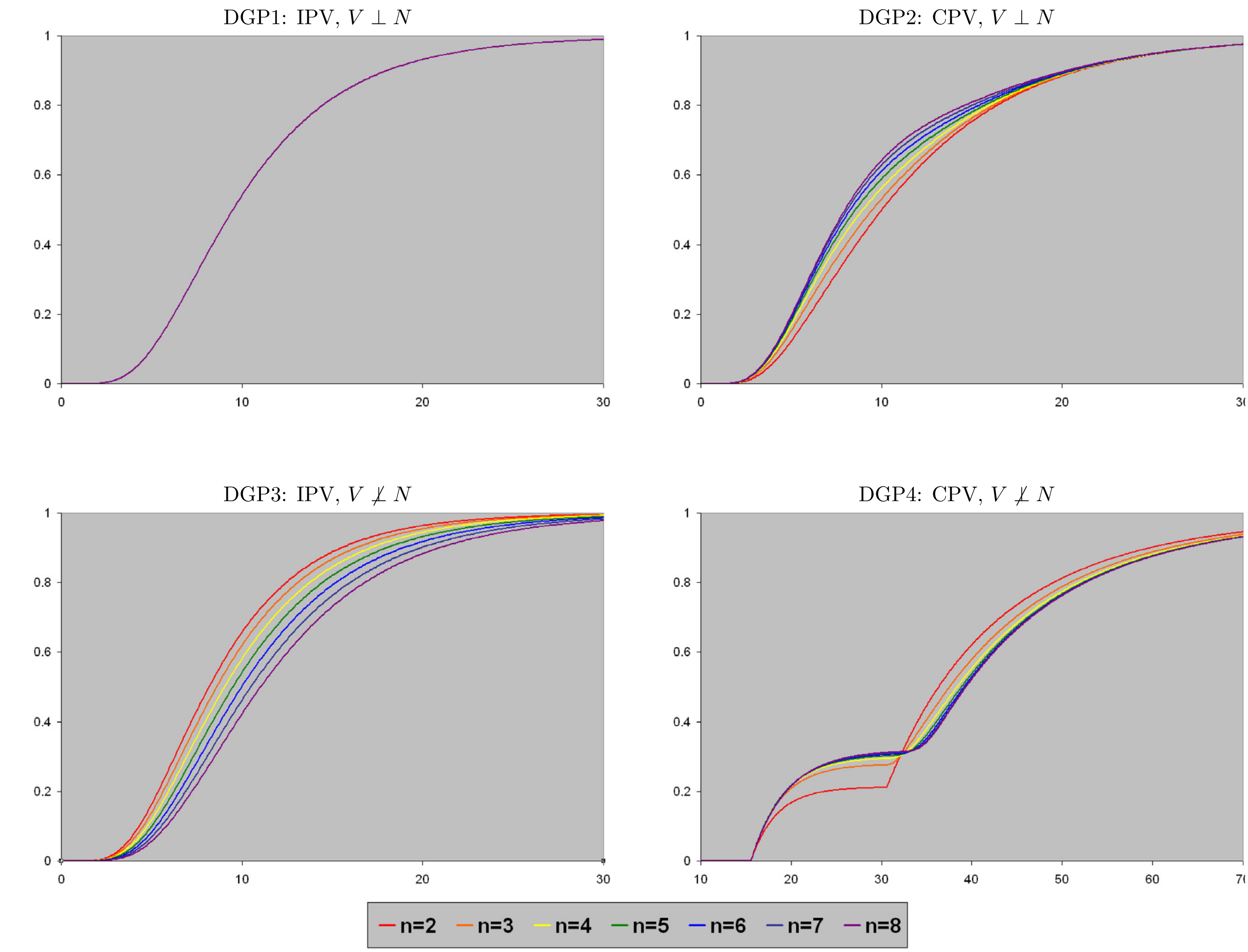

Fig. 1. $\psi^{-1}_{n-1:n}(G_{n:n}(v))$ against v under four scenarios.

5.1.1. Kernels and bandwidths

We have $r = 1$ (one continuous observable X). The smallest order of the kernel that can satisfy Assumption 3.3 is $M = 2r+1 = 3$. We employed a kernel of the type

$$K(\psi) = \sum_{\ell=1}^{2} c_\ell \cdot \left(s^2 - \psi^2\right)^{2\ell} \cdot \mathbb{1}\{|\psi| \leq s\},$$

with $s = 5 \cdot r = 5$ (analogously to Section 4.4 of Aradillas-López et al., 2013). The coefficients c_1, c_2 were chosen to ensure that $K(\cdot)$ was a kernel of order $M = 4$. Our bandwidths are given as (10), with $\epsilon_h = \frac{9}{10} \cdot \frac{1}{4 \cdot r \cdot (2 \cdot r+1)}$ and $\epsilon_b = \frac{9}{10} \cdot \epsilon_b$ ($r = 1$ as described previously). This yielded $\alpha_h \approx 0.28$ and $\alpha_b \approx 0.32$.

Choice of tuning coefficients $\boldsymbol{c_h}$, $\boldsymbol{c_b}$ *and* $\boldsymbol{c_\kappa}$

For simplicity we fixed $c_\kappa = 10^{-1}$ in all cases and we focused on analyzing the sensitivity of our results to the choices of c_h and c_b. We applied our tests in two different ways:

(i) **Different individual combinations of** $(\boldsymbol{c_b}, \boldsymbol{c_h})$. We obtained results for each of the following combinations: $(c_b, c_h) \in \{0.01, 0.10, 0.50, 1.0\} \times \{0.20, 0.40, 0.60\}$. In total this includes 12 different combinations and covers a relatively wide range of values for these bandwidth coefficients.

(ii) **Jackknifed-bandwidth approach**. We also applied the Jackknife approach outlined in Section 3.7.2. Take each of the $j = 1, \ldots, 12$ combinations (c_{b_j}, c_{h_j}) described above and let $\widehat{t_{L_j}}$ denote the corresponding test-statistic. For a collection of non-negative weights $(w_j)_{j=1}^{12}$ such that $\sum_{j=1}^{12} w_j = 1$ we constructed a Jackknifed test-statistic as $\ddot{t}_L = \sum_{j=1}^{12} w_j \cdot \widehat{t_{L_j}}$ in two different ways. First we used uniform weights: $w_j = 1/12$ for each j. Then we tried weights aimed at increasing the power of our procedure by giving more weight to combinations $\left(c_{b_j}, c_{h_j}\right)$ that produced larger values of $\widehat{t_{L_j}}$. To achieve this we used

$$w_j = \frac{\Phi(\widehat{t_{L_j}})^2}{\sum_{k=1}^{12} \Phi(\widehat{t_{L_k}})^2}, \quad \text{where } \Phi(\cdot) = \text{Standard Normal cdf.} \quad (16)$$

We applied our tests for samples of size $L = 100$, 250, 500 and 1000.

Testing range

We let $\mathcal{N} = \{2, \ldots, 12\}$ and $\mathcal{Z} = \left[B_{(0.01)}, B_{(0.99)}\right]$, where $B_{(p)} = p$th quantile of transaction price observed in the data. Let

Table 2
Asymptotic rejection probabilities predicted for our test.

	DGP(A)	DGP(B)	DGP(C)
$\lim_{L\to\infty} Pr$ (Reject (14))	0	1	1
$\lim_{L\to\infty} Pr$ (Reject (15))	1	0	1

These asymptotic predictions are valid at any significance level.

$\widehat{f}_X^{(.005)} = .005$th quantile of $\widehat{f}_X(\cdot)$. We used

$$\mathcal{X} = \left\{x: \widehat{f}_X(x) \geq \widehat{f}_X^{(.005)} \text{ and } 10^{-4} \leq \widehat{G}_{k:n}(z|x) \leq 1 - 10^{-4} \right.$$
$$\left. \forall (n,z) \in \mathcal{N} \times \mathcal{Z} \text{ and each } 2 \leq k \leq n \right\}.$$

5.1.2. Experiment results for auction models

Step-by-step details of the construction of our test statistics can be found in our online Supplementary Appendix, at http://www.personal.psu.edu/aza12/testing_auctions_supplement.pdf. Our results are detailed in Appendix B. At very small sample sizes ($L = 100$) our tests had relatively little power, but power grows very quickly, and our results approach the asymptotic predictions summarized in Table 2 starting at $L = 250$. Our results are sensitive to the bandwidth choice, and we find that our Jackknife appears to be a very good solution to this, as it yielded excellent finite-sample results in all cases. By combining the information from different bandwidths, our Jackknifed-based test-statistics are better able to detect whether there is evidence against the null hypothesis; furthermore, this does not appear to come at the cost of size distortions. Choosing Jackknife weights according to (16) led to dramatic power improvements, but even the naive uniform weights produced good results. Choosing an "optimal" bandwidth may very well lead to superior finite-sample results than our Jackknife procedure, but what constitutes an optimal choice in our context is a complicated problem and is left for future research. However, we are greatly encouraged by the simplicity and performance of the proposed Jackknife approach. As we will see below, this simple approach also led to excellent results when we compare our procedure against competing tests.

5.2. Comparison of finite-sample performance with existing methods

Here, we analyze the performance of our approach for the Monte Carlo designs analyzed in Andrews and Shi (2013) and later in Lee et al. (2014). We will compare our results against those of Andrews and Shi (2013, Section 10.3) (hereafter AS), (Chernozhukov et al., 2013) (hereafter CLR), (Lee et al., 2013) (LSW1), and Lee et al. (2014) (LSW2). The null hypothesis is simply $H_0 \;:\; E\left[Y-\theta \middle| X=x\right] \leq 0$ for each $x \in \mathcal{X}$, for a fixed θ. The DGP for (Y,X) is given by $Y = f(X) + U$, where $X \sim \text{Unif}\left[-2,2\right]$ and U is a truncated normal of the form $U = \min\left\{\max\left\{-3,\widetilde{U}\right\},3\right\}$ with $\widetilde{U} \sim \mathcal{N}(0,1)$. AS considered two different functions for f: $f_{AS1}(x) = D\cdot\phi\left(x^{10}\right)$, and $f_{AS2}(x) = D\cdot\max\left\{\phi\left((x-1.5)^{10}\right),\phi\left((x+1.5)^{10}\right)\right\}$, where ϕ is the standard normal PDF. Both of these functions are characterized by steep slopes, with f_{AS1} and f_{AS2} having single and double-plateau shapes respectively. AS applied their test to the values $D = 1$ and $D = 5$ and compared their test with CLR and LSW1. More recently these results were compared by LSW2 against their test. Here we collect the results in AS, CLR, LSW1 and LSW2 and compare them against ours. As in LSW2, our testing range is $\mathcal{X} = [-1.8, 1.8]$. As weighing function we use $\omega(x) = f_X(x)$ (i.e., we use a density-weighted version of our test-statistic).

Unlike our auction designs, this simple experiment has a key feature we can exploit. By its simple nature, if we reject H_0 for some θ_0 we should automatically reject it for any $\theta < \theta_0$. Thus the asymptotic probability of rejecting H_0 should be nonincreasing in θ, suggesting that letting b_L and κ_L decrease as θ decreases might boost the power of our procedure[21] (see footnote 4 and the related discussion in Section 3.7). For example, we can use $c_b(\theta) = g(\theta)\cdot c_b$ and $c_\kappa(\theta) = g(\theta)\cdot c_\kappa$, with $g(\cdot)$ being a monotonically increasing function. This can help our finite-sample rejection probabilities be nonincreasing in θ, as should be the case asymptotically. Below we perform the test for two values $\theta_1 > \theta_2$ and as part of our experiments we tried $g(\theta_1) = 1$ and $g(\theta_2) = 0.10$. Following AS and LSW2 we tested H_0 for two different values of θ: $\theta_1 = \sup_{x\in\mathcal{X}} f(x)$ and $\theta_2 = \sup_{x\in\mathcal{X}} f(x) - 0.02$. By construction, H_0 is satisfied for $\theta = \theta_1$ and violated for $\theta = \theta_2$. As mentioned previously, our procedure is properly comparable to LSW1 and LSW2. We expect to observe the following: (i) Our test should have better power properties than LSW1, which is more conservative by construction. (ii) Our test should have power properties comparable to LSW2, which like ours relies on the properties of contact sets.

The results are summarized in Tables B.3 and B.4, with additional results included in our online Supplementary Appendix. We find that our Jackknifed-bandwidth approach produced excellent results. When compared to the other tests we confirm that ours performs better in all cases than LSW1, and that it is comparable to LSW2. Our results were slightly better than LSW2 (in terms of power) in cases where $D = 5$, while LSW2 performed slightly better in the case $f = f_{AS1}$ and $D = 1$. Our results were quite similar in the case $f = f_{AS2}$ and $D = 1$. We also found that our method performed particularly well in very small sample sizes ($L = 100$ and $L = 250$). As the sample size grows above 250 both procedures essentially produce equivalent results. Our results are also comparable to AS in all cases except $f = f_{AS1}$ and $D = 5$, where ours (along with LSW2) produced superior results. As LSW2 point out, the relative poor performance of CLR can be attributed to the plateau-shaped feature of the function f in our designs. If f were sharply peaked, CLR would perform better than AS and (as confirmed in unreported simulations by Lee et al. (2014)) CLR would also perform better than LSW2. This is reasonable since CLR is based on the sup-norm statistic while LSW2 (and ours) is based on one-sided L_p-statistics. Therefore we conjecture that CLR would also perform better than our test in such cases. Overall we conclude that AS, CLR, LSW2 and our approach complement each other.

6. Application to USFS timber data

6.1. Timber auctions

Finally, we apply our tests to data from timber auctions run by the United States Forest Service. A number of other papers have studied Forest Service auctions empirically. Nearly all have done so within the framework of independent private values.[22]

[21] Also note that if we reject H_0 for some θ_0, then $\mathbb{1}\{E[Y|X]-\theta \geq \theta\} = \mathbb{1}\{\theta_0 \geq E[Y|X]-\theta \geq \theta\}$ for any $\theta < \theta_0$. This information can be used in the construction of our influence function and the corresponding estimator of its variance, σ_L.

Table 3
Test results on ascending auction timber data.

	Eq. (12)	Test of IPV	N Fixed	Bidding assumptions Haile-and-Tamer bidding	
Results					
	$c_b = 0.01$				
	$c_h = 0.6$	$c_h = 0.8$	$c_h = 1.0$	$c_h = 1.2$	$c_h = 1.4$
$\widehat{t}$ Outcome*	18.2437 *Reject*	28.2473 *Reject*	32.1576 *Reject*	35.9888 *Reject*	38.07608 *Reject*
	Eq. (14)	Test of $V \perp N$	N Variable	Bidding assumptions $B_{n:n} = V_{n-1:n}$	
Results					
	$c_b = 0.01$				
	$c_h = 0.6$	$c_h = 0.8$	$c_h = 1.0$	$c_h = 1.2$	$c_h = 1.4$
$\widehat{t}$ Outcome*	0.2745 *Fail to reject*	0.0228 *Fail to reject*	0.0008 *Fail to reject*	0.0043 *Fail to reject*	0.0003 *Fail to reject*
	Eq. (15)	Test of IPV	N Variable	Bidding assumptions $B_{n:n} = V_{n-1:n}$	
Results					
	$c_b = 0.01$				
	$c_h = 0.6$	$c_h = 0.8$	$c_h = 1.0$	$c_h = 1.2$	$c_h = 1.4$
$\widehat{t}$ Outcome*	11.5319 *reject*	13.6534 *reject*	14.2701 *reject*	14.4783 *reject*	14.5934 *reject*

* Critical values for rejection are 1.645 for $\alpha = 5\%$ and 2.326 for $\alpha = 1\%$.

Two recent papers, however, have found indirect evidence of correlation among valuations. Athey et al. (2011) estimate a model allowing for unobserved heterogeneity on data from first-price auctions. In Aradillas-López et al. (2013), we estimate a model allowing for correlated values on English auction data; we find the estimates (of expected profit as a function of reserve price) differ significantly from estimates made under the assumption of independence. Thus, while independence is a standard assumption in empirical work, both in general and applied to these particular auctions, there is some recent evidence to suggest this assumption might be worrisome.

6.2. Data

Data on all USFS timber auctions held between 1978 and 1996 was made available to us by Phil Haile. We focus on the auctions held between 1982 and 1990, as the reserve price policy in place was stable during that period, and the reserve prices used were generally recognized not to be binding, allowing us to infer the number of potential bidders from the number who submitted bids.[23] We use auctions from Region 6 (mostly Oregon), which relative to other regions provides a large sample of English auctions. We use the same conventions as Haile and Tamer (2003) to select auctions most likely to satisfy the assumption of private values.

We control for *six* auction covariates which have been emphasized in the previous literature as being relevant demand shifters: the density of timber (timber volume over acres in the tract, which we label X^1); the government's appraisal value of the timber (which we label X^2); the estimated profit from manufacturing the timber (sales value minus manufacturing cost, X^3); the estimated harvesting cost (per unit of timber, X^4); the species concentration (the HHI (Herfindahl index) computed as a function of the volume of various species present, X^5); and the total volume of timber sold in the six months prior to each auction (as a measure of the bidding firms' existing inventory, X^6). Bids and monetary covariates are all measured in 1983 dollars. We let $X = (X^1, X^2, X^3, X^4, X^5, X^6)$ refer to the vector of covariates, and $X_i = (X_i^1, X_i^2, X_i^3, X_i^4, X_i^5, X_i^6)$ the data corresponding to the ith auction. X was treated as a continuously distributed random vector. We drop auctions with $N = 1$ (since there is no second-highest bidder to whose value we can link the transaction price) and $N = 12$ (as this appears to be top-coding for "more than 11"). Thus, our range for N is $\mathcal{N} = \{2, 3, \dots, 11\}$, which leaves us with a sample of $L = 2034$ auctions.

6.3. Kernels, bandwidths and testing range

We used the same types of kernels as in our Monte Carlo experiments in Section 5.1.1 given $r = 6$. We employed a kernel of order $M = 14$ of a multiplicative form $K(\psi_1, \dots, \psi_6) = \prod_{\ell=1}^{6} k(\psi_\ell)$, where

$$k(\psi) = \sum_{\ell=1}^{7} c_\ell \cdot \left(s^2 - \psi^2\right)^{2\ell} \cdot \mathbb{1}\{|\psi| \le s\}.$$

Like in our Monte Carlo experiments the support was set to $s = 5 \cdot r$ ($s = 30$ in this case). This is also the same type of kernel used in Aradillas-López et al. (2013, Section 4.4). Our bandwidths h_L, b_L and c_L were chosen using the expressions in (10), with $\epsilon_h = \frac{9}{10} \cdot \frac{1}{4 \cdot r \cdot (2 \cdot r + 1)}$ and $\epsilon_b = \frac{9}{10} \cdot \epsilon_b$ (and $r = 6$ in this case). Our Monte Carlo experiments for auctions models suggested that very small bandwidths could lead to size distortions or power loss, which

22 See, e.g., Baldwin et al. (1997); Haile (2001); Haile et al. (2003); Lu and Perrigne (2008); Athey and Levin (2001); and Haile and Tamer (2003).

23 Campo et al. (2002) write, "It is well known that this reserve price does not act as a screening device to participating", and perform analysis that confirms that "the possible screening effect of the reserve price is negligible" (p. 33). See also Haile (2001), Froeb and McAfee (1988), and Haile and Tamer (2003).

could likely be exacerbated in this case with $r = 6$ continuous conditioning variables. To avoid these potential issues we used $c_h \geq 0.6$ in our results. Next we describe our testing range. We use $\mathcal{N} = \{2, 3, \ldots, 11\}$ and $\mathcal{Z} = \left[B_{(0.01)}, B_{(0.99)}\right]$, where $B_{(p)}$ = pth-quantile of the observed transaction price. Let $\widehat{f}_X^{(.005)}$ = .005th quantile of $\widehat{f}_X(\cdot)$. We use

$$\mathcal{X} = \left\{x: \widehat{f}_X(x) \geq \widehat{f}_X^{(.005)} \text{ and } 10^{-4} \leq \widehat{G}_{k:n}(z|x) \leq 1 - 10^{-4} \right.$$
$$\left. \forall 2 \leq k \leq n, n \in \mathcal{N}, z \in \mathcal{Z}\right\}.$$

6.4. Results

We computed our test for different values of c_b and c_h, fixing $c_\kappa = 0.10$ throughout. Table 3 shows the results for the tests of (12), (14) and (15) for all the values of c_h studied and for $c_b = 0.01$; for the sample sizes we had our results were very robust to changes in c_b. The accept/reject results are consistent across all the tuning parameters used.[24]

These results paint a consistent picture of the timber data. Both testing methods – comparing winning to losing bids in auctions of the same size, and comparing transaction prices across auctions of different sizes – allow us to reject independence of valuations, and instead give strong evidence of positive correlation among valuations.[25] On the other hand, we fail to reject a model of correlated values which are independent of N; thus, the exclusion restriction appears plausible in the ascending auction data.

7. Conclusion

In this paper, we considered testing of economic models whose testable implications involve inequality comparisons between nonlinear transformations of nonparametric conditional moments. Our motivating example was specification tests in ascending auctions, but this setup extends to multiple examples of interest. Because many commonly-used models in economics fit this description, it is important to have econometric tools capable of testing these restrictions in a computationally feasible way in the presence of rich covariate data. The test we propose satisfies these requirements while improving upon existing, conservative methods by depending asymptotically on the properties of contact sets (the regions where the inequalities are binding) instead of relying on least-favorable configurations. As such, our econometric procedure complements the approach of Lee et al. (2014) (LSW2), which also exploits the contact sets. Our contribution is to introduce a test that asymptotically adapts to the contact sets instead of relying on an estimate for such sets like the test in LSW2. The asymptotic properties of our test are comparable to those in LSW2 and they both improve upon existing, conservative approaches based on least-favorable configurations. As our Monte Carlo experiments suggest, they also complement other existing procedures, performing better in some settings. Our main economic application (and motivation) involved testing for independence in bidders' private values in ascending auctions. Applying our test to data from the United States Forest Service timber auctions, we found clear evidence to reject the IPV model in favor of a model of correlated private values. Because the IPV assumption is at the heart of key auction theory results, this finding has significant policy implications, which are analyzed in detail in Aradillas-López et al. (2013).

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Appendix A. Econometrics

A.1. Proof of Theorem 1

A step-by-step detailed proof is found in our online Supplemental Appendix, which can be found at http://www.personal.psu.edu/aza12/testing_auctions_supplement.pdf. Here we summarize the main steps. Let

$$\begin{aligned}\widetilde{v}_i^\ell(x,z,n;h) &= \left(\frac{S^\ell(Y_i,x,z,n) - s^\ell(x,z,n)}{f_{X,N}(x,n)}\right) \\ &\quad \cdot \mathcal{K}(X_i - x; h) \cdot \mathbb{1}\{N_i = n\}, \\ v_i(x,z,n,n';h) &= \left(\widetilde{v}_i(x,z,n;h)', \widetilde{v}_i(x,z,n';h)'\right)',\end{aligned}$$

and $v_i^q(x,z,n,n';h) = \nabla_s m^q\left(x,z;n,n'\right)' v_i(x,z,n,n';h)$. In the first step of the proof we show that, under the conditions in Theorem 1,

$$\begin{aligned}&Pr\left(\sup_{(x,z,n,n')\in\mathcal{W}}\left|\widehat{R}^q(x,z;n,n') - R^q(x,z;n,n')\right| \geq b_L\right) \\ &\leq \overline{K}_1 \cdot \exp\left\{-\sqrt{L} \cdot h_L^r\left(\overline{K}_2 \cdot b_L - \overline{K}_3 \cdot h_L^M\right)\right\}, \qquad (A.1)\end{aligned}$$

for some positive constants $\overline{K}_1$, $\overline{K}_2$ and $\overline{K}_3$. And

$$\begin{aligned}&\widehat{R}^q(x,z;n,n') - R^q(x,z;n,n') \\ &= \frac{1}{L \cdot h_L^r}\sum_{i=1}^L v_i^q\left(x,z,n,n';h_L\right) + \xi_L^q(x,z,n,n'), \\ &\text{where } \sup_{(x,z,n,n')\in\mathcal{W}}\left\|\xi_L^q(x,z,n,n')\right\| = O_p\left(\frac{\log(L)^2}{L \cdot h_L^r}\right). \qquad (A.2)\end{aligned}$$

From here we study the properties of $\widehat{\mathcal{T}}_{n,n'}^q(z) = \frac{1}{L}\sum_{i=1}^L \widehat{R}^q(X_i,z;n,n') \cdot \mathbb{1}\left\{\widehat{R}^q(X_i,z;n,n') \geq -b_L\right\} \cdot \mathbb{I}_{\mathcal{X}}(X_i)$. Using (A.1) and (A.2), we show that

$$\begin{aligned}\widehat{\mathcal{T}}_{n,n'}^q(z) &= \frac{1}{L}\sum_{i=1}^L \max\left\{R^q(X_i,z;n,n'),0\right\} \cdot \mathbb{I}_{\mathcal{X}}(X_i) \\ &\quad + \frac{1}{L}\sum_{i=1}^L\left(\widehat{R}^q(X_i,z;n,n') - R^q(X_i,z;n,n')\right) \\ &\quad \cdot \mathbb{1}\left\{R^q(X_i,z;n,n') \geq 0\right\} \cdot \mathbb{I}_{\mathcal{X}}(X_i) + \zeta_L^q(z,n,n'), \qquad (A.3)\end{aligned}$$

[24] Since we reject (12) and (15) and fail to reject (14) for every bandwidth combination tried, there was no need to apply the Jackknife procedure described above, as any weighted average of the bandwidth-dependent test statistics would lead to the same accept/reject decision.

[25] As we noted previously, rejection of (15) implies automatically that (13) is rejected, too.

where $\sup_{(z,n,n')\in\mathcal{W}}\left|\zeta_L^q(z,n,n')\right| = O_p\left(L^{-1/2-\epsilon}\right)$ for some $\epsilon > 0$. The first term on the right hand side of (A.3) is a U-statistic and we proceed to study its asymptotic properties using its Hoeffding decomposition (Serfling, 1980) and results from U-process theory (Sherman, 1994). Let $\varepsilon(y,x,z,n) = S(y,x,z,n) - s(x,z,n)$, and let

$$\widetilde{\varepsilon}(u_2,x_1,z;n,n') = \left(\frac{\varepsilon(y_2,x_1,z,n)'\cdot\mathbb{1}\{n_2=n\}}{f_{X,N}(x_1,n)}, \frac{\varepsilon\left(y_2,x_1,z,n'\right)'\cdot\mathbb{1}\left\{n_2=n'\right\}}{f_{X,N}(x_1,n')}\right)',$$

$$\phi^q\left(u_2,x_1,z;n,n'\right) = \left[\nabla_s m^q(x_1,z;n,n')'\widetilde{\varepsilon}(u_2,x_1,z;n,n')\right]\cdot\mathbb{1}\left\{R^q(x_1,z;n,n')\geq 0\right\},$$

$$f^q(x_1,u_2,z,n,n';h) = \phi^q\left(u_2,x_1,z;n,n'\right)\cdot\mathbb{I}_{\mathcal{X}}(x_1)\cdot\frac{1}{h^r}\mathcal{H}(x_2-x_1;h).$$

Next, let

$$\Delta_L^q\left(u,z,n,n'\right) = E_X\left[f^q\left(X,u,z,n,n';h_L\right)\right],$$

$$\lambda_L^q\left(U_i,z;n,n'\right) = \left(\max\left\{R^q(X_i,z;n,n'),0\right\}\cdot\mathbb{I}_{\mathcal{X}}(X_i) - \mathcal{T}_{n,n'}^q(z)\right) + \left(\Delta_L^q\left(U_i,z,n,n'\right) - E_U\left[\Delta_L^q\left(U,z,n,n'\right)\right]\right),$$

(recall that $\mathcal{T}_{n,n'}^q(z) = E_X\left[\max\left\{R^q(X,z;n,n'),0\right\}\cdot\mathbb{I}_{\mathcal{X}}(X)\right]$). Under the conditions of Theorem 1, the Hoeffding decomposition of the U-Statistic on the right hand side of (A.3) yields

$$\widehat{\mathcal{T}}_{n,n'}^q(z) = \mathcal{T}_{n,n'}^q(z) + \frac{1}{L}\sum_{i=1}^{L}\lambda_L^q\left(U_i,z;n,n'\right) + \xi_L^q(z,n,n'), \tag{A.4}$$
where $\left|\xi_L^q(n,n')\right| = O_p\left(L^{-1/2-\epsilon}\right)$ for some $\epsilon > 0$.

Examining the structure of $\lambda_L^q\left(U_i,z;n,n'\right)$ it is easy to see that for each $z\in\mathcal{Z}$,

(i) $E\left[\lambda_L^q\left(U_i,z;n,n'\right)\right] = 0$.
(ii) If $P_X\left(R^q(X,z;n,n') < 0 \middle| X\in\mathcal{X}\right) = 1$, then $\lambda_L^q\left(U_i,z;n,n'\right) = 0$ w.p.1. That is, if the contact set for z has measure zero then $\lambda_L^q\left(U_i,z;n,n'\right) = 0$ almost surely.

Using the previous results we obtain the asymptotic properties of $\widehat{\mathcal{T}}_{n,n'}^q$ described Theorem 1. Recall that $\widehat{\mathcal{T}}_{n,n'}^q = \int\widehat{\mathcal{T}}_{n,n'}^q(z)d\mathcal{P}(z)$ and $\mathcal{T}_{n,n'}^q = \int\mathcal{T}_{n,n'}^q(z)d\mathcal{P}(z)$. Let

$$\varphi^q\left(u_2,x_1;n,n'\right) = \int_{z\in\mathcal{Z}}\left[\nabla_s m^q(x_1,z;n,n')'\widetilde{\varepsilon}(u_2,x_1,z;n,n')\right]\cdot\mathbb{1}\left\{R^q(x_1,z;n,n')\geq 0\right\}d\mathcal{P}(z),$$

$$f^q(x_1,u_2,n,n';h) = \varphi^q\left(u_2,x_1;n,n'\right)\cdot\mathbb{I}_{\mathcal{X}}(x_1)\cdot\frac{1}{h^r}\mathcal{H}(x_2-x_1;h),$$

$$\Delta^q(u_2,n,n';h) = E_X\left[f^q(X,u_2,n,n';h)\right],$$

$$\Delta^q(u_2,n,n';h_L) \equiv \Delta_L^q(u_2,n,n'),$$

and

$$\mathcal{D}_{n,n'}^q(X) = \int_{z\in\mathcal{Z}}\max\left\{R^q(X,z;n,n'),0\right\}d\mathcal{P}(z),$$

$$\lambda_L^q(U_i;n,n') = \Big(\mathcal{D}_{n,n'}^q(X_i)\cdot\mathbb{I}_{\mathcal{X}}(X_i) - \underbrace{E_X\left[\mathcal{D}_{n,n'}^q(X)\cdot\mathbb{I}_{\mathcal{X}}(X)\right]}_{=\mathcal{T}_{n,n'}^q}\Big) + \left(\Delta_L^q(U_i,n,n') - E_U\left[\Delta_L^q(U,n,n')\right]\right). \tag{A.5}$$

From (A.4), we have

$$\widehat{\mathcal{T}}_{n,n'}^q = \mathcal{T}_{n,n'}^q + \frac{1}{L}\sum_{i=1}^{L}\lambda_L^q\left(U_i;n,n'\right) + \xi_L^q(n,n'),$$

where $\left|\xi_L^q(n,n')\right| = O_p\left(L^{-1/2-\epsilon}\right)$ for some $\epsilon > 0$.

Note that

(i) $E\left[\lambda_L^q\left(U_i;n,n'\right)\right] = 0$.
(ii) If $P_X\left(R^q(X,z;n,n') < 0\middle|X\in\mathcal{X}\right) = 1$ for a.e $z\in\mathcal{Z}$ (i.e., if the contact set has measure zero), then $\lambda_L^q\left(U_i;n,n'\right) = 0$ w.p.1.

This proves Theorem 1. □

Appendix B. Appendix—Monte Carlo experiment results

Tables B.1 and B.2 describe our results for the auction design experiments. Table B.1 summarizes the results from our bandwidth-Jackknife approach. For comparison, Table B.1 presents the results for *each* one of the individual combinations of constants (c_b, c_h) used to construct the bandwidths. Tables B.3 and B.4 summarize the Jackknife-bandwidth results from the Monte Carlo experiments used to compare our method against competing approaches, as described in Section 5.2. Our online Supplemental Appendix includes the full set of results of our test for *each* one of the individual combinations of constants (c_b, c_h) used to construct the bandwidths.

Table B.1
Monte Carlo results for our auction designs using the Jackknifed-bandwidth approach.

Rejection rates												
L	Jackknife with uniform weights						Jackknife with weights as in (16)					
	DGP(A)		DGP(B)		DGP(C)		DGP(A)		DGP(B)		DGP(C)	
	(14)	(15)	(14)	(15)	(14)	(15)	(14)	(15)	(14)	(15)	(14)	(15)
100	.000	.000	.000	.000	.000	.108	.000	.030	.000	.000	.052	.518
250	.000	.026	.000	.000	.944	.980	.000	.520	.000	.002	.986	.994
500	.000	.932	.020	.000	1.00	1.00	.000	.994	.418	.024	1.00	1.00
1000	.000	1.00	.994	.000	1.00	1.00	.000	1.00	1.00	.046	1.00	1.00

Results from 500 simulations.

Table B.2
Monte Carlo results for our auction designs for different combinations of the bandwidth constants (c_b, c_h).

Rejection rates

	DGP(A)		DGP(B)		DGP(C)		DGP(A)		DGP(B)		DGP(C)		DGP(A)		DGP(B)		DGP(C)	
	(14)	(15)	(14)	(15)	(14)	(15)	(14)	(15)	(14)	(15)	(14)	(15)	(14)	(15)	(14)	(15)	(14)	(15)
L	$c_b = .01$ $c_h = .20$						$c_b = .01$ $c_h = .40$						$c_b = .01$ $c_h = .60$					
100	.000	.000	.000	.000	.160	.442	.000	.004	.000	.002	.256	.416	.000	.032	.000	.004	.242	.316
250	.000	.018	.000	.000	.794	.882	.014	.414	.010	.038	.708	.792	.014	.316	.040	.018	.664	.656
500	.022	.568	.006	.028	.986	.986	.042	.722	.284	.070	.960	.950	.026	.640	.434	.014	.882	.890
1000	.066	.986	.336	.164	1.00	1.00	.088	.946	.930	.056	1.00	.994	.096	.898	.986	.024	.994	.972
	$c_b = .10$ $c_h = .20$						$c_b = .10$ $c_h = .40$						$c_b = .10$ $c_h = .60$					
100	.000	.000	.000	.000	.146	.474	.000	.014	.000	.000	.240	.336	.000	.026	.000	.002	.230	.294
250	.000	.024	.000	.000	.808	.868	.016	.398	.018	.036	.732	.744	.010	.330	.024	.018	.598	.644
500	.014	.618	.018	.032	.986	.974	.032	.752	.272	.062	.944	.926	.008	.604	.408	.024	.870	.856
1000	.086	.986	.318	.116	1.00	1.00	.076	.954	.928	.058	.998	.994	.074	.884	.980	.004	.992	.976
	$c_b = .50$ $c_h = .20$						$c_b = .50$ $c_h = .40$						$c_b = .50$ $c_h = .60$					
100	.000	.000	.000	.000	.096	.390	.000	.010	.000	.002	.188	.326	.002	.048	.000	.002	.174	.264
250	.000	.018	.000	.002	.748	.826	.010	.424	.014	.046	.678	.648	.008	.310	.030	.008	.554	.522
500	.022	.570	.006	.046	.962	.970	.026	.722	.318	.078	.914	.844	.012	.568	.364	.008	.858	.808
1000	.068	.968	.342	.152	1.00	1.00	.058	.916	.904	.062	.996	.994	.034	.848	.980	.012	.982	.978
	$c_b = 1.0$ $c_h = .20$						$c_b = 1.0$ $c_h = .40$						$c_b = 1.0$ $c_h = .60$					
100	.000	.002	.000	.000	.078	.316	.000	.012	.000	.002	.142	.244	.000	.020	.000	.000	.116	.206
250	.000	.026	.000	.002	.638	.776	.016	.400	.020	.038	.538	.612	.004	.318	.042	.012	.434	.492
500	.018	.562	.006	.040	.932	.958	.036	.676	.312	.080	.850	.842	.008	.564	.322	.012	.736	.738
1000	.078	.962	.296	.122	1.00	1.00	.018	.888	.926	.050	.980	.996	.022	.782	.972	.002	.934	.942

Results from 500 simulations.

Table B.3
Results of the test "$H_0 : E[Y - \theta|X = x] \leq 0\ \forall x \in \mathcal{X}$", for $\theta = \theta_1 \equiv \max\{f(x): x \in \mathcal{X}\}$ (i.e., H_0 is satisfied).

Rejection rates. Nominal level $\alpha = 0.05$.

DGP	L	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)
		AS		CLR		LSW1	LSW2			AGQ							
		CvM	KS	Series	Local Linear					$c_b = 0.01$			$c_b = 1.0$			Jackknifed bandwidths	
							$C_{CS} =$			$c_h =$			$c_h =$			Uniform weights	Weights as (16)
							0.4	0.5	0.6	0.2	0.4	0.6	0.2	0.4	0.6		
$f = f_{AS1}$ $D = 1$	100	.014	.014	.293	.196	.000	.020	.010	.001	.000	.000	.002	.000	.000	.000	.000	.000
	250	.025	.027	.195	.107	.000	.049	.040	.029	.020	.004	.002	.004	.004	.000	.000	.000
	500	.025	.030	.128	.075	.000	.032	.024	.023	.038	.022	.034	.008	.006	.006	.000	.000
	1000	.029	.034	.091	.065	.000	.038	.029	.027	.066	.020	.020	.024	.014	.014	.000	.000
$f = f_{AS1}$ $D = 5$	100	.000	.000	.606	.287	.000	.004	.001	.000	.000	.000	.000	.000	.000	.000	.000	.000
	250	.000	.000	.317	.144	.000	.047	.037	.025	.000	.000	.000	.000	.000	.000	.000	.000
	500	.000	.000	.167	.092	.000	.037	.028	.024	.000	.000	.000	.000	.000	.000	.000	.000
	1000	.000	.000	.100	.073	.000	.035	.032	.032	.000	.000	.000	.000	.000	.000	.000	.000
$f = f_{AS2}$ $D = 1$	100	.030	.031	.380	.279	.000	.013	.009	.007	.000	.000	.000	.000	.000	.000	.000	.000
	250	.031	.036	.238	.146	.000	.048	.035	.027	.000	.000	.000	.000	.000	.000	.000	.000
	500	.037	.043	.146	.100	.000	.034	.029	.024	.000	.000	.000	.000	.000	.000	.000	.000
	1000	.031	.037	.099	.073	.000	.051	.043	.038	.000	.000	.000	.000	.000	.000	.000	.000
$f = f_{AS2}$ $D = 5$	100	.002	.001	.679	.345	.000	.002	.001	.000	.000	.000	.000	.000	.000	.000	.000	.000
	250	.003	.002	.388	.174	.000	.048	.035	.024	.000	.000	.000	.000	.000	.000	.000	.000
	500	.006	.006	.192	.110	.000	.036	.029	.027	.000	.000	.000	.000	.000	.000	.000	.000
	1000	.006	.009	.107	.082	.000	.057	.050	.042	.000	.000	.000	.000	.000	.000	.000	.000

• Figures in columns (1)–(5) come from Table V in Andrews and Shi (2013) and are also reproduced in Table 3 of Lee et al. (2014).
• Figures in columns (6)–(8) are from Table 3 in Lee et al. (2014). Columns (9)–(16) present our results based on 500 simulations.
• LSW1 refers to the test in Lee et al. (2013) based on conservative Standard Normal critical values from the least favorable configuration.
• LSW2 refers to the test in Lee et al. (2014) which constructs bootstrap critical values based on estimates of the contact sets.
• The tuning parameter C_{CS} is used in the estimation of the contact sets in LSW2 (see Lee et al., 2014 for details).

Table B.4
Results of the test "$H_0 : E[Y - \theta|X = x] \leq 0\ \forall x \in \mathcal{X}$", for $\theta = \theta_2 \equiv \max\{f(x): x \in \mathcal{X}\} - 0.02$ (i.e., H_0 is violated).

Rejection rates. Nominal level $\alpha = 0.05$.

DGP	L	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)
		AS		CLR		LSW1	LSW2			AGQ							
		CvM	KS	Series	Local Linear					$c_b = 0.01$			$c_b = 1.0$			Jackknifed bandwidths	
							$C_{CS} =$			$c_h =$			$c_h =$			Uniform weights	Weights as (16)
							0.4	0.5	0.6	0.2	0.4	0.6	0.2	0.4	0.6		
$f = f_{AS1}$ $D = 1$	100	.160	.110	.120	.170	.020	.190	.100	.050	.648	.318	.148	.584	.238	.138	.086	.280
	250	.430	.330	.180	.310	.080	.560	.510	.460	.752	.376	.190	.664	.320	.184	.258	.556
	500	.750	.630	.280	.500	.300	.830	.820	.800	.790	.438	.270	.740	.322	.210	.448	.746
	1000	.970	.930	.430	.740	.750	.980	.980	.980	.896	.516	.338	.820	.472	.276	.746	.930
$f = f_{AS1}$ $D = 5$	100	.000	.000	.090	.110	.010	.060	.020	.000	.732	.476	.284	.628	.364	.284	.400	.712
	250	.000	.000	.150	.270	.040	.520	.460	.380	.894	.610	.552	.818	.570	.458	.938	.988
	500	.030	.010	.230	.440	.180	.810	.790	.770	.972	.860	.746	.932	.786	.702	1.00	1.00
	1000	.300	.110	.390	.670	.600	.970	.970	.970	.998	.960	.950	.982	.928	.934	1.00	1.00
$f = f_{AS2}$ $D = 1$	100	.300	.210	.110	.160	.100	.310	.210	.140	.694	.374	.212	.608	.326	.182	.212	.538
	250	.700	.540	.170	.340	.350	.730	.680	.650	.854	.490	.344	.790	.402	.220	.722	.898
	500	.940	.850	.300	.530	.740	.940	.940	.920	.938	.660	.448	.880	.490	.364	.974	.992
	1000	1.00	.990	.450	.770	.980	1.00	1.00	1.00	.968	.706	.526	.924	.592	.388	.992	1.00
$f = f_{AS2}$ $D = 5$	100	.050	.010	.090	.120	.050	.110	.050	.030	.576	.352	.350	.584	.332	.288	.306	.558
	250	.340	.170	.140	.300	.250	.700	.650	.580	.862	.722	.726	.818	.646	.618	.986	1.00
	500	.770	.580	.260	.490	.640	.930	.920	.910	.956	.888	.894	.932	.818	.866	1.00	1.00
	1000	.990	.960	.410	.710	.960	1.00	1.00	1.00	.994	.976	.986	.986	.938	.970	1.00	1.00

• Figures in columns (1)–(5) come from Table V in Andrews and Shi (2013) and are also reproduced in Table 4 of Lee et al. (2014).
• Figures in columns (6)–(8) are from Table 4 in Lee et al. (2014). Columns (9)–(16) present our results based on 500 simulations.
• As reported in Table 4 of Lee et al. (2014), figures in columns (1)–(5) are "CP-corrected", while those in columns (6)–(8) are not "CP-corrected".
• See the notes in Table B.3.

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